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Approximately 2-Absorbing Submodule

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ABSTRACT

In this paper present and study the notion of approximately 2-Absorbing submodules as generalization of 2-Absorbing submodules. Many new properties of this class obtain.

MSC..

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1. Introduction

Throughout this paper R is a commutative ring with $1 \neq 0$ and M is a unitary R-module. A. Y. Darainm and F. Soheilnia in [1] introduced the concept of 2-Absorbing submodule where a proper submodule N of M is called 2-absorbing submodule of M if whenever a, b $\in R, m \in M$ and $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N: M)$.

Our concern in this paper is define and study the concept approximately 2-Absorbing submodule and give many new properties and characterizations.

2. Approximately 2-Absorbing Submodule

<u>Definition 2.1</u>: let N < M , N is called Approximately 2-Absorbing Submodule if whenever a, b ∈ R, m ∈ M, abm ∈ N, then either $a^2m \in N$ or $b^2m \in N$ or $(ab)^2 \in (N: M)$.

<u>Remarks and Example 2.2</u>:

1. every 2- absorbing submodule $\Longrightarrow$ Approximately 2-Absorbing Submodule

   proof:

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let a, b ∈ R , abm ∈ N ⟶ am ∈ N or bm ∈ N or ab ∈ (N:M). Then we get

a^2m ∈ N or b^2m ∈ N or $(a\ b)^2$ ∈ (N:M) but the converse is not true for example

M = Z_{24} asZ - Module let N = <8> N is not 2 – absorbing by 2.2.2 ∈ N but 2.2 ∉ (N: M) = 8Z and 2.2 ∉ M. Now N is approximately 2- absorbing

2.2.2 ∈ N we get $2^2$2=8 ∈ N or 2.4.1 ∈ N we get 4^2.1=16 ∈ N or 2^2.2 ∈ N or $(2.2)^2$ = 16 ∈ 8 Z = (N: M) or 4.2.2 ∈ N or 4^2.2 ∈ N or 8.1.1 ∈ N or 8^2.1 ∈ N.

2. "The intersection of each pair of distinct prime submodules of R-module M is Approximately 2-Absorbing Submodule "[1].
3. It is clear that every prime submodule is Approximately 2-Absorbing Submodule. However, the converse is not truein general, for example: Consider $z_6$as Z-module, $< 0 >$ not prime submodule of $z_6$since 2.3 = 0 but 3∉ $< 0 >$ and 2 ∉ $(< 0 >: z_6)$ but $< 0 >=< 2 > \cap < 3 >$ and is approximately 2- absorbing sub module of z_6 as Z-module by part 1.
4. Let N be Approximately 2-Absorbing Submodule of M, then for each A ⊆ M , either A ⊆N or A ∩N is

   Approximately 2-Absorbing Submodule of A

Proof: Suppose A ⊈ N then to prove A ∩ N is T of A

Let abx ∈ A ∩ N with N is approximately 2 absorbing and x∈ M , a, b ∈ R

We get a^2 x∈ N or b^2 x ∈ N or $(ab)^2$ ∈ (N:M) then a^2 x ∈ A ∩ N or b^2 x ∈ A ∩ N or $(ab)^2$ ∈ (A ∩ N : A)

5. Let N ,W be two submodules of an R-module M and N < W . If N is Approximately 2-Absorbing Submodule of M , then N is Approximately 2-Absorbing Submodule of W

proof: If W = M nothing to prove. Let abx ∈ N, a, b, ∈ R, x ∈ w $then$ x ∈ M. But N is T of M so either a^2x ∈ N or b^2x ∈ N or $(ab)^2$ ∈ (N: M) and since N <W implies (N: M) ≤ (N:W) then either a^2x ∈ N or b^2x ∈ N or $(ab)^2$ ∈ (N:W) hence N is Approximately 2-Absorbing Submodule of W

6. sine every quasi-prime submodule is 2-absorbing, (where a proper submodule N of M is called quasi-prime submodule if whenever a, b ∈ R, m∈M and abm∈ N , then am ∈ N or bm ∈ N)[2] hence it is Approximately 2-Absorbing Submodule. However Approximately 2-Absorbing Submodule may not be quasi-prime , as the following example shown : Consider the Z-module Z. The submodule N =4Z is Approximately 2-Absorbing Submodule of Z since, if a,b,c ∈ Z with abc ∈ 4Z = N , then at least two of a,b,c are even . Hence either $(ab)^2$ ∈ N or a^2c∈ N or b^2c ∈ N. But 4Z is not quasi-prime , since 2.2.1 ∈4Z but 2.1∉4Z.
7. Let Nand W be two submodules of an R-module M such that N ≅ W

If N is Approximately 2-Absorbing Submodule, it is not necessary that W is as the following example explains this :

Consider the Z -module Z , the submodule 2Z is T but 2Z ≅ 30Z and 30Z is not T since 2.3.5 = 30 ∈ 30Z but 2^2.5 ∉ 30$Zand$$3^2$.5 ∉ 30Z and $(2.3)^2$ = 36∉ 30Z.

8. The intersection of two Approximately 2-Absorbing Submodule need not be approximately 2-Absorbing Submodule for example : 6 Z and 5 Z are Approximately 2-Absorbing Submodulein the Z -module Z , but 6 Z ∩ 5 Z = 30 Z which is not approximately 2-absorbing.

Proposition 2.3 : Let $\varphi: M \longrightarrow M'$ be an R-epimorphism . If W is Approximately 2-Absorbing Submodule of M' , then φ^{-1}(W) is Approximately 2-Absorbing Submodule of M .

proof: Let φ : M⟶ M' epimorphism ,If w is Approximately 2-Absorbing Submodule then φ^{-1}(W) is Approximately 2-Absorbing Submodule of $M < M'$ then $\varphi^{-1}(W) < M$ since φ epimorphism,let bam ∈ φ^{1}(W) a, b ∈ R, m∈ M a b φ (m) ∈ W and as is W is T then either b^2 θ (m) ∈ w or $(ab)^2 \in (W: M')$ or a^2 φ (m) ∈ W if a^2 φ (m) ∈ W *then* a^2 m ∈ φ^{-1}(W) if

$b^2 \varphi(m) \in w$ *then* b^2 m ∈ $(\varphi^{-1}W)$ if $(ab)^2 \in (W: M')$ *then* $(ab)^2 M' \subseteq W$ but

φ (M) ⊆ M' *so* $(ab)^2$ φ (M) ⊆ W that is $(ab)^2 M \subseteq \varphi^{-1}$ (W)

$(ab)^2 \in (\varphi^{-1}(W)\ M:)$ φ^{-1} (W) is Approximately 2-Absorbing Submodule of M.

Proposition 2.4: f : M ⟶ M′ be an epimorphism , N < M such that kerf ⊆ N then N is T of M if and only if f(N) is Approximately 2-Absorbing Submodule of M′.

proof: Let abm′ ∈ f (N) , m′ ∈ M′ , a , b ∈ R m′ = f(m) for some m ∈ M

abf(m) ∈ f(N) *so* abf (m) = f (n) , for some n ∈ N f (abm) −f (n)= 0

$we\ get\ abm - n \in kerf \subseteq N$ implies that a b m – n ∈ N but (N is T) so either $a^2m \in N$ or $b^2m \in N$ or $(ab)^2 \in (N: M)$ if $a^2m \in N$ *Then* f (m) ∈ f (N) that is a^2 f(m) ∈ f (N) so $a^2m' \in$ f (N) similarly if $b^2m \in N$ *then* f (b^2m) ∈ f (N) implies that b^2m' ∈f (N) if $(ab)^2 \in (N:M)$ *then* $(ab)^2 M \subseteq N$ and so f (($(ab)^2$ M) ⊆ f (N) implies that $(a\,b)^2 M' \subseteq$ f (N)and we get $(a\,b)^2 \in$ (f (N): M′)Then f (N) is Approximately 2-Absorbing Submodule of M′.

⟸ *let* a b m ∈ N then f (abm) ∈ f (N) so abf (m) ∈ f (N) sine f(N) is Approximately 2-Absorbing Submodule of M′ we get

1. a^2 f (m) ∈ f (N) *then* f (a^2m) = f (n), for some n ∈ N hence a^2 m – n ∈ kerf ⊆ N so a^2 m ∈ N
2. b^2 f(m) ∈ f(N) then similarly b^2 m ∈ N
3. if $(ab)^2 \in$ (f(N): M′) *then* $(ab)^2$ M′ ⊆ f (N) so $(ab)^2$ f (x) ∈ f (N),∀x ∈ M so that f (($(ab)^2$ x) = f (n), for some n ∈ N and hence $(ab)^2$ x ∈ N for each x ∈ M thus$(ab)^2 \in$ (N:R M), N is Approximately 2-Absorbing Submodule of M.

Corollary2.5: R be a ring Let , M an R- module and N , K submodules of M with K ≤ N . Then N is Approximately 2-Absorbing Submodule of M if and only if $\frac{N}{K}$ is Approximately 2-Absorbing Submodule of $\frac{M}{K}$.

Proof: Let N is Approximately 2-Absorbing Submodule of M

Let ab (m + K) ∈ $\frac{N}{K}$ then abm ∈ N with N is Approximately 2-Absorbing Submodule of M then $a^2\, m \in$ N or $b^2m \in$ N or $(ab)^2 \in (N_R:M)$ we get a^2 (m+k) ∈ $\frac{N}{K}$ or b^2 (m+k) ∈ $\frac{N}{K}$ or $(ab)^2 \in (\frac{N}{K}:_R \frac{M}{K})$

⟸ *let* abm ∈ N ⟹ abm + K ∈ $\frac{N}{K}$ With $\frac{N}{K}$ is Approximately 2-Absorbing Submodule of $\frac{M}{K}$ then a^2m+ K ∈ $\frac{N}{K}$ or b^2 (m+K) ∈ $\frac{N}{K}$ or$(ab)^2 \in (\frac{N}{K} :_R \frac{M}{K})$

We get $a^2m \in N$ or $b^2m \in N$ or $(ab)^2 \in (N :_R M)$

Proposition 2.6 : Let N a proper submodule of an R-module M . Then N is Approximately 2-Absorbing Submodule of M if and only if abK ⊆N for some a , b ∈R , K⊆M . implies $(ab)^2$∈(N:M) , or a^2K ⊆N or b^2K ⊆N Then N is Approximately 2-Absorbing Submodule of M.

proof: Let N be Approximately 2-Absorbing Submodule and abK ⊆ N suppose that $(ab)^2 \notin$ (N:M), a^2K ⊈ N and b^2K ⊈N .Then there exists m_1 , m_2 in K *such that* $a^2m_1 \notin$ N and $b^2m_2 \notin$ N,Since $abm_1 \in$ N and $(ab)^2 \notin$ (N:M), $a^2m_1 \notin$ N then $b^2m_1 \in$ N also $abm_2 \in$ N and $(ab)^2 \notin$ (N:M), $b^2 m_2 \notin$ N then $a^2 m_2 \in$ N.Now $(ab)^2 (m_1 + m_2) \in$ N and $(ab)^2 \notin$ (N:M).We have $a^2 (m_1 + m_2) \in$ N , *i.e.* $a^2 m_1 + a^2 m_2 \in$ N we get $a^2m_1 \in$ N contradiction ! If $b^2 (m_1 + m_2) \in$ N , *i.e.* $b^2 m_1 + b^2 m_2 \in$ N so $b^2m_2 \in$ N contradiction !then either $(ab)^2 \in$ (N:M) or a^2 K ⊆ N or b^2 K ⊆ N.

Theorem2.7: Let N a proper submodule of an R-module M , then the following statement are equivalent :

1. N is Approximately 2-Absorbing Submodule of M
2. If IJK ⊆ N , for some ideal I and J of R and some submodule K of M then either I^2K ⊆ N or J^2K⊆ N or $(IJ)^2$ ⊆(N:M) .

proof: (1) ⟹ (2)

Suppose N is Approximately 2-Absorbing Submodule of M and IJK ⊆ N for some ideals I and J of R and some submodule K of M and $(IJ)^2$ ⊈ (N:M) .

let$(IJ)^2$ ⊈ (N:M) to show that I^2 K ⊆ N or J^2 K ⊆ N suppose I^2 K ⊈ N and J^2 K ⊈ N there exist a_1, a_2 show that $a_1 \in$ I and $a_2 \in$ J and $a^2{}_1$ K ⊈ N and $a^2{}_2$ K ⊈ N have $a_1^2\, a_2^2$ K ⊆ N since N is T so we have $(a_1a_2)^2 \in$ (N:M) Since $(IJ)^2$ ⊈ (N:M) , ∃ $b_1 \in$ I, $b_2 \in$ J such that $(b_1b_2)^2 \notin$ (N:M) , But b_1 b_2 K ⊆ N We have $b_1{}^2$ K ⊆ N or $b_2{}^2$ K ⊆ N

Case (1) $b^2{}_1$ K ⊆ N and $b^2{}_2$ K ⊈ N Since $a^2{}_1$ $b^2{}_2$ K ⊆ N and $b^2{}_2$ K ⊈ N and $a^2{}_1$ K ⊈ N so $(a_1 b_2)^2 \in$ (N:M) by Proposition 2.6 Since $b_1{}^2$ K ⊆ N and $a_1{}^2$ K ⊈N we get $(a_1{}^2 + b_1{}^2)$ K ⊈ N ⟹ $(a_1{}^2 + b_1{}^2)$ $b_2{}^2$ K ⊆ N neither$(a_1{}^2 + b_1{}^2)$ K ⊆ N nor $b_2{}^2$ K ⊆ N We get $((a_1{}^2 + b_1{}^2)\, b_2{}^2) \in$ (N:M) by Proposition 2.1.6 but $(a_1{}^2 + b_1{}^2)\, b_2{}^2 = a_1{}^2\, b_2{}^2 + b_1{}^2\, b_2{}^2\, a_1{}^2\, b_2{}^2 \in$ (N:M) ⟹ $(b_1\, b_2)^2 \in$ (N:M) contradiction

Case 2) $b^2{}_2$ K ⊆ N and $b^2{}_1$ K ⊈ N . By a similar argument of case(1),

we reach to a contradiction

Case 3) $b_1{}^2$ K ⊆ N and $b_2{}^2$ K ⊆ N Since $b_2{}^2$ K ⊆ N and $a_2{}^2$ K ⊈ N we conclude $(a_2{}^2 + b_2{}^2)$ K ⊈ N but $a_1{}^2\, (a_2{}^2 + b_2{}^2)$ K ⊆ N and Neither $a_1{}^2$ K ⊆ N nor $(a_2{}^2 + b_2{}^2)$ K ⊆ N hance $a_1{}^2\, (a_2{}^2 + b_2{}^2) \in$ (N:M) by Proposition 2.6 since $(a_1a_2)^2 \in$ (N:M) and $a_1{}^2\, a_2{}^2 + a_1{}^2\, b_2{}^2 \in$ (N:M) we have $a_1{}^2\, b_2{}^2 \in$ (N:M) $(a_1{}^2 + b_1{}^2)\, a_2{}^2$ K ⊆ N ⟹ $a_2{}^2$ K ⊆ N or $(a_1{}^2 + b_1{}^2)$ K ⊆ N or $(a_1{}^2 + b_1{}^2)\, a_2{}^2 \in$ (N:M) by Proposition 2.6.But $(a_1{}^2 + b_1{}^2)\, a_2{}^2 = a_1{}^2\, a_2{}^2 + b_1{}^2\, a_2{}^2$ So $= a_1{}^2\, a_2{}^2 + b_1{}^2\, a_2{}^2 \in$ (N:M) we get $b_1{}^2\, a_2{}^2 \in$ (N:M) Now, $(a_1{}^2 + b_1{}^2)\, (a_2{}^2 + b_2{}^2)$ K ⊆ N we Have $a_1{}^2\, a_2{}^2 + a_1{}^2\, b_2{}^2 + b_1{}^2\, a_2{}^2 + b_1{}^2\, b_2{}^2 \in$ (N:M) by Proposition 2.6 so $b_1{}^2\, b_2{}^2 \in$ (N:M) is contradiction !

Theorem 2.8: If N is Approximately 2-Absorbing Submodule of M , then $(N:_R M)$ is approximately 2-absorbing ideal of R.

proof: Let abc ∈ $(N:_RM)$, abc ∈ R $a^2c \notin (N:M)$, $b^2 c \notin (N:M)$, show that $(ab)^2 \in (N:M)$ there exist m_1 , m_2 ∈ M such that $a^2cm_1 \notin N$, $b^2cm_2 \notin N$ But ab $(cm_1 + cm_2) \in N$ since N is approximately 2-absorbing so $a^2 (cm_1 + cm_2) \in N$ or $b^2 (cm_1 + cm_2)$ or $(a^2 b^2) \in (N:R M)$ we get $a^2cm_1+a^2cm_2 \in N$ then $a^2cm_1 \in N$ which is contradiction !if $b^2cm_1+b^2cm_2 \in N$ then $b^2cm_1 \in N$ which is contradiction then $(ab)^2 \in (N:M)$.

Theorem 2.9: Let N be a submodule of a multiplication R-module M such that $(N :_R M)$ is approximately 2-absorbing ideal of R. Then N is Approximately 2-Absorbing Submodule of M .

Proof: As $(N :_R M) \neq R$, N ≠ M . Let a, b ∈ R, m ∈ M , and abm ∈ N . Since M is a multiplication R-module, there exists an ideal I of R such that Rm = IM . Thus abIM ⊆ N . Hence, abI ⊆ $(N :_R M)$. Now by assumption, $(ab)^2 \in (N :_R M)$ or $a^2I \subseteq (N :_R M)$ or $b^2I \subseteq (N :_R M)$. Therefore, $(ab)^2 \in (N :_R M)$ or $a^2IM \subseteq N$ or $b^2IM \subseteq N$. Thus $(ab)^2 \in (N :_R M)$ or $a^2m \in N$ or $b^2m \in N$ .by[3]

Proposition 2.10: Let R be a commutative ring , M is a cyclic R-module and N is a submodule of M . Then N is Approximately 2-Absorbing Submodule of M if and only if $(N:_RM)$ is approximately 2-absorbing ideal of R.

Proof: ⟹ Let N is T of M Assume M = Rm , m ∈ M . Let abc ∈ $(N:_R M)$ when a,b,c ∈R but $(ab)^2 \notin (N:M)$ and $b^2c \notin (N:_R M)$,

prove $a^2c \in (N:_RM)$ it following from abcm ∈ N, and $(ab)^2 m \notin N$ $b^2cm \notin N$.That , $a^2c\, m \in N$ *then* $a^2c \in (N:M)$

⟸ let $(N:_RM)$ is approximately 2-absorbing ideal of R, let abx ∈ N , x ∈ M , x = cm, c ∈ R so abc m ∈ N ⟹ abc∈ $(N:_R m)$ = (N:M) by $(N :_R M)$ is approximately 2-absorbing ideal of R $a^2 c \in (N: M)$ or $b^2 c \in (N:M)$ or $(ab)^2 \in (N:M)$ $a^2 cm \in N$ or $b^2cm \in N$ or $(ab)^2 \in$ N:M) $a^2x \in N$ or $b^2 x \in N$ or $(ab)^2 \in (N:M)$ N is Approximately 2-Absorbing Submodule of M .

Proposition2.11 : Suppose M is a finitely generated multiplication R-module . If I approximately is 2-absorbing ideal of R such that annM ⊆I , then IM is Approximately 2-Absorbing Submodule of M.

Proof: Let abm ∈ IM where a,b ∈ R,m ∈ M ,hence ab(m) ⊆ IM . Since M is multiplication then (m)=JM for some ideal J of R . Thus abJM ⊆ IM

abJ ⊆ I+annM = I by [3]. But I is approximately 2- absorbing ideal of R , so either $(ab)^2 \in$ I or $a^2J \subseteq I$ or $b^2J \subseteq I$, it follows that$(ab)^2 \in (IM :_R M)$

or $a^2JM \subseteq IM$ or $b^2JM \subseteq IM$; that is either $(ab)^2 \in$ (IM:R M) or a^2 (m) ⊆ IM orb^2 (m) ⊆ IM Thus $(ab)^2 \in$ (IM : R M) or $a^2m \in IM$ or$b^2m \in IM$ and so IM is Approximately 2-Absorbing Submodule of M.

Corollary 2.12 : Suppose M is a faithful finitely generated multiplication R-module . If I is approximately 2-absorbing ideal of R, then IM is Approximately 2-Absorbing Submodule of M.

Proof: It follows directly by Proposition 2.11.

Corollary 2.13: Let M be a faithful finitely generated multiplication R-module . Then every proper submodule of M is Approximately 2-Absorbing Submodule if and only if every proper ideal of R is approximately 2-absorbing .

Proof : ⟸ It follows directly by Corollary (2.12) .

⟹ Let I be a proper ideal of R . Then $N= IM$ is a proper submodule of M

So it is Approximately 2-Absorbing Submodule and hence by Theorem (2.8) , (N:M) is approximately 2-absorbing ideal . But M is faithful finitely generated multiplication R-module , so (N:M)=I by [3,Theorem.3.1].

Proposition 2.14: Let M be an R-module , N a proper submodule of M ,if N is approximately 2-absorbing then (N $:_R$ <m>) is approximately 2-absorbing ideal for each m ∈M-N.

Proof: Let abc ∈(N$:_R$<m>) then abcm∈ N , but N is approximately 2-absorbing submodule then a^2(cm) ∈ N or b^2(cm) ∈ N or $(ab)^2$ ∈ (N $:_R$ M), so that a^2 cm ∈ N or b^2cm ∈ N or $(ab)^2$ M ⊆ N that is a^2c ∈ (N : (m)) or b^2c∈ (N $:_R$(m)) or $(ab)^2$∈ (N $:_R$ (m)) hence (N$:_R$ (m)) is approximately 2-absorbing ideal.

Proposition 2.15: Let N be a proper submodule of an R-module M . The following statements are equivalent:

1. N is Approximately 2-Absorbing Submodule of M
2. (N $:_M$ I) is approximately 2-absorbing , for each ideal I of R with IM ⊈N
3. (N $:_M$ (r)) is approximately 2-absorbing submodule for each r ∈R with rM ⊈N

Proof: (1)⟹(2) Let I be an ideal of R with IM ⊈N is a proper submodule of M

Let abm ∈(N $:_M$ I) where a , b ∈ R , m ∈ M, then ab(Im) ⊆ N But N is approximately 2-absorbing submodule of M , so by Proposition(2.6) , either a^2(Im) ⊆ N or b^2(Im) ⊆ N or $(ab)^2$ ∈ (N : R M) ⊆ ((N : I): M) . Hence either a^2m ∈ (N : I) or b^2m ∈(N : I) or $(ab)^2$ ∈((N : M I): M) . Thus (N $:_M$ I) is approximately 2-absorbing submodule .

(2)⟹(3) It is clear .

(3)⟹(1) Take r =1 then (N : (1))= N , so N is approximately 2-absorbing.

Proposition 2.16: Let N is Approximately 2-Absorbing Submodule of an R-module M. For any a, b, c ∈ R, m ∈ M , abcm ∈ N implies either$(ab)^2$ m ∈ N ora^2c m ∈ N orb^2c m ∈ N

Proof: Let abm ∈ N, then ab(cm) ∈ N Since N is T then a^2 (cm) ∈N or b^2(cm) ∈N or $(ab)^2$ ∈(N:M) .So a^2c m ∈N or $b^2c\ m$ ∈N or $(ab)^2$m ∈N .

Theorem2.1.17: Let N be a proper submodule of an R-module M . Consider the following statements:

1. N is Approximately 2-Absorbing Submodule of M
2. For each a,b ∈ R , m ∈ M . If a^2b^2m ∉ N , then (N: abm) ⊆(N:a^2m)∪(N:b^2m).
3. For each a,b ∈ R , m∈M if a^2b^2m ∉N then (N: abm)⊆(N : a^2m) or

   (N: abm) ⊆(N:b^2m) Then (1) ⟹(2) ⟹(3) and if M is cyclic , then (3) ⟹(1)

Proof : (1) ⟹ (2) Let c ∈ (N : abm) then cabm ∈ N so abcm ∈ N . Since N is T, we get either a^2cm ∈ N or b^2cm ∈ N or a^2b^2M ⊆ N.but a^2b^2m ∉ N(by hypotheses), Then only c ∈ (N : a^2 m) or c ∈ (N : b^2m)

Thus (N : ab m) ⊆ (N : a^2m) ∪ (N : b^2m) .

1. **(3)** It is clear.
2. **(1)** Now suppose that M=(m_1) for some m_1∈M.Let abm ∈ N. But m= m_1r for some r ∈ R then abrm_1∈ N and suppose that a^2 b^2∉ (N :M) so, a^2 b^2 m_1 ∉ N and r ∈(N : $ab$$m_1$) . Hence either r ∈(N : $a^2$$m_1$) or r ∈(N : $b^2$$m_1$) by condition(3). And so a^2rm_1∈ N or b^2r m_1∈ N , that is either a^2m ∈ N or b^2m ∈ N . Hence N is Approximately 2-Absorbing Submodule.

N of M is called a pure submodule of an R-module M if IM ∩ N = IN or any ideal I of R [4].

<u>Proposition 2.18 :</u> Let N be a proper pure submodule of an R-module M , If (0) is Approximately 2-Absorbing Submodule of M , then N is Approximately 2-Absorbing Submodule.

Proof: Let abm ∈ N where a ,b ∈ R , m ∈ M Put I=(ab) then abm ∈ IM ∩ N , but IM ∩ N = IN , so abm = abn for some n ∈ N , then ab(m – n) = 0 , but (o) is approximately 2-absorbing then a^2 (m – n) = 0 or b^2(m – n) = 0 or $(ab)^2$∈ annM ⊆ (N : M) . So we get a^2m=a^2n ∈ N or b^2m=b^2n ∈N or$(ab)^2$ ∈ (N : M) , Thus N is Approximately 2-Absorbing Submodule.

<u>proposition 2.19 :</u> Let N be a proper submodule of an R-module M and S multiplicative subset of R, then S^{-1}N is T of S^{-1} R-submodule of S^{-1}M if N is Approximately 2-Absorbing Submodule.

proof: Let $\bar{a}$,$\bar{b}$ ∈ S^{-1} R, $\bar{x}$ ∈ S^{-1}M ,then $\bar{a}=\frac{a_1}{s_1}$, $\bar{b}=\frac{b_1}{s_2}$, $\bar{x}=\frac{x_1}{s_3}$ for some

$a_1, b_1,$ ∈R, $x_1 \in M$ s_1, s_2, s_3 ∈ S $\frac{a_1}{s_1}\frac{b_1}{s_2}\frac{x_1}{s_3}$ ∈ S^{-1}N then $\frac{a_1b_1x_1}{t}$ ∈ S^{-1}N where s_1 $s_2s_3 = t$ ∈ S. Then there exists t_1 ∈S such that $t_1a_1b_1x_1$ ∈ N since N is T we get $t_1{a_1}^2x_1$ ∈ N or ${b_1}^2x_1$ ∈ N or $t_1(a_1b_1)^2$ ∈ (N:M) If $t_1{a_1}^2x_1$ ∈ N, then $\frac{t_1{a_1}^2x_1}{t_1s_1s_3}$ ∈ S^{-1}N, so $\frac{{a_1}^2}{s_1}\frac{x_1}{s_3}$ ∈ S^{-1}N. If ${b_1}^2x_1$ ∈ N, then $\frac{{b_1}^2x_1}{s_2\,s_3}$ ∈ S^{-1}N so $\frac{{b_1}^2}{s_2}\frac{x_1}{s_3}$ ∈ S^{-1}N. If $t_1(a_1b_1)^2$ ∈ (N:M) then $\frac{t_1(a_1b_1)^2}{t_1s_1s_2}$ ∈ S^{-1}N *so* $\frac{{a_1}^2}{s_1}\frac{{b_1}^2}{s_2}$ ∈ S^{-1}(N: M) . But S^{-1} (N:M) ⊆ (S^{-1} N: S^{-1} M) then $\frac{{a_1}^2}{s_1}\frac{{b_1}^2}{s_2}$ ∈ (S^{-1} N: S^{-1} M). Hence S^{-1} N is Approximately 2-Absorbing Submodule.

Let φ: M⟶ S^{-1}M defined by φ(m) $=\frac{m}{1}$ for each m∈ M. It is clear that φ is an R-homomorphism

References

[1] A .Y Darani, F.Soheilnia, "2-Absorbing and Weakly 2-Absorbing Submodules" , www.math.science.cmu.ac.th/thaijournal,2011.

[2] H.M.Abdul-Razak "Quasi-Prime Modules and Quasi-Prime Submo.m.,.,,mhn,.dules", M.Sc.Thesis, College of Education Ibn Al-Haitham, University of Baghdad, ,(1999).

[3] Z.A. El- Bast,. and P.F .Smith, "Multiplication Modules" , Comm. in Algebra, 16(1988), 755-779.

[4] E. W.Andeson and K..Fulle. “ings and ategories of Modules”Springer –Verlage,New York, 1992.