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Some Properties of Fuzzy Ideals of Unit Regular Semigroups

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ABSTRACT

This paper introduces the concept of unit-regular fuzzy ideals of semi groups, which is a generalization of regular fuzzy ideals. Several of their significant related properties have been examined. According to the unit-regular semi group, the fuzzy-interior-ideal, fuzzy-ideal, fuzzy-bi-ideal, and fuzzy-generalized-bi-ideal have all been described.

MSC..

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1-Introduction :

In 1965, L. A. Zadeh brought the idea of fuzzy units that have been applied in many fields, Medical technological know-how, theoretical physics, robotics, pc technological know-how, manage engineering, records technology, measure concept, logic, set theory, topology, and similar disciplines can all be located. Rosenfeld changed into the primary to ponder the case wherein S is a groupoid in 1971 [5]. The definition of fuzzy subgroups and fuzzy left (proper, sided) ideal of S turned into given by means of him. Kuroki [4] defined a fuzzy semi group and special types of fuzzy ideal in semi groups

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in 1979 and characterized them. Stienfeld explored the concept of securing quasi-ideals with rings and semi groups[2]. The idea of quasi-ideal represents a generalization of left and proper ideals, and bi-ideal represent a generalization of quasi-ideal.

1- PRELIMINARIES

"Given a semi group S, Let U and V be sub sets of S.The multiplication ofU and V is defined as follows:
UV={uv∈S;u∈U and v∈V}.A non-empty sub set U of S is referred to as a sub-semi group of S if UU⊆U, A non-empty sub set U of S is referred to as an ideal of the left (right) S if SU⊆U (US⊆U). Further,U is known as a 2-sided ideal of S if it's far left and right ideals, respectively of S.A non-empty sub set U of S is referred to as an interior-ideal(int-ideal) of S if SUS⊆U,, and a quasi-ideal(Q-ideal) of S if US∩SU⊆U. A sub semi-group U of S is called a bi-ideal of S if USU⊆U.A sub set that is not empty U is called a gen-bi-ideal of S if USU⊆U [3]**."**

Definition1.1 :[6]

If there is a unit element u in a semi-group S such that x=xux , ∀x in S, then the semi-group is said to be **unit regular.**

Example: $(Z_{10}, .)$ is unit regular semi-groups

Definition1.2 : [1]

Consider two fuzzy-sub sets(Fu-sub set) of S, m and n. The product k∘s is defined by

$$(k \circ s)_{(x)} = \begin{cases} \bigvee_{x=yz} \{k(y) \wedge s(z), & if\ \exists\ y,z\ \in S\ such\ that\ x = yz \\ 0 & otherwise \end{cases}$$

Definition 1.3: [7]

A Fu-sub set m for S is refered to as a Fu-sub-semi group of S if

m(pq) ≥ m(p)∧m(q) for all p, q ∈ S ,and refered to as a Fu-left ideal (right) of S when

m(pq) ≥ m(q) (m(pq) ≥ m(p)) for all p, q ∈ S .

A Fu sub set m is known as a Fu-2-sided ideal (Fu-ideal) of S if it's both a Fu-left, and a Fu-right ideal of S.

"consider m be a subset of S.Recall that we denote by means of C_m the characteristic function of m" [3]

Lemma 1.4 :[5]

If B is a nonempty sub set of a semi-group S.Then the following properties are true:

1) $C_B$is a fu- sub semi-group of S⇔B is a sub semi-group of S.
2) $C_B$is a fu-left (right, 2-sided) ideal of S⇔ B is a left (right, 2-sided) ideal of S.

Lemma 1.5: [5]

The following characteristics are present when k is a Fu-sub set of S.

1) k∘k ⊆k ⇔ k is Fu-sub-semi-group of S.
2) S∘k ⊆k⇔k is Fu-left ideal of S.
3) k∘S ⊆k⇔k is Fu-right-ideal of S.
4) S∘k⊆k and k∘S⊆k⇔k is Fu-2-sided ideal of S.

Definition 1.6: [1]

A Fu- sub set m of a semi-group S is said a Fu-Bi -ideal of S if m(pqs) ≥ m(p) ∧m(s) ∀p ,q and s of S .

Lemma 1.7:[4]

Let U be a nonempty sub set of a semi -group S,then U is a Bi -ideal of S⇔ C_U of U is a fuzzy Bi-ideal of S.

Definition1.8: [1]

A Fu-sub-set m of S is called Fu-int-ideal (Fu-int-ideal) of S if m(paq)≥ m(a) ∀p,a and q of S.

Lemma 1.9:[4]

Consider m to be a Fu-sub-set of S. Then m is Fu-int-ideal of S⇔S∘m∘S⊆m

Definition 1.10: [4]

A fuzzy- Quasi -ideal (Fu-Q -ideal) m of S is defined as (m∘S)⊆(S∘m)⊆m

Note: "Any Fu-2-sided -ideal of S is a Fu-Q -ideal of S and any Fu-Q -ideal of S is a Fu-Bi -ideal of S. The opposite often does not apply.[3]"

Lemma 1.11:[3]

"Let $\phi \neq A$ sub set of a semi group S.Then A is a Fu-Q -ideal of S ⇔ C_A is Fu-Q -ideal of S."

Definition 1.12:[1]

Let ϕ ≠U⊆S, U is called a generalized-Bi-ideal of S iff USU⊆U.A Fu-sub-set m of S is said a **Fu-G-Bi -ideal** of S if m(paq) ≥ m(p)⋀m(q) ∀p,a and q ∈S .

Note: "Every Fu-Bi-ideal of a semi group S is a Fu-G- Bi-ideal of S. But the converse of this statement does not hold in general.[3]"

Definition 1.13: [3]

Assume that m is a Fu-sub-set of S. A Fu-Bi-ideal of S created by m is the smallest Fu-Bi-ideal of S containing m; it is represented by the symbol $\langle \mathbf{m} \rangle_{\mathbf{B}}$

Theorem1.14:[3]

Let m be a monoid S Fu-sub-set.Then for any a ∈ S , $\langle m \rangle_B$= j where j(a) = ∨{m(x_1) ∧m(x_3)\ a= $x_1 x_2 x_3$, x_1, x_2, x_3 ∈ S }

Examples :

1- Consider S as a semi -group of 4 elements {k,l,r,s} with the following multiplication table

·	k	l	r	s
k	k	k	k	k
l	k	k	k	k
r	k	k	l	k
s	k	k	l	l

Let Abe a fu -subset of S such that A(k)=0.7, A(l)=0,A(r)=0,A(s)=0,then A is a fu-int -ideal of S which is not a fu-two-sided -ideal of S , A(abc)=A(k)=0.7>= A(b) ∀a, b, c ∈S, thus A is a fu-int-ideal of S but since A(sr)=A(q)=0<= 0.3 =A(r) , A is not a fu-left -ideal of S.

2- Consider S as a semi -group of 4 elements {0,q,r,s} with the following multiplication table

·	0	q	r	s
0	0	0	0	0
q	0	q	r	0
r	0	0	0	0
s	0	s	0	0

Then Q={0,q} is Q -ideal of S and is not ideal of .Define the fu-sub set A of S as follows A(0)=A(q)=0.7 and A(r) =A(s)=0,then A is fu-Q -ideal of S but is not fu -ideal of S.

3- Consider S as a semi -group of 4 elements {0,q,r,s} with the following multiplication table

·	0	q	r	s
0	0	0	0	0
q	0	0	0	0
r	0	0	0	q
s	0	0	q	r

, then B={0,r} is Bi-ideal of S but is not Q-ideal of S , define the fu-sub set A of S as A(0)=A(r)=0.7 and A(q)=A(s)=0,then A is fu-Bi -ideal of S but is not a fu-Q -ideal of S.

4- Consider S as a semi -group of 4 elements {k,q,r,s} with the following multiplication table

·	k	q	r	s
k	k	k	k	k
q	k	k	k	k
r	k	k	q	k
s	k	k	q	q

Let A be a fu -sub set of S such that A(k)=0.5, A(q)=0, A(r)=0.2 and A(s)=0,A is a fu-G-bi -ideal of S but not a bi -ideal of S .

2- Fuzzy Ideals of Unit Regular Semi groups

The properties of Fu-ideals of unit regular semi-groups are discussed in this section.

Theorem 2.1: Let L left ideal and R right ideal of a semi-group S, if S is unit regular,then R∩L=RL.

Proof: Let RandL be the right and left ideals, respectively of S, and assume that S is unit regular, we need to show that $R \cap L = RL$. (i.e) $R \cap L \subseteq RL$ and $RL \subseteq R \cap L$). Let x∈ $R \cap L$ this means that x in both R and L therefore x∈ RL (since RL⊆ R and $RL \subseteq$ L). therefore $R \cap L \subseteq RL$ and let x∈ RL,let x=ab where a∈ R and $b \in L$ .since S is unit regular there exists u $\in S$ such that x=xux (let xu=a and x=b)we have x=ab also x=xux .we have x∈ R and $x \in L$ this means that x∈ $R \cap L$ therefore $RL \subseteq R \cap L$ .Thas, $R \cap L = RL$.

Lemma 2.2: Given a unit regular semi-group S, let m be Fu-sub set. Then, the subsequent circumstances are comparable:

(1) m is a Fu-ideal of S .
(2) m is a Fu-int-ideal of S.

Proof: Lemma 1.5 shows that (2) entails (1), which is sufficient to establish. Let a and c be any members of S. Since S is a unit regular, ∃ elements $u_1, u_2 \in S$ such that a=a$u_1 a$, c= c$u_2 c$ thus we have :

m(ac) =((a$u_1 a)c$) =m((au_1)ac)=m($w_1$ac) ≥m(a) and

m(ac)= m(a(c$u_2 c$)) =m(ac($u_2 c$))=m(acw_2) ≥ $m(c)$ therefore m is a Fu-2 -sided -ideal of S.

Lemma2.3:A unit regular semi -group S has Fu-Bi -ideal ∀ Fu-G-Bi -ideal of S.

Proof: Let t and c be any components of S,and let m be any Fu-G-Bi -ideal of S. As a result, we have m(tc) = m(t(cxc))= m(t(cx)c)> m(t)∧ m(c) since S is a unit regular, ∃ an element x ∈ S such that c=cxc. This suggests that m is Fu-Bi -ideal of S since it is Fu -sub semi -group of S.

Theorem2.4: Let m be a Fu-sub set of a monoid S. If S is unit regular semi -group,then

$\langle m \rangle_B$=j where j(a)= $\vee\{m(x_1) \wedge m(x_3) \backslash a= x_1x_2x_3 , x_1, x_2, x_3 \in S\}$ $\forall a \in S$.

Proof : All that is required to establish j(a) ≥ m(a) for every a ∈S is to use theorem 1.14.Moreover, for any a in S, j(a)=$\vee\{m(x_1) \wedge m(x_3) \backslash a= x_1x_2x_3 , x_1, x_2, x_3 \in S\}$

≥ j(a) = $\vee\{m(a) \wedge m(a) \backslash a= axa \in S\}$=m(a)

Definition2.5 :[3] Let a∈S and t∈ [0,1].Define the Fu-sub set a_t of S as follows:

$$a_t(x) = \begin{cases} t & if\ x = a , \\ 0 & otherwise \end{cases} \quad \forall x \in S .$$

Then, a_t is referred to as a Fu-singleton or Fu-point. Assume m =x_t.The following outcome can then be obtained using Theorem 2.4.

Corollary 2.6:_ Assume that S is a monoid and that $x_t$is a Fu-singleton of S,if the unit regular semi-group S is. Consequently, $\langle x_t \rangle$B= j, where

$$j(a)=\begin{cases} t & \text{if there exists some } y \in S \text{ such that } a = yxy\,, \\ 0 & \text{otherwise} \end{cases}$$

$\forall a \in S.$

Theorem 2.7: "[5] The cases that follow are comparable for a semi-group S.

1)S is regular

2) For each right-ideal R of S,and each left-ideal L of S,R∩L= RL.

3)For any Q –ideal A of S,A=ASA.

Theorem 2.8: The subsequent cases are similar for a semigroup S.

1) S unit regular.

2) For any Fu-Q -ideal m of S, m=m∘S∘m.

3)For any Fu-G-Bi -ideal m of S, m=m∘S∘m.

proof: Let m be any Fu-G-Bi-ideal of S and a ∈S. Assuming (1) holds, ∃ a unit element u ∈S such that c=cuc because S is a unit regular. Therefore, we have

$$(m \circ S \circ m\ )_{(c)} = \vee_{\,c=yc}\{(m \circ S)_{(y)} \wedge m(c)\}$$

$$\geq \{(m \circ S)_{cu} \wedge m(c)\}$$

$$= \{m(c) \wedge m(c) \wedge m(c)\}$$

$$\geq\ m(c) \wedge m(c)$$

$$= m(c)$$

And so m⊆m∘S∘m since m is G-Bi -ideal of S , m∘S∘m⊆m .Thus m=m∘S∘m

That [1] →[2] →[3] is evident. Presently (2) →(1) Assuming that (2) is true, we get ASA⊆A since, given any Q -ideal of S, A C_A is Fu- Q -ideal by lemma 2.2

$$\vee_{\,a=ya}\{(C_A \circ S)_{(y)} \wedge C_A(a)\}=(C_A \circ S) \circ (C_A \circ S)$$

$$= C_A(a) = 1.$$

This suggests that C_A (a)=1 and $(C_A \circ S)$=1 exist for elements t and v of S.

Using a=tv.Therefore, $\vee_{\,t=pq}\ \{(C_A)_{(p)} \wedge S(q)\} = (C_A \circ S)_t = 1.$

According to this,elements d and c of S must exist for $C_A(d)$=1 and S(e)=1.

Using b=de.Because d,c∈ A and e∈S, a=bc=(de)c∈ASA. Consequently,A=ASA since A⊆ASA. Therefore, since S is unit regular, (2) implies (1) based on theorem 2.7.

Theorem 2.9: A unit regular semi-group S, a Fu-2-sided ideals are all idempotent.

Proof: Let m be the ideal of a unit regular semi group S that is Fu-2-sided. Lemma 1.5 therefore gives us m ∘ S ∘ m ⊆ m. Since S is unit regular, m is a Fu-G-Bi-ideal of S. According to Theorem 2.8, we may deduce that m= m∘S∘m=m∘(S∘m)⊆m∘S⊆m, and as a result, m∘m=m.

Definition 2.10: [3]

When the left and right duos combine to form a semi group S, it is referred to as duo. A semi-group S is called left (right) duo if each An ideal of S that has 2-sides is its left (right) ideal.

Theorem2.11: A unit regular semi group S is left (right) duo exclusively in the case of the Fu-left (right) duo.

Proof: Let S be the left-duo.Since the left ideal Sa is a 2-sided-ideal of S,let m be any Fu-left -ideal of S and a,b∈S. Furthermore, because S is unit regular,we get ab∈(aSa)b⊆(Sa)S⊆Sa. This suggests that there is an element a∈S such that ab = xa.

Hence, m(ab)=m(xa)≥m(a) since m is Fu-left -ideal of S.As a result, S is Fu-left-duo since m is Fu-right -ideal of S and m is a Fu-2-sided -ideal.

On the other hand, let's say S is the Fu-left duo.Lemma 1.4 states that the characteristic function image of an arbitrary left-ideal A is a Fu-left ideal of S. Next, presuming

Corollary2.12: A unit regular semi group is duo ⇔it is Fu-duo.

Theorem 2.13: For a unit regular semi group S , the sub sequent situations are equal.

1)A right (left, 2-sided) ideal of S is a bi-ideal of S in all cases.

2)As for Fu-right (left, 2-sided) ideals of S, all Fu-Bi-ideals of S are Fu-right.

Proof: Let m be any Fu-Bi-ideal of S with t,u∈S. Assume that (1) holds. As the set tSt is assumed to be a right -ideal of S, it is bi-ideal of S.We have tu∈(tSt)S⊆tSt because S is unit regular. This means that there is an element v ∈S such that yu=tvt.Since f is a fu-bi-ideal of S,

m(tu)=m(tvt) ≥ m(t)∧ m(t) = m(t),which means m is a Fu-right ideal of S.Consequently, we obtain (2).

On the other hand, if (2) is true and we let A be any Bi-ideal of S, then theorem 1.7 indicates that the characteristic function C_A is a Fu-Bi -ideal of S, hence, we can assume that it is Fu-right -ideal of S, as shown by (lemma1.4).

Theorem 2.14*:* For a semi -group S the subsequent situations are similar.

1) S is unit regular.
2) The equation m∩n=m∘n holds ∀ Fu-left ideal n and each Fu-right ideal m of S.

proof: If (1) is correct, then let m be Fu-right -ideal and n be any left -ideal of S.Then,(by lemma 1.5) we have $m\circ n\subseteq m\circ S\subseteq m$ and $m\circ n\subseteq S\circ n\subseteq n$ thus $m\circ n\subseteq m\cap n$,let $a\in S$. Since S is unit regular, ∃ an element $u\in S$ such that a=aua. Therefore, we get

$$(m\circ n)_{(a)} = \vee_{a=yz}\{m(y)\wedge n(z)\}$$

$$\geq m(au)\wedge n(a)$$

$$\geq m(a)\wedge n(a)$$

$$= (m\cap n)_{(a)}$$

Thus, $m\circ n\supseteq m\cap n$, meaning that $m\circ n=m\cap n$, and thus (1) ⟶ in contrast, if (2) is true, then let M and N represent any right and left ideals of S,respectively,in order to determine that $M\cap N\subseteq MN$ holds.If an is true, then (by lemma1.4) the characteristic functions C_M and C_N of R and L are, respectively, a Fu-right ideal and a Fu-left -ideal of S .Consequently ,we have

$$C_{MN}(a) = (C_M\circ C_N)_{(a)}$$

$$= (C_M\cap C_N)_{(a)}$$

$$C_{M\cap N}(a)$$

$$= 1$$

Therefore, given that the inclusion in the other direction is always true, we get that $M\cap N=MN$.This means that since S is unit regular according to Theorem 2. 8, (2)⟶(1). And therefore,$a\in MN$, and thus $M\cap N\subseteq MN$.

Theorem 2.15: For a semi group S The ensuing prerequisites are comparable.

(1) S is unit regular
(2) $m\cap n=m\circ n\circ m$ ∀ Fu-Q -ideal m and ∀ Fu-2-sided -ideal n of S
(3) $m\cap n=m\circ n\circ m$ ∀ Fu − Q_ideal m and ∀ Fu- int -ideal n of S
(4) $m\cap n=m\circ n\circ m$ ∀ Fu-Bi-ideal m and ∀ Fu- 2-sided -ideal n of S
(5) $m\cap n=m\circ n\circ m$ ∀ Fu-Bi-ideal m and ∀ Fu-int -ideal n of S
(6) $m\cap n=m\circ n\circ m$ ∀ Fu- G- Bi-ideal m and ∀ Fu- 2-sided -ideal n of S
(7) $m\cap n=m\circ n\circ m$ ∀ Fu- G- Bi-ideal m and ∀ Fu-int -ideal of S .

Proof: Assuming (1) is true, let m and n be any Fu-G-Bi -ideal and Fu-int -ideal of S,respectively, Then, $m\circ n\circ m\subseteq m\circ S\circ m\subseteq m$ and $m\circ n\circ m\subseteq S\circ n\circ S\subseteq n$, and so $m\circ n\circ m\subseteq m\cup n$.

Let k be a member of S.Since S is unit regular,∃ an element u in S such that k = kuk = (kukuk). Therefore, since n is a Fu-int -ideal of S,we get

$$(m\circ n\circ m)_{(k)} = \vee_{k=yk}\{m(y)\wedge(n\circ m)(k)\}$$

$$\geq m(k)\wedge(g\circ m)(ukuk)$$

$$= m(k)\wedge\vee_{ukuk=pq}\{n_{(p)}\wedge m(q)\}$$

$\geq m(k) \wedge (n(uku) \wedge m(k))$

$\geq m(k) \wedge n(k)$

$= (m \cap n)(k)$

Thus, $m \circ n \circ m \subseteq m \cap n$, and thus, (1) →(7), $m \circ n \circ m = m \cap n$.

That is evident from (7)→(5)→(3)→(2) and (7)→(6)→(4) →(2)

Assuming (2) to be true, let m be any Fu-Q -ideal of S.Since S is a Fu-2-sided ideal of S, we get

$m = m \cap S = m \circ S \circ m$.This shows that S is unit regular, which is implied (2) by theorem 2.9.

References

[1] Mohammed, A. S., Asaad, S. H., & Ahmed, I. S. (2022). New properties of strongly regular left almost semi-group. *Journal of Discrete Mathematical Sciences and Cryptography*, *25*(2), 615-622.

[2] Howie, J. M. (1976). An introduction to semigroup theory, Academic Press, New YORK .

[3] Mordeson, J. N., Malik, D. S., & Kuroki, N. (2012). *Fuzzy semigroups* (Vol. 131). Springer.

[4] Kuroki, N. (1981). On fuzzy ideals and fuzzy bi-ideals in semigroups. *Fuzzy sets and systems*, *5*(2), 203-215.

[5] Rosenfeld, A. (1971). Fuzzy groups. *Journal of mathematical analysis and applications*, *35*(3),512-517.

[6] McFadden, R. B. (1984). Unit-regular orthodox semigroups. *Glasgow Mathematical Journal*, *25*(2), 229-240.

[7] Asaad, S. H., Ahmed, I. S., & Mohammed, A. S. (2020, November). Generalization Fuzzy Completely Intra Γ-ideals in Γ-Semi Groups. In *Journal of Physics: Conference Series* (Vol. 1664, No. 1, p. 012028). IOP Publishing.

[8] Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, *8*(3), 338-353.