Journal of Al-Qadisiyah for Computer Science and Mathematics Vol. 15(4) 2023 , pp math. 11–20

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)

On Sandwich Results of Univalent Functions Defined by Generalized Abbas-Atshan Operator

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ARTICLE INFO

Article history:
Received: 13 /12/2023
Rrevised form: 28 /12/2023
Accepted : 29 /12/2023
Available online: 30 /12/2023

Keywords:

Analytic function, Integral operator, Differential subordination, Superordination. Sandwich theorem.

ABSTRACT

In the present paper, we obtain sandwich theorems for univalent functions by using some results of differential subordination and superordination for univalent functions involving integral operator.

https:// 10.29304/jqcsm.2023.15.41350

1. **Introduction**

Let $H = H(D)$ be the class of analytic functions in the open unit disk $D = \{z \in \mathbb{C}: |z| < 1\}$. For $n \in N$ and $a \in \mathbb{C}$. Let $H[a, n]$ be the subclass of H of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (a \in \mathbb{C}).$$

Let A denote the subclass of H of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n , (z \in D), \tag{1.1}$$

which are analytic in the open unit disk $D = \{z \in \mathbb{C}: |z| < 1\}$. Let f and g are analytic functions in H, f is said to be subordinate to g, or g is said to be superordinate to f in D and write $f \prec g$, if there exists a Schwarz function K in D,

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which with $K(0)=0$, and $|K(z)|<1$, $(z\in D)$ where $f(z)=g(K(z))$. In such a case we write $f\prec g$ or $f(z)\prec g(z)$ $(z\in D)$. If g is univalent in D, then $f\prec g$ if and only if $f(0)=g(0)$ and $f(D)\subset g(D)$. ([12,13]).

Definition(1)[12]: Let $\emptyset:\mathbb{C}^3\times D\to\mathbb{C}$ and $h(z)$ be univalent in D. If $p(z)$ is analytic in D and satisfies the second-order differential subordination:

$$\emptyset(p(z), z\,p'(z), z^2p''(z); z)\prec h(z), \tag{1.2}$$

then $p(z)$ is called a solution of the differential subordination (1.2), and the univalent function $q(z)$ is called a dominant of the solution of the differential subordination (1.2), or more simply dominant if $p(z)\prec q(z)$ for all $p(z)$ satisfying (1.2). A univalent dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z)\prec q(z)$ for all dominant $q(z)$ of (1.2) is said to be the best dominant is unique up to a relation of D.

Definition(2) [12] : Let $p,h\in A$ and $\emptyset(r,s,t;z):\mathbb{C}^3\times D\to\mathbb{C}$. If p and $\emptyset(p(z), z\,p'(z), z^2p''(z); z)$ are univalent functions in D and if p satisfies:

$$h(z)\prec\emptyset(p(z), z\,p'(z), z^2p''(z); z), \tag{1.3}$$

then p is called a solution of the differential superordination (1.3). An analytic function $q(z)$, which is called a subordinat of the solutions of the differential superordination (1.3), or more simply a subordinant if $p\prec q$ for all the functions p satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies $q\prec\tilde{q}$ for all the subordinates q of (1.3) is said to be the best subordinat.

Several authors [7,12] obtained sufficient conditions on the functions h, p and $\emptyset$ for which the following implication holds

$$h(z)\prec\emptyset(p(z), zp'(z), z^2p''(z); z),$$

$$\text{then } q(z)\prec p(z). \tag{1.4}$$

Using the results (see [2,3,4,5,9,10,13]) to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z)\prec\frac{zf'(z)}{f(z)}\prec q_2(z),$$

where q_1 and q_2 are given univalent functions in D and $q_1(0)=q_2(0)=1$. Also, several authors (see[2,4,5,6,8]) derived some differential subordination and superordination results with some sandwich theorems.

For $f\in A$, Abbas-Atshan Operator[1] defined the following generalized integral operator:

$$\mathcal{J}_{\alpha,\beta,s}f(z)=\frac{s^{\alpha}(\ln(\beta))^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}v^{\alpha-2}\beta^{-sv}f(zv)dv, \tag{1.5}$$

where $\alpha,s\in N$ and $\beta\geq 2$.

For $f(z)\in A$ given by (1.1), we have

$$\mathcal{J}_{\alpha,\beta,s}f(z)=z+\sum_{n=2}^{\infty}\left(\frac{\Gamma(\alpha+n-1)}{(\ln(\beta))^{n-1}\Gamma(\alpha)s^{n-1}}\right)a_nz^n. \tag{1.6}$$

From (1.6), we note that

$$z\left(\mathcal{J}_{\alpha,\beta,s}f(z)\right)'=\alpha\mathcal{J}_{\alpha+1,\beta,s}f(z)-(\alpha-1)\mathcal{J}_{\alpha,\beta,s}f(z). \tag{1.7}$$

The main object of the present investigation is to find sufficient conditions for certan normalized analytic function f to satisfy:

$$q_1(z) \prec \left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}} \prec q_2(z).$$

and

$$q_1(z) \prec \left[\frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)}\right]^{\frac{1}{\mu}} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in D with $q_1(0) = q_2(0) = 1$.

In this paper, we derive some sandwich theorems, involving the operator $\mathcal{J}_{\alpha,\beta,s}f(z)$.

2-Preliminaries

We need the following definitions and lemmas to prove our results.

Definition 2.1 [12]. Denote by Q the set of all functions q that are analytic and injective on $\overline{D}\backslash E(q)$, where $\overline{D} = D \cup \{z \in \partial D\}$, and

$$E(q) = \left\{\varepsilon \in \partial D: \lim_{z\to\varepsilon} q(z) = \infty\right\}$$

and are such that $q'(\varepsilon) \neq 0$ for $\varepsilon \in \partial D\backslash E(q)$. Further, let the subclass of Q for which $q(0) = a$ be denoted by Q(a), and $Q(0) = Q_0, Q(1) = Q_1 = \{q \in Q: q(0) = 1\}$.

Lemma 2.1 [13]. Let q be a convex univalent function in D and let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}\backslash\{0\}$ with

$$\mathrm{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\mathrm{Re}(\frac{\alpha}{\beta})\right\}.$$

If p is analytic in D and

$$\alpha p(z) + \beta z p'(z) \prec \alpha q(z) + \beta z q'(z), \tag{2.1}$$

then $p \prec q$ and q is the best dominant of (2.1).

Lemma 2.2 [3]. Let q be univalent in the unit disk D and let θ and ϕ be analytic in a domain D containing q(D) with $\phi(w) \neq 0$,when $w \in q(D)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta\big(q(z)\big) + Q(z)$. Suppose that

(i) Q(z) is starlike univalent in D,

(ii) $\mathrm{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in D$.

If p is analytic in D, with $p(0) = q(0)$, $p(D) \subseteq D$ and

$$\theta\big(p(z)\big) + zp'(z)\phi\big(p(z)\big) \prec \theta\big(q(z)\big) + zq'(z)\phi\big(q(z)\big), \tag{2.2}$$

then $p \prec q$ and q is the best dominant of (2.2).

Lemma 2.3 [13]. Let q be a convex univalent in D and let $\beta \in \mathbb{C}$, that $\mathrm{Re}(\beta) > 0$. If $p \in H[q(0), 1] \cap Q$ and $p(z) + \beta z p'(z)$ is univalent in D, then

$$q(z) + \beta z q'(z) \prec p(z) + \beta z p'(z), \tag{2.3}$$

which implies that $q \prec p$ and q is the best subordinant of (2.3).

Lemma 2.4 [11]. Let q be univalent in the unit disk D and let θ and ϕ be analytic in a domain D containing q(D). Suppose that

(i) $\mathrm{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ for $z \in D$,

(ii) $Q(z) = zq'(z)\phi(q(z))$ is starlike univalent in D.

If $p \in H[q(0), 1] \cap Q$, with $p(D) \subset D$, $\theta(p(z)) + zp'(z)\phi(p(z))$ is univalent in D and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)), \tag{2.4}$$

then $q \prec p$ and q is the best subordinant of (2.4).

3-Differential Subordination Results

Here, we introduce some differential subordination results by using the Abbas-Atshan Operator.

Theorem 3.1. Let q be convex univalent function in D with $q(0) = 1, 0 \neq \varepsilon \in \mathbb{C}, \mu > 0$ and suppose that q satisfies:

$$\mathrm{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\mathrm{Re}\left(\frac{1}{\varepsilon\mu}\right)\right\}. \tag{3.1}$$

If $f \in A$ satisfies the subordination condition :

$$\left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}} + \varepsilon\left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}}\left[\alpha\left(\frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)} - 1\right)\right] \prec q(z) + \varepsilon\mu zq'(z) \tag{3.2}$$

then

$$\left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}} \prec q(z), \tag{3.3}$$

and q is the best dominant of (3.2).

Proof. Define the function p by

$$p(z) = \left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}}, \tag{3.4}$$

then the function p(z) is analytic in D and $p(0) = 1$, therefore, differentiating (3.4) with respect to z ,we get

$$\frac{zp'(z)}{p(z)} = \frac{1}{\mu}\left[\frac{z\left(\mathcal{J}_{\alpha,\beta,s}f(z)\right)'}{\mathcal{J}_{\alpha,\beta,s}f(z)} - 1\right]. \tag{3.5}$$

Now, by using the identity (1.7) in (3.5) ,we get

$$\frac{zp'(z)}{p(z)} = \frac{1}{\mu}\left[\alpha\left(\frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)} - 1\right)\right].$$

Therefore,

$$\mu zp'(z) = \left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}}\left[\alpha\left(\frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)} - 1\right)\right].$$

The subordination (3.2) from the hypothesis becomes

$$p(z) + \varepsilon\mu z p'(z) \prec q(z) + \varepsilon\mu z q'(z).$$

An application of lemma (2.1) with $\beta = \varepsilon\mu$ and $\alpha = 1$, we obtain (3.3).

Putting $q(z) = \left(\frac{1+z}{1-z}\right)$ in Theorem (3.1), we obtain the following corollary:

Corollary 3.1. Let $0 \neq \varepsilon \in \mathbb{C}$, $\mu > 0$ and

$$\mathrm{Re}\left\{1 + \frac{2z}{1-z}\right\} > \max\left\{0, -\mathrm{Re}\left(\frac{1}{\varepsilon\mu}\right)\right\}.$$

If $f \in A$ satisfies the subordination

$$\left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}} + \varepsilon\left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}}\left[\alpha\left(\frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)} - 1\right)\right] \prec \left(\frac{1 - z^2 + 2\varepsilon\mu z}{(1-z)^2}\right),$$

then

$$\left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}} \prec \left(\frac{1+z}{1-z}\right)$$

and $q(z) = \left(\frac{1+z}{1-z}\right)$ is the best dominant.

Theorem 3.2. Let $q(z)$ be convex univalent function in D with $q(0) = 1$, $q'(z) \neq 0$ $(z \in D)$ and assume that q satisfies

$$\mathrm{Re}\left\{1 + \frac{q(z)}{\varepsilon} - z\frac{q'(z)}{q(z)} + z\frac{q''(z)}{q'(z)}\right\} > 0, \tag{3.6}$$

where $\varepsilon \in \mathbb{C}\backslash\{0\}$ and $z \in D$. Suppose that $z\frac{q'(z)}{q(z)}$ is starlike univalent in D. If $f \in A$ satisfies

$$\Psi(\mu, \alpha, \beta, s, \varepsilon; z) \prec t + q(z) + \varepsilon z\frac{q'(z)}{q(z)}, \tag{3.7}$$

where,

$$\Psi(\mu, \alpha, \beta, s, \varepsilon; z) = t + \left[\frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)}\right]^{\frac{1}{\mu}} + \frac{\varepsilon}{\mu}\left[\alpha\left(\frac{\mathcal{J}_{\alpha+2,\beta,s}f(z)}{\mathcal{J}_{\alpha+1,\beta,s}f(z)} - \frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)}\right)\right], \tag{3.8}$$

then

$$\left[\frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)}\right]^{\frac{1}{\mu}} \prec q(z) \tag{3.9}$$

and q is the best dominant of (3.7).

Proof. Define the function p by

$$p(z) = \left[\frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)}\right]^{\frac{1}{\mu}}. \tag{3.10}$$

Then the function $p(z)$ is analytic in D and $p(0) = 1$ differenitating (3.10) with respect to z and using the identity (1.7), we get,

$$\frac{zp'(z)}{p(z)} = \frac{1}{\mu}\left[\alpha\left(\frac{\mathcal{J}_{\alpha+2,\beta,s}f(z)}{\mathcal{J}_{\alpha+1,\beta,s}f(z)} - \frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)}\right)\right].$$

By setting $\theta(w) = t + w$ and $\phi(w) = \frac{\varepsilon}{w}$, $w \neq 0$,

we see that $\theta(w)$ is analytic in $\mathbb{C}$ and $\phi(w)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = \varepsilon z\frac{q'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = t + q(z) + \varepsilon z\frac{q'(z)}{q(z)}.$$

We see that $Q(z)$ is starlike univalent in D, we gat

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{1 + \frac{q(z)}{\varepsilon} - z\frac{q'(z)}{q(z)} + z\frac{q''(z)}{q'(z)}\right\} > 0.$$

Through simple calculation, we find that

$$\Psi(\mu,\alpha,\beta,s,\varepsilon;z) = t + p(z) + \varepsilon z\frac{p'(z)}{p(z)}, \tag{3.11}$$

where $\Psi(\mu,\alpha,\beta,s,\varepsilon;z)$ is given by (3.8).

From (3.7) and (3.11), we have

$$t + p(z) + \varepsilon z\frac{p'(z)}{p(z)} \prec t + q(z) + \varepsilon z\frac{q'(z)}{q(z)}. \tag{3.12}$$

Therefore, by Lemma(2.2), we get $p(z) \prec q(z)$. By using (3.10), we obtain the result.

Putting $q(z) = \left(\frac{1+Az}{1+Bz}\right)$, $(-1 \le B < A \le 1)$ in Theorem(3.2), we obtain the following corollary:

Corollary 3.2. Let $-1 \le B < A \le 1$ and

$$\operatorname{Re}\left\{1 + \frac{1+Az}{\varepsilon(1+Bz)} - \frac{z(A-B)}{(1+Bz)(1+Az)} - \frac{2z}{1+Bz}\right\} > 0,$$

where $\varepsilon \in \mathbb{C}\backslash\{0\}$ and $z \in D$, if $f \in A$ satisfies

$$\Psi(\mu,\alpha,\beta,s,\varepsilon;z) \prec t + \left(\frac{1+Az}{1+Bz}\right) + \varepsilon z\frac{A-B}{(1+Az)(1+Bz)},$$

where is given $\Psi(\mu,\alpha,\beta,s,\varepsilon;z)$ by (3.8), then

$$\left[\frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)}\right]^{\frac{1}{\mu}} \prec \left(\frac{1+Az}{1+Bz}\right)$$

and $q(z) = \left(\frac{1+Az}{1+Bz}\right)$ is the best dominant.

4-Differential Superordination Results

Theorem 4.1. Let q be convex univalent function in D with $q(0) = 1$, $\mu > 0$ and $\text{Re}\{\varepsilon\} > 0$. Let $f \in A$ satisfies

$$\left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}} \in H[q(0),1] \cap Q$$

and

$\left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}} + \varepsilon\left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}}\left[\alpha\left(\frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)} - 1\right)\right]$ be univalent in D. If

$$q(z) + \varepsilon\mu z q'(z) \prec \left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}} + \varepsilon\left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}}\left[\alpha\left(\frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)} - 1\right)\right], \tag{4.1}$$

then

$$q(z) \prec \left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}} \tag{4.2}$$

and q is the best subordinant of (4.1).

Proof. Define the function p by

$$p(z) = \left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}}. \tag{4.3}$$

Differentiating (4.3) with respect to z , we get

$$\frac{zp'(z)}{p(z)} = \frac{1}{\mu}\left[\frac{z\left(\mathcal{J}_{\alpha,\beta,s}f(z)\right)'}{\mathcal{J}_{\alpha,\beta,s}f(z)} - 1\right]. \tag{4.4}$$

After some computations and using (1.7), from (4.4), we obtain

$$\left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}} + \varepsilon\left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}}\left[\alpha\left(\frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)} - 1\right)\right] = p(z) + \varepsilon\mu z p'(z),$$

and now, by using Lemma(2.3), we get the desired result.

Putting $q(z) = \left(\frac{1+z}{1-z}\right)$ in Theorem (4.1), we obtain the following corollary:

Corollary 4.1. Let $\mu > 0$ and $\text{Re}\{\varepsilon\} > 0$. If $f \in A$ satisfies

$$\left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}} \in H[q(0),1] \cap Q$$

and

$\left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}} + \varepsilon\left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}}\left[\alpha\left(\frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)} - 1\right)\right]$ be univalent in D. If

$$\left(\frac{1 - z^2 + 2\varepsilon\mu z}{(1-z)^2}\right) \prec \left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}} + \varepsilon\left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}}\left[\alpha\left(\frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)} - 1\right)\right],$$

then

$$\left(\frac{1+z}{1-z}\right) \prec \left[\frac{\mathcal{J}_{\alpha,\beta,s}f(z)}{z}\right]^{\frac{1}{\mu}}$$

and $q(z) = \left(\frac{1+z}{1-z}\right)$ is the best subordinant.

Theorem 4.2. Let q be convex univalent function in D with $q(0) = 1, q'(z) \neq 0$ and assume that q satisfies

$$\operatorname{Re}\left\{\frac{q(z)}{\varepsilon}q'(z)\right\} > 0, \tag{4.5}$$

where $\varepsilon \in \mathbb{C}\backslash\{0\}$ and $z \in D$.

Suppose that $z\frac{q'(z)}{q(z)}$ is starlike univalent in D. Let $f \in A$ satisfies

$$\left[\frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)}\right]^{\frac{1}{\mu}} \in H[q(0), 1] \cap Q,$$

and $\Psi(\mu, \alpha, \beta, s, \varepsilon; z)$ is univalent in D, where is given $\Psi(\mu, \alpha, \beta, s, \varepsilon; z)$ by (3.8). If

$$t + q(z) + \varepsilon z\frac{q'(z)}{q(z)} \prec \Psi(\mu, \alpha, \beta, s, \varepsilon; z), \tag{4.6}$$

then

$$q(z) \prec \left[\frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)}\right]^{\frac{1}{\mu}} \tag{4.7}$$

and q is the best subordinant of (4.6).

Proof. Define the function p by

$$p(z) = \left[\frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)}\right]^{\frac{1}{\mu}}. \tag{4.8}$$

Differentiating (4.8) with respect to z, we get

$$\frac{zp'(z)}{p(z)} = \frac{1}{\mu}\left[\alpha\left(\frac{\mathcal{J}_{\alpha+2,\beta,s}f(z)}{\mathcal{J}_{\alpha+1,\beta,s}f(z)} - \frac{\mathcal{J}_{\alpha+1,\beta,s}f(z)}{\mathcal{J}_{\alpha,\beta,s}f(z)}\right)\right].$$

By setting $\theta(w) = t + w$ and $\phi(w) = \frac{\varepsilon}{w}$, $w \neq 0$,

we see that $\theta(w)$ is analytic in $\mathbb{C}$ and $\phi(w)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0, w \in \mathbb{C}\backslash\{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = \varepsilon z\frac{q'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in D,

$$\mathrm{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} = \mathrm{Re}\left\{\frac{q(z)}{\varepsilon} q'(z)\right\} > 0.$$

By a straightforward computation, we obtain

$$\Psi(\mu, \alpha, \beta, s, \varepsilon; z) = t + p(z) + \varepsilon z \frac{p'(z)}{p(z)}, \tag{4.9}$$

where $\Psi(\mu, \alpha, \beta, s, \varepsilon; z)$ is given by (3.8).

From (4.6) and (4.9), we have

$$t + q(z) + \varepsilon z \frac{q'(z)}{q(z)} \prec t + p(z) + \varepsilon z \frac{p'(z)}{p(z)}. \tag{4.10}$$

Therefore, by Lemma(2.4), we get $q(z) \prec p(z)$.

5-Sandwich Results

Theorem 5.1. Let q_1 be convex univalent function in D with $q_1(0) = 1$, $\mu > 0$ and $\mathrm{Re}\{\varepsilon\} > 0$ and q_2 be univalent D, $q_2(0) = 1$ and satisfies (3.1). Let $f \in A$ satisfies

$$\left[\frac{\mathcal{J}_{\alpha,\beta,s} f(z)}{z}\right]^{\frac{1}{\mu}} \in H[1,1] \cap Q$$

and

$$\left[\frac{\mathcal{J}_{\alpha,\beta,s} f(z)}{z}\right]^{\frac{1}{\mu}} + \varepsilon \left[\frac{\mathcal{J}_{\alpha,\beta,s} f(z)}{z}\right]^{\frac{1}{\mu}} \left[\alpha\left(\frac{\mathcal{J}_{\alpha+1,\beta,s} f(z)}{\mathcal{J}_{\alpha,\beta,s} f(z)} - 1\right)\right]$$

be univalent in D. If

$$q_1(z) + \varepsilon\mu z q_1{}'(z) \prec \left[\frac{\mathcal{J}_{\alpha,\beta,s} f(z)}{z}\right]^{\frac{1}{\mu}} + \varepsilon \left[\frac{\mathcal{J}_{\alpha,\beta,s} f(z)}{z}\right]^{\frac{1}{\mu}} \left[\alpha\left(\frac{\mathcal{J}_{\alpha+1,\beta,s} f(z)}{\mathcal{J}_{\alpha,\beta,s} f(z)} - 1\right)\right] \prec q_2(z) + \varepsilon\mu z q_2{}'(z),$$

then

$$q_1(z) \prec \left[\frac{\mathcal{J}_{\alpha,\beta,s} f(z)}{z}\right]^{\frac{1}{\mu}} \prec q_2(z)$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

Theorem 5.2. Let q_1 be convex univalent in D with $q_1(0) = q_2(0) = 1$. Suppose q_1 satisfies (4.5) and q_2 satisfies (3.6). Let $f \in A$ satisfies

$$\left[\frac{\mathcal{J}_{\alpha+1,\beta,s} f(z)}{\mathcal{J}_{\alpha,\beta,s} f(z)}\right]^{\frac{1}{\mu}} \in H[1,1] \cap Q$$

and $\Psi(\mu, \alpha, \beta, s, \varepsilon; z)$ is univalent in D, where is given $\Psi(\mu, \alpha, \beta, s, \varepsilon; z)$ by (3.8). If

$$t + q_1(z) + \varepsilon z \frac{q_1{}'(z)}{q_1(z)} \prec \Psi(\mu, \alpha, \beta, s, \varepsilon; z) \prec t + q_2(z) + \varepsilon z \frac{q_2{}'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left[\frac{\mathcal{J}_{\alpha+1,\beta,S} f(z)}{\mathcal{J}_{\alpha,\beta,S} f(z)}\right]^{\frac{1}{\mu}} \prec q_2(z)$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

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