On Sandwich Results of Univalent Functions Defined by Generalized Abbas-Atshan Operator

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\textbf{ABSTRACT}

In the present paper, we obtain sandwich theorems for univalent functions by using some results of differential subordination and superordination for univalent functions involving integral operator.

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which with \( K(0) = 0 \), and \(|K(z)| < 1, (z \in D) \) where \( f(z) = g(K(z)) \). In such a case we write \( f < g \) or \( f(z) < g(z) (z \in D) \). If \( g \) is univalent in \( D \), then \( f < g \) if and only if \( f(0) = g(0) \) and \( f(D) \subset g(D) \) ([12,13]).

**Definition (1) [12]:** Let \( \emptyset : \mathbb{C}^3 \times D \to \mathbb{C} \) and \( h(z) \) be univalent in \( D \). If \( p(z) \) is analytic in \( D \) and satisfies the second-order differential subordination:

\[
\emptyset(p(z), z, p'(z), z^2p''(z); z) < h(z),
\]

**Definition (2) [12]:** Let \( p, h \in A \) and \( \emptyset(r, t, s; z) : \mathbb{C}^3 \times D \to \mathbb{C} \). If \( p \) and \( \emptyset(p(z), z, p'(z), z^2p''(z); z) \) are univalent functions in \( D \) and if \( p \) satisfies:

\[
h(z) < \emptyset(p(z), z, p'(z), z^2p''(z); z),
\]

then \( p(z) \) is called a solution of the differential superordination (1.2), and the univalent function \( q(z) \) is called a dominant of the solution of the differential subordination (1.2), or more simply dominant if \( p(z) < q(z) \) for all \( p(z) \) satisfying (1.2). A univalent dominant \( \bar{q}(z) \) that satisfies \( \bar{q}(z) < q(z) \) for all dominant \( q(z) \) of (1.2) is said to be the best dominant is unique up to a relation of \( D \).

Several authors [7,12] obtained sufficient conditions on the functions \( h, p \) and \( \emptyset \) for which the following implication holds

\[
h(z) < \emptyset(p(z), z, p'(z), z^2p''(z); z),
\]

then \( q(z) < p(z) \).

Using the results (see [2,3,4,5,9,10,13]) to obtain sufficient conditions for normalized analytic functions to satisfy:

\[
q_1(z) < \frac{zf'(z)}{f(z)} < q_2(z),
\]

where \( q_1 \) and \( q_2 \) are given univalent functions in \( D \) and \( q_1(0) = q_2(0) = 1 \). Also, several authors (see [2,4,5,6,8]) derived some differential subordination and superordination results with some sandwich theorems.

For \( f \in A \), Abbas-Atshan Operator [1] defined the following generalized integral operator:

\[
J_{a,b,s}f(z) = \frac{s^{\alpha}(\ln(\beta))^{\alpha}}{\Gamma(\alpha)} \int_0^s \nu^{\alpha-\beta}z^\nu f(zv)dv,
\]

where \( \alpha, s \in \mathbb{N} \) and \( \beta \geq 2 \).

For \( f(z) \in A \) given by (1.1), we have

\[
J_{a,b,s}f(z) = z + \sum_{n=2}^\infty \left( \frac{\Gamma(\alpha+n-1)}{\ln(\beta)^n n! \Gamma(\alpha)^n} \right) a_n z^n.
\]

From (1.6), we note that

\[
z\left(J_{a,b,s}f(z)\right)' = \alpha J_{a+1,b,s}f(z) - (\alpha - 1)J_{a,b,s}f(z).
\]

The main object of the present investigation is to find sufficient conditions for certain normalized analytic function \( f \) to satisfy:
\[
q_1(z) \prec \left[ \frac{J_{a,RFS}(z)}{z} \right]^{\frac{1}{a}} < q_2(z),
\]
and
\[
q_1(z) \prec \left[ \frac{J_{a+1,RFS}(z)}{J_{a,RFS}(z)} \right]^{\frac{1}{a}} < q_2(z),
\]
where \(q_1\) and \(q_2\) are given univalent functions in \(D\) with \(q_1(0) = q_2(0) = 1\).

In this paper, we derive some sandwich theorems, involving the operator \(J_{a,RFS}(z)\).

### 2. Preliminaries

We need the following definitions and lemmas to prove our results.

**Definition 2.1** [12]. Denote by \(Q\) the set of all functions \(q\) that are analytic and injective on \(D \setminus E(q)\), where \(D = D \cup \{z \in \partial D\}\), and

\[
E(q) = \left\{ \varepsilon \in \partial D : \lim_{z \to \varepsilon} q(z) = \infty \right\}
\]
and are such that \(q'(\varepsilon) \neq 0\) for \(\varepsilon \in \partial D \setminus E(q)\). Further, let the subclass of \(Q\) for which \(q(0) = a\) be denoted by \(Q(a)\), and \(Q(0) = Q_0, Q(1) = Q_1 = \{q \in Q : q(0) = 1\}\).

**Lemma 2.1** [13]. Let \(q\) be a convex univalent function in \(D\) and let \(\alpha \in \mathbb{C}, \beta \in \mathbb{C} \setminus \{0\}\) with

\[
\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max \left\{ 0, -\Re \left( \frac{\alpha}{\beta} \right) \right\},
\]
If \(p\) is analytic in \(D\) and

\[
\alpha p(z) + \beta zp'(z) < \alpha q(z) + \beta zq'(z),
\]
then \(p < q\) and \(q\) is the best dominant of (2.1).

**Lemma 2.2** [3]. Let \(q\) be univalent in the unit disk \(D\) and let \(\theta\) and \(\phi\) be analytic in a domain \(D\) containing \(q(D)\) with \(\phi(w) \neq 0\), when \(w \in q(D)\). Set \(Q(z) = zq'(z)\phi(q(z))\) and \(h(z) = \theta(q(z)) + Q(z)\). Suppose that

(i) \(Q(z)\) is starlike univalent in \(D\),
(ii) \(\Re \left\{ \frac{\partial h(q(z))}{\partial q(z)} \right\} > 0\) for \(z \in D\).

If \(p\) is analytic in \(D\), with \(p(0) = q(0), p(D) \subseteq D\) and

\[
\theta(p(z)) + zp'(z)\phi(p(z)) < \theta(q(z)) + zq'(z)\phi(q(z)),
\]
then \(p < q\) and \(q\) is the best dominant of (2.2).

**Lemma 2.3** [13]. Let \(q\) be a convex univalent in \(D\) and let \(\beta \in \mathbb{C}\), that \(\Re(\beta) > 0\). If \(p \in H[q(0), 1] \cap Q\) and \(p(z) + \beta zp'(z)\) is univalent in \(D\), then

\[
q(z) + \beta zq'(z) < p(z) + \beta zp'(z),
\]
which implies that \(q < p\) and \(q\) is the best subordinant of (2.3).
Lemma 2.4 [11]. Let \( q \) be univalent in the unit disk \( D \) and let \( \theta \) and \( \phi \) be analytic in a domain \( D \) containing \( q(D) \). Suppose that

\[
(i) \quad \text{Re} \left( \frac{\theta'(q(z))}{\phi(q(z))} \right) > 0 \quad \text{for} \quad z \in D,
\]

\[
(ii) \quad Q(z) = zq'(z)\phi(q(z)) \text{ is starlike univalent in } D.
\]

If \( p \in H[q(0), 1] \cap Q \), with \( p(D) \subset D \), \( \theta(p(z)) + zp'(z)\phi(p(z)) \) is univalent in \( D \) and

\[
\theta(q(z)) + zq'(z)\phi(q(z)) < \theta(p(z)) + zp'(z)\phi(p(z)),
\]

then \( q < p \) and \( q \) is the best subordinant of (2.4).

3-Differential Subordination Results

Here, we introduce some differential subordination results by using the Abbas-Atshan Operator.

Theorem 3.1. Let \( q \) be convex univalent function in \( D \) with \( q(0) = 1 \), \( 0 \neq \varepsilon \in \mathbb{C} \), \( \mu > 0 \) and suppose that \( q \) satisfies:

\[
\text{Re} \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0, -\text{Re} \left( \frac{1}{q'(z)} \right) \right\}.
\]

(3.1)

If \( f \in A \) satisfies the subordination condition:

\[
\left[ \frac{J_{a,\beta,S}f(z)}{z} \right]^\frac{1}{\mu} + \varepsilon \left[ \frac{J_{a,\beta,S}f(z)}{z} \right]^\frac{1}{\mu} \left[ a \left( \frac{J_{a+1,\beta,S}f(z)}{J_{a,\beta,S}f(z)} - 1 \right) \right] < q(z) + \varepsilon \mu q'(z)
\]

(3.2)

then

\[
\left[ \frac{J_{a,\beta,S}f(z)}{z} \right]^\frac{1}{\mu} < q(z),
\]

(3.3)

and \( q \) is the best dominant of (3.2).

Proof. Define the function \( p \) by

\[
p(z) = \left[ \frac{J_{a,\beta,S}f(z)}{z} \right]^\frac{1}{\mu},
\]

(3.4)

then the function \( p(z) \) is analytic in \( D \) and \( p(0) = 1 \), therefore, differentiating (3.4) with respect to \( z \), we get

\[
\frac{zp'(z)}{p(z)} = \frac{1}{\mu} \left[ z \left( \frac{J_{a,\beta,S}f(z)}{J_{a,\beta,S}f(z)} \right)' \right] - 1.
\]

(3.5)

Now, by using the identity (1.7) in (3.5) , we get

\[
\frac{zp'(z)}{p(z)} = \frac{1}{\mu} \left[ a \left( \frac{J_{a+1,\beta,S}f(z)}{J_{a,\beta,S}f(z)} \right) \right].
\]

Therefore,

\[
\mu z p'(z) = \left[ \frac{J_{a,\beta,S}f(z)}{z} \right]^\frac{1}{\mu} \left[ a \left( \frac{J_{a+1,\beta,S}f(z)}{J_{a,\beta,S}f(z)} - 1 \right) \right] .
\]
The subordination (3.2) from the hypothesis becomes
\[ p(z) + \varepsilon \mu z p'(z) < q(z) + \varepsilon \mu q'(z). \]

An application of lemma (2.1) with \( \beta = \varepsilon \mu \) and \( \alpha = 1 \), we obtain (3.3).

Putting \( q(z) = \left( \frac{1+z}{1-z} \right) \) in Theorem (3.1), we obtain the following corollary:

**Corollary 3.1.** Let \( 0 \neq \varepsilon \in \mathbb{C}, \mu > 0 \) and
\[ \Re \left\{ 1 + \frac{2z}{1-z} \right\} > \max \left\{ 0, -\Re \left( \frac{1}{\varepsilon \mu} \right) \right\}. \]

If \( f \in A \) satisfies the subordination
\[ \left[ \frac{J_{\alpha, \beta, S}f(z)}{z} \right]^{1/\mu} + \varepsilon \left[ \frac{J_{\alpha, \beta, S}f(z)}{z} \right]^{1/\mu} \left( \frac{J_{\alpha+1, \beta, S}f(z)}{J_{\alpha, \beta, S}f(z)} - 1 \right) < \left( \frac{1-z^2 + 2\varepsilon \mu z}{(1-z)^2} \right), \]
then
\[ \left[ \frac{J_{\alpha, \beta, S}f(z)}{z} \right]^{1/\mu} < \left( \frac{1+z}{1-z} \right) \]
and \( q(z) = \left( \frac{1+z}{1-z} \right) \) is the best dominant.

**Theorem 3.2.** Let \( q(z) \) be convex univalent function in \( D \) with \( q(0) = 1 \), \( q'(z) \neq 0 \) (\( z \in D \)) and assume that \( q \) satisfies
\[ \Re \left\{ 1 + \frac{q(z)}{\varepsilon} - z \frac{q'(z)}{q(z)} + z^2 \frac{q''(z)}{q(z)} \right\} > 0, \quad (3.6) \]
where \( \varepsilon \in \mathbb{C} \setminus \{0\} \) and \( z \in D \). Suppose that \( z \frac{q'(z)}{q(z)} \) is starlike univalent in \( D \). If \( f \in A \) satisfies
\[ \Psi(\mu, \alpha, \beta, s, \varepsilon; z) < t + q(z) + \varepsilon z \frac{q'(z)}{q(z)}, \quad (3.7) \]
where,
\[ \Psi(\mu, \alpha, \beta, s, \varepsilon; z) = t + \left[ \frac{J_{\alpha+1, \beta, S}f(z)}{J_{\alpha, \beta, S}f(z)} \right]^{1/\mu} + \varepsilon \left[ \frac{J_{\alpha+2, \beta, S}f(z)}{J_{\alpha+1, \beta, S}f(z)} - \frac{J_{\alpha+1, \beta, S}f(z)}{J_{\alpha, \beta, S}f(z)} \right], \quad (3.8) \]
then
\[ \left[ \frac{J_{\alpha+1, \beta, S}f(z)}{J_{\alpha, \beta, S}f(z)} \right]^{1/\mu} < q(z), \quad (3.9) \]
and \( q \) is the best dominant of (3.7).

**Proof.** Define the function \( p \) by
\[ p(z) = \left[ \frac{J_{\alpha+1, \beta, S}f(z)}{J_{\alpha, \beta, S}f(z)} \right]^{1/\mu}. \quad (3.10) \]
Then the function \( p(z) \) is analytic in \( \mathbb{D} \) and \( p(0) = 1 \) differenitating (3.10) with respect to \( z \) and using the identity (1.7), we get,

\[
\frac{zp'(z)}{p(z)} = \frac{1}{\mu} \left[ \alpha \frac{J_{\mu+2,\beta,s}(z)}{J_{\mu+1,\beta,s}(z)} - \frac{J_{\mu+1,\beta,s}(z)}{J_{\alpha,\beta,s}(z)} \right].
\]

By setting \( \theta(w) = t + w \) and \( \phi(w) = \frac{\epsilon}{w}, w \neq 0, \)

we see that \( \theta(w) \) is analytic in \( \mathbb{C} \) and \( \phi(w) \) is analytic in \( \mathbb{C}\setminus \{0\} \) and that \( \phi(w) \neq 0, w \in \mathbb{C}\setminus \{0\} \). Also, we get

\[
Q(z) = zq'(z)\phi(q(z)) = \epsilon z \frac{q'(z)}{q(z)}
\]

and

\[
h(z) = \theta(q(z)) + Q(z) = t + q(z) + \epsilon z \frac{q'(z)}{q(z)}
\]

We see that \( Q(z) \) is starlike univalent in \( \mathbb{D} \), we get

\[
\text{Re} \left( \frac{zh'(z)}{Q(z)} \right) = \text{Re} \left( 1 + \frac{q(z)}{\epsilon} - z \frac{q'(z)}{q(z)} + z \frac{q''(z)}{q'(z)} \right) > 0.
\]

Through simple calculation, we find that

\[
\Psi(\mu, \alpha, \beta, s, \epsilon; z) = t + p(z) + \epsilon z \frac{p'(z)}{p(z)},
\]

where \( \Psi(\mu, \alpha, \beta, s, \epsilon; z) \) is given by (3.8).

From (3.7) and (3.11), we have

\[
t + p(z) + \epsilon z \frac{p'(z)}{p(z)} < t + q(z) + \epsilon z \frac{q'(z)}{q(z)}. \tag{3.12}
\]

Therefore, by Lemma(2.2), we get \( p(z) < q(z) \). By using (3.10), we obtain the result.

Putting \( q(z) = \frac{1+A z}{1+ B z} \), \((-1 \leq B < A \leq 1)\) in Theorem(3.2), we obtain the following corollary:

**Corollary 3.2.** Let \(-1 \leq B < A \leq 1\) and

\[
\text{Re} \left( 1 + \frac{1 + A z}{\epsilon (1 + B z)} - \frac{z(A - B)}{(1 + B z)(1 + A z)} - \frac{2z}{1 + B z} \right) > 0,
\]

where \( \epsilon \in \mathbb{C}\setminus \{0\} \) and \( z \in \mathbb{D} \), iff \( z \in A \) satisfies

\[
\Psi(\mu, \alpha, \beta, s, \epsilon; z) < t + \left( \frac{1 + A z}{1 + B z} \right) + \epsilon z \frac{A - B}{(1 + A z)(1 + B z)},
\]

where is given \( \Psi(\mu, \alpha, \beta, s, \epsilon; z) \) by (3.8), then

\[
\left[ \frac{J_{\alpha+1,\beta,s}(z)}{J_{\alpha,\beta,s}(z)} \right]^{\frac{1}{2}} < \frac{1 + A z}{1 + B z}
\]
and q(\(z\)) = \(\frac{1+4z}{1+4z^2}\) is the best dominant.

### 4-Differential Superordination Results

**Theorem 4.1.** Let \(q\) be convex univalent function in \(D\) with \(q(0) = 1, \mu > 0\) and \(\text{Re}\{\varepsilon\} > 0\). Let \(f \in A\) satisfies

\[
\left[ \frac{J_{a,b,s}f(z)}{z} \right]^\frac{1}{\mu} \in H[q(0),1] \cap Q
\]

and

\[
\left[ \frac{J_{a,b,s}f(z)}{z} \right]^\frac{1}{\mu} + \varepsilon \left[ \frac{J_{a,b,s}f(z)}{z} \right]^\frac{1}{\mu} \left[ \alpha \left( \frac{J_{a+1,b,s}f(z)}{J_{a,b,s}f(z)} - 1 \right) \right]
\]

be univalent in \(D\). If

\[
q(z) + \varepsilon \mu q'(z) < \left[ \frac{J_{a,b,s}f(z)}{z} \right]^\frac{1}{\mu} + \varepsilon \left[ \frac{J_{a,b,s}f(z)}{z} \right]^\frac{1}{\mu} \left[ \alpha \left( \frac{J_{a+1,b,s}f(z)}{J_{a,b,s}f(z)} - 1 \right) \right],
\]

then

\[
q(z) \prec \left[ \frac{J_{a,b,s}f(z)}{z} \right]^\frac{1}{\mu}
\]

and \(q\) is the best subordinant of (4.1).

**Proof.** Define the function \(p\) by

\[
p(z) = \left[ \frac{J_{a,b,s}f(z)}{z} \right]^\frac{1}{\mu}.
\]

Differentiating (4.3) with respect to \(z\), we get

\[
\frac{zp'(z)}{p(z)} = \frac{1}{\mu} \left[ z \left( \frac{J_{a,b,s}f(z)}{z} \right)' - 1 \right].
\]

After some computations and using (1.7), from (4.4), we obtain

\[
\left[ \frac{J_{a,b,s}f(z)}{z} \right]^\frac{1}{\mu} + \varepsilon \left[ \frac{J_{a,b,s}f(z)}{z} \right]^\frac{1}{\mu} \left[ \alpha \left( \frac{J_{a+1,b,s}f(z)}{J_{a,b,s}f(z)} - 1 \right) \right] = p(z) + \varepsilon \mu p'(z),
\]

and now, by using Lemma(2.3), we get the desired result.

Putting \(q(z) = \left( \frac{1+z}{1-z} \right)\) in Theorem (4.1), we obtain the following corollary:

**Corollary 4.1.** Let \(\mu > 0\) and \(\text{Re}\{\varepsilon\} > 0\). If \(f \in A\) satisfies

\[
\left[ \frac{J_{a,b,s}f(z)}{z} \right]^\frac{1}{\mu} \in H[q(0),1] \cap Q
\]

and
\[
\left[\frac{\partial_{\alpha, \beta, s} f(z)}{z} \right]^\mu + \epsilon \left[\frac{\partial_{\alpha, \beta, s} f(z)}{z} \right]^\mu \left(1 + \frac{\partial_{\alpha, \beta, s} f(z)}{\partial_{\alpha+1, \beta, s} f(z)} - 1\right) \right] \text{ be univalent in } D. \text{ If }
\left(1 - \frac{z^2 + 2 \epsilon \mu z}{(1 - z)^2}\right) < \left[\frac{\partial_{\alpha, \beta, s} f(z)}{z} \right]^\mu + \epsilon \left[\frac{\partial_{\alpha, \beta, s} f(z)}{z} \right]^\mu \left(1 + \frac{\partial_{\alpha, \beta, s} f(z)}{\partial_{\alpha+1, \beta, s} f(z)} - 1\right),
\]
then
\[
\frac{1 + z}{1 - z} < \left[\frac{\partial_{\alpha, \beta, s} f(z)}{z} \right]^\mu
\]
and \(q(z) = \frac{1 + z}{1 - z}\) is the best subordinant.

**Theorem 4.2.** Let \(q\) be convex univalent function in \(D\) with \(q(0) = 1, q'(z) \neq 0\) and assume that \(q\) satisfies
\[
\text{Re}\left\{\frac{q(z)}{\epsilon} q'(z)\right\} > 0, \quad (4.5)
\]
where \(\epsilon \in \mathbb{C}\setminus\{0\}\) and \(z \in D\).

Suppose that \(z\frac{q'(z)}{q(z)}\) is starlike univalent in \(D\). Let \(f \in A\) satisfies
\[
\left[\frac{\partial_{\alpha+1, \beta, s} f(z)}{\partial_{\alpha, \beta, s} f(z)} \right]^\mu \in H[q(0), 1] \cap Q,
\]
and \(\Psi(\mu, \alpha, \beta, s, \epsilon; z)\) is univalent in \(D\), where is given \(\Psi(\mu, \alpha, \beta, s, \epsilon; z)\) by (3.8). If
\[
t + q(z) + \epsilon z \frac{q'(z)}{q(z)} < \Psi(\mu, \alpha, \beta, s, \epsilon; z), \quad (4.6)
\]
then
\[
q(z) < \left[\frac{\partial_{\alpha+1, \beta, s} f(z)}{\partial_{\alpha, \beta, s} f(z)} \right]^\mu \quad (4.7)
\]
and \(q\) is the best subordinant of (4.6).

**Proof.** Define the function \(p\) by
\[
p(z) = \left[\frac{\partial_{\alpha+1, \beta, s} f(z)}{\partial_{\alpha, \beta, s} f(z)} \right]^\mu. \quad (4.8)
\]
Differentiating (4.8) with respect to \(z\), we get
\[
\frac{zp'(z)}{p(z)} = \frac{1}{\mu} \left(1 + \frac{\partial_{\alpha+2, \beta, s} f(z)}{\partial_{\alpha+1, \beta, s} f(z)} - \frac{\partial_{\alpha+1, \beta, s} f(z)}{\partial_{\alpha, \beta, s} f(z)} \right).
\]
By setting \(\theta(w) = t + w\) and \(\phi(w) = \frac{\epsilon}{w}, w \neq 0,\)
we see that \(\theta(w)\) is analytic in \(\mathbb{C}\) and \(\phi(w)\) is analytic in \(\mathbb{C}\setminus\{0\}\) and that \(\phi(w) \neq 0, w \in \mathbb{C}\setminus\{0\}\). Also, we get
\[
Q(z) = zq'(z)\phi(q(z)) = \epsilon zq'(z) q(z).
\]
It is clear that $Q(z)$ is starlike univalent in $D$,

$$\text{Re}\left\{\frac{\psi(z)}{\psi'(z)}\right\} = \text{Re}\left\{\frac{q(z)}{q'(z)}\right\} > 0.$$ 

By a straightforward computation, we obtain

$$\Psi(\mu, \alpha, \beta, s, \varepsilon; z) = t + p(z) + \varepsilon z \frac{p'(z)}{p(z)},$$

(4.9)

where $\Psi(\mu, \alpha, \beta, s, \varepsilon; z)$ is given by (3.8).

From (4.6) and (4.9), we have

$$t + q(z) + \varepsilon z \frac{q'(z)}{q(z)} < t + p(z) + \varepsilon z \frac{p'(z)}{p(z)}.$$ 

(4.10)

Therefore, by Lemma(2.4), we get $q(z) < p(z)$.

5-Sandwich Results

**Theorem 5.1.** Let $q_1$ be convex univalent function in $D$ with $q_1(0) = 1$, $\mu > 0$ and $\text{Re}(\varepsilon) > 0$ and $q_2$ be univalent in $D$, $q_2(0) = 1$ and satisfies (3.1). Let $f \in A$ satisfies

$$\left[\frac{j_{\alpha, \beta, s} f(z)}{z}\right]^\frac{1}{\mu} \in \mathbb{H}[1,1] \cap Q$$

and

$$\left[\frac{j_{\alpha, \beta, s} f(z)}{z}\right]^\frac{1}{\mu} + \varepsilon \left[\frac{j_{\alpha, \beta, s} f(z)}{z}\right]^\frac{1}{\mu} \left[\alpha \left(\frac{j_{\alpha+1, \beta, s} f(z)}{j_{\alpha, \beta, s} f(z)} - 1\right)\right]$$

be univalent in $D$. If

$$q_1(z) + \varepsilon \mu q_1'(z) < \left[\frac{j_{\alpha, \beta, s} f(z)}{z}\right]^\frac{1}{\mu} + \varepsilon \left[\frac{j_{\alpha, \beta, s} f(z)}{z}\right]^\frac{1}{\mu} \left[\alpha \left(\frac{j_{\alpha+1, \beta, s} f(z)}{j_{\alpha, \beta, s} f(z)} - 1\right)\right] < q_2(z) + \varepsilon \mu q_2'(z),$$

then

$$q_1(z) < \left[\frac{j_{\alpha, \beta, s} f(z)}{z}\right]^\frac{1}{\mu} < q_2(z)$$

and $q_1$ and $q_2$ are respectively the best subordinant and the best dominant.

**Theorem 5.2.** Let $q_1$ be convex univalent in $D$ with $q_1(0) = q_2(0) = 1$. Suppose $q_1$ satisfies (4.5) and $q_2$ satisfies (3.6). Let $f \in A$ satisfies

$$\left[\frac{j_{\alpha, \beta, s} f(z)}{z}\right]^\frac{1}{\mu} \in \mathbb{H}[1,1] \cap Q$$

and $\Psi(\mu, \alpha, \beta, s, \varepsilon; z)$ is univalent in $D$, where is given $\Psi(\mu, \alpha, \beta, s, \varepsilon; z)$ by (3.8). If

$$t + q_1(z) + \varepsilon z \frac{q_1'(z)}{q_1(z)} < \Psi(\mu, \alpha, \beta, s, \varepsilon; z) < t + q_2(z) + \varepsilon z \frac{q_2'(z)}{q_2(z)}.$$
then

\[ q_1(z) < \left[ \frac{\gamma_{a+1, \beta, S\alpha f(z)}}{\gamma_{a, \beta, S\alpha f(z)}} \right]^{\frac{1}{\mu}} < q_2(z) \]

and \( q_1 \) and \( q_2 \) are respectively the best subordinant and the best dominant.

References