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New Sandwich-Type Results of Meromorphic Multivalent Functions Defined by Rafid operator

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ABSTRACT

this study aims to ascertain the outcomes differantial subordnation and superordnation for meromorphic p-valent functions given by the Rafid operetor within a punctared open unit disc. We acquire multiple results that bear a resemblance to sandwiches.

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1. Introduction

Letting $\sum_{\wp}$ represent a collection of anallytic functions that may be exprassed in the following:

$$\mathfrak{L}(z) = \frac{1}{z^{\wp}} + \sum_{\mathcal{K}=1}^{\infty} \mathfrak{v}_{\mathcal{K}} z^{\mathcal{K}}, \tag{1.1}$$

These are meromorphic functions that are analytic and have multivalant within a puntured open dick $\mathfrak{U}^* = (z: z \in \mathbb{C}, 0 < |z| < 1\ )$. Multiple writers have conducted research on meromorphic functions in various classes and under different settings, as documented in references [8, 9, 21]. $\mathfrak{K}$ represents a linear space comprising each analytic functions in $\mathfrak{U}$. Given positive integar $\mathfrak{n}$ and complex number $\mathfrak{v}$, we define

$$\mathfrak{K}[\mathfrak{v}, \mathfrak{n}] = \{\mathfrak{L} \in \mathfrak{K} : \mathfrak{L}(z) = \mathfrak{v} + \mathfrak{v}_{\mathfrak{n}} z^{\mathfrak{n}} + \mathfrak{v}_{\mathfrak{n}+1} z^{\mathfrak{n}+1} + \cdots\} \quad (\mathfrak{v} \in \mathbb{C}).$$

A variables $\mathfrak{I}$, $\mathfrak{L}$ are analytic fanctions within $\mathfrak{K}$, we assert that $\mathfrak{L}$ is subordnate to $\mathfrak{I}$ in $\mathfrak{U}$, or $\mathfrak{I}$ superordnate to $\mathfrak{L}$ in $\mathfrak{U}$ writes $\mathfrak{L}(z) \prec \mathfrak{I}(z)$, if a Schiwarz function exists $\mathfrak{w}$ within $\mathfrak{U}$ which, accompanied by $\mathfrak{w}(0) = 0$, also $|\mathfrak{w}(z)| < 1$, $(z \in \mathfrak{U})$, where $\mathfrak{L}(z) = \mathfrak{I}\,(\mathfrak{w}(z))$.

Furthermore, assuming $\mathfrak{I}$ is a univalent function in $\mathfrak{U}$, we possess the subsequent equivalency reletionship, as indicated by the references [10,11,15,16]:

$$\mathfrak{L}(z) \prec \mathfrak{I}(z) \leftrightarrow \mathfrak{L}(0) = \mathfrak{I}(0),\ \mathfrak{L}(\mathfrak{U}) \subset \mathfrak{I}(\mathfrak{U}), (z \in \mathfrak{U}).$$

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Definition (1.1) ([15], also see[21]): Letting $\mathfrak{X}:\mathbb{C}^3\times\mathfrak{U}\to\mathbb{C}$ with $\mathfrak{Y}(z)$ be analytic within $\mathfrak{U}$. If $p(z)$ with $\mathfrak{X}(p(z),zp'(z),z^2p''(z);z)$ be univalant functions in $\mathfrak{U}$, when $p(z)$ has to fulfill the sacond-order differantial superordnation:

$$\mathfrak{Y}(z)\prec\mathfrak{X}(p(z),zp'(z),z^2p''(z);z), \tag{1.2}$$

therefore, $p(z)$ is referred to as a selution of a differantial superordnation (1.2). A subordinant function $q(z)$ is an analytic function that is associated with a solutions of a differential superordnation (1.2), in simpler terms, a subordnant if $q\prec f$ for each the fanctions p fulfills (1.2). A univalant subordnant $\check{q}$ which fulfills $q\prec\check{q}$ as each the subordnants q the superiority in (1.2) as a best subordnant.

Definition (1.2) [15]: Letting $\mathfrak{X}:\mathbb{C}^3\text{x }\mathfrak{U}\to\mathbb{C}$ with $\mathfrak{Y}(z)$ be univalent within $\mathfrak{U}$, when $p(z)$ is analytic within $\mathfrak{U}$ and fulfills a condition of being sacond-order differantialy subordnated:

$$\mathfrak{X}(p(z),zp'(z),z^2p''(z);z)\prec\mathfrak{Y}(z), \tag{1.3}$$

further, the function $p(z)$ is referred to as a differantial subordnation solutian (1.3), while $q(z)$ is refarred to as a dominent of the differantial subordnation (1.3) or, to express it clearly, a dominant if $p(z)\prec q(z)$.

For every $p(z)$ that fulfil equation (1.3), a univalent dominating function $\check{q}(z)$ that fulfils $\check{q}(z)\prec q(z)$ for every dominent $q(z)$ of (1.3) it's claimed to obtain best dominent.

Millier, Mocaanu [16] and more authors [1,2,3,4,5,6,7,8,9,10,12] and also [13,14,18,20,21,24,25] established necessary conditions on the functions $\mathfrak{Y}$, p, and $\mathfrak{X}$ in order to obtain the following conclusion:

$$\mathfrak{Y}(z)\prec\mathfrak{X}(p(z),zp'(z),z^2p''(z);z)\to q(z)\prec p(z)(z\in\mathfrak{U}). \tag{1.4}$$

Assuming that $\mathfrak{L}\in\sum_p$ it is defined by (1.1) furthermore, $\mathfrak{F}\in\sum_p$ called

$$\mathfrak{F}(z)=\frac{1}{z^p}+\sum_{\mathcal{K}=1}^{\infty}b_{\mathcal{K}}z^{\mathcal{K}}.$$

The Hadamaard product, also known as convalution, of $\mathfrak{L}$ and $\mathfrak{F}$ is given by:

$$(\mathfrak{L}*\mathfrak{F})(z)=\sum_{\mathcal{K}=1}^{\infty}a_{\mathcal{K}}b_{\mathcal{K}}z^{\mathcal{K}}=(\mathfrak{F}*\mathfrak{L})(z),\quad (z\in\mathfrak{U}).$$

Through the utilisation of conclusions, (see [2,4,5,6,9,13,14,18,20,22,23,24,25]) to fulfil necessary conditions for satisfying of normailzed analytic functions

$$q_1(z)\prec\frac{z\,\mathcal{G}'(z)}{\mathcal{G}(z)}\prec q_2(z)$$

when q_1, q_2 include univalent functions in $\mathfrak{U}$ and $q_1(0)=q_2(0)=1$. Shanmugm et al. [22][23], and also Goyal et al. [12], new research has yielded recent discoveries regarding the outcomes of sandwiches for classes of analytic functions (Refer to [1,3,4,6,11]).

Letting $0\le\lambda\le1$; $0\le\gamma\le1$; $p\in\mathcal{N}$ and $\mathfrak{L}\in\sum_p$, Salah et al. [19] applied multivalent Rafid operetor $S_{\lambda,p}^{\gamma}:\sum_p\to\sum_p$, defined by

$$S_{\lambda,p}^{\gamma}\,\mathfrak{L}(z)=\frac{1}{(1-\lambda)^{\gamma+1}\Gamma(\gamma+1)}\int_0^{\infty}t^{\gamma+p}e^{\left(\frac{-t}{1-\lambda}\right)}\mathfrak{L}(zt)dt, \tag{1.5}$$

then

$$S_{\lambda,p}^{\gamma}\,\mathfrak{L}(z)=\frac{1}{z^p}+\sum_{\mathcal{K}=1}^{\infty}L(\lambda,k,\gamma)a_{\mathcal{K}}z^{\mathcal{K}}, \tag{1.6}$$

where

$$L(\lambda,\mathcal{K},\gamma)=(1-\lambda)^k(\gamma+1)_{\mathcal{K}},$$

and $(u)_{\mathcal{K}}$ represents the Pochammer symboli such that:

$$(u)_k=\frac{\Gamma(u+\mathcal{K})}{\Gamma(u)}=\begin{cases}1 & \text{if } k=0,\\ u(u+1)\ldots(u+k-1) & \text{if } \kappa\in\mathcal{N}.\end{cases} \tag{1.7}$$

Note that, if $p=1$ in (1.5), the Rafid operator was introduced by Rossy and Varma [17]. By applying equation (1.6), it becomes clear that

$$S_{\lambda,p}^{\gamma}\left(z\,\mathfrak{L}'(z)\right)=z\left(S_{\lambda,p}^{\gamma}\mathfrak{L}(z)\right)',$$

thus

$$z\left(S_{\lambda,p}^{\gamma}\mathfrak{L}(z)\right)'=(1+\gamma)S_{\lambda,p}^{\gamma+1}\mathfrak{L}(z)-(p+1+\gamma)S_{\lambda,p}^{\gamma}\mathfrak{L}(z). \tag{1.8}$$

The fundamental objective of this definition is to identify the appropriate condetions when specific normalized analytic functions can be satisfied:

$$q_1(z)\prec\left[z^p\,S_{\lambda,p}^{\gamma}\mathfrak{L}(z)\right]^{\tau}\prec q_2(z),$$

and

$$q_1(z)\prec\left[\frac{vz^p\,S_{\lambda,p}^{\gamma+1}\mathfrak{L}(z)+\eta v^p\,S_{\lambda,p}^{\gamma}\mathfrak{L}(z)}{v+\eta}\right]^{\frac{1}{\tau}}\prec q_2(z),$$

whenever univalent functions $q_1(z)$, $q_2(z)$ are provided in $\mathfrak{U}$ with $q_1(z)=q_2(z)=1$.

2. Preliminaries

The provided defintions and lemmas will aid us in demonstrating our fundamental conclusions.

Definition (2.1) [15]: Letting $\mathfrak{S}$ represent the collection of each functions q that are both injctive and analytic on $\overline{\mathfrak{U}} \setminus \mathfrak{Z}(q)$, when $\overline{\mathfrak{U}} = \mathfrak{U} \cup \{z \in \partial\mathfrak{U}\}$, where

$$\mathfrak{Z}(q) = \left\{\varepsilon \in \partial\mathfrak{U}: \lim_{z\to\varepsilon} q(z) = \infty\right\}, \tag{2.1}$$

as well as being $q'(z) \neq 0$ for $\varepsilon \in \partial\mathfrak{U} \setminus \mathfrak{Z}(q)$. Also, consider the subclass of $\mathfrak{S}$ that is $q(0) = 1$ be indicated by $\mathfrak{S}(\mathfrak{a})$, and $\mathfrak{S}(0) = \mathfrak{S}_0$, $\mathfrak{S}(1) = \mathfrak{S}_1 = \{q \in \mathfrak{S}: q(0) = 1\}$.

Lemma (2.1) [16]: Letting $q(z)$ be a convex univalant function in $\mathfrak{U}$ with $\mathscr{f} \in \mathbb{C}, \mathfrak{P} \in \mathbb{C}\backslash\{0\}$ with

$$\mathcal{R}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > max\left\{0, -\mathcal{R}\left(\frac{\mathscr{f}}{\mathfrak{P}}\right)\right\}.$$

Assuming p is analytic in $\mathfrak{U}$, with

$$\mathscr{f}p(z) + \mathfrak{P}zp'(z) \prec \mathscr{f}q(z) + \mathfrak{P}zq'(z), \tag{2.2}$$

consequently, q be the bested dominent of (2.2) and $p(z) \prec q(z)$.

Lemma (2.2) [11]: Consider the function $q(z)$ to be univalant in $\mathfrak{U}$, assume $\Phi, \wp$ are analytic within a domein $\mathfrak{Q}$ including $q(\mathfrak{U})$ also $w \neq 0$, $w \in q(\mathfrak{U})$. Establish $\mathfrak{S}(z) = zq'(z)\Phi(q(z))$ and $\mathfrak{Y}(z) = \wp(q(z)) + \mathfrak{S}(z)$. Consider:

a- $\mathfrak{S}(z)$ be starlike univalent in $\mathfrak{U}$,

b- $\mathcal{R}\left\{\frac{z\mathfrak{Y}'(z)}{\mathfrak{S}(z)}\right\} > 0, (z \in \mathfrak{U})$.

When p is analytics function in $\mathfrak{U}$, also $p(0) = q(0)$, $p(\mathfrak{U}) \subseteq \mathfrak{Q}$, with

$$\wp(p(z)) + zp'(z)\Phi(p(z)) \prec \wp(q(z)) + zq'(z)\Phi(q(z)), \tag{2.3}$$

consequently, q be a best dominent of (2.3) and $p \prec q$.

Lemma (2.3) [16]: Letting $q(z)$ be a convex univalent in $\mathfrak{U}$ with $q(0) = 1$. Assume $\mathfrak{P} \in \mathbb{C}$, so $\mathcal{R}(\mathfrak{P}) > 0$, when $p(z) \in \mathfrak{R}[q(o),1] \cap \mathfrak{S}$ and $p(z) + \mathfrak{P}zp'(z)$ is univalant in $\mathfrak{U}$, thus

$$aq(z) + \mathfrak{P}zq'(z) \prec ap(z) + \mathfrak{P}zp'(z), \tag{2.4}$$

consequantly, q be the bested dominent of (2.4) and $p(z) \prec q(z)$.

Lemma (2.4) [16]: Consider $q(z)$ as a convax univalent function within $\mathfrak{U}$ with $\Phi, \wp$ are analytic in a domein $\mathfrak{D}$ including $q(\mathfrak{U})$. Assume that

a- $\mathfrak{S}(z) = zq'(z)\phi(q(z))$ is starlike univalent in $\mathfrak{U}$,

b- $\mathcal{R}e\left\{\frac{\wp'(q(z))}{\Phi(q(z))}\right\} > 0$, $(z \in \mathfrak{U})$.

When $p \in \mathfrak{R}[q(o),1] \cap \mathfrak{S}$, with $p(\mathfrak{U}) \subset \mathfrak{D}$, $\wp(p(z)) + zp'(z)\Phi(p(z))$ is univalant in $\mathfrak{U}$ and

$$\wp(q(z)) + zq'(z)\Phi(q(z)) \prec \wp(p(z)) + zp'(z)\Phi(p(z)), \tag{2.5}$$

consequently, q be the bested dominent of (2.5) and $q \prec p$.

3. Results of Differential Subordinations

Now, let us engage in a discussion. The Rafid operator $S^{\gamma}_{\lambda,p}$ can yield various differential subordnation outcomes.

Theorem 3.1 Consider $q(z)$ a convax univalent in $\mathfrak{U}$, adding $q(0) = 1$, with $q'(z) \neq 0$, to each $z \in \mathfrak{U}$. Assume that $\varrho, \tau \in \mathbb{C}\backslash\{0\}$, and

$$\mathcal{R}\left\{1 + \frac{z\,q''(z)}{q'(z)}\right\} > max\left\{0, -\mathcal{R}\left(\frac{\tau}{\varrho}\right)\right\}. \tag{3.1}$$

considering that $\mathfrak{L} \in \sum_p$ fulfils the subordnation condition:

$$\psi(z) \prec q(z) + \frac{\varrho}{\tau} zq''(z), \tag{3.2}$$

where

$$\psi(z) = \varrho(\gamma + 1)\left[z^{p}\ S^{\gamma}_{\lambda,p}\mathfrak{L}(z)\right]^{\tau}\left[\left(\frac{S^{\gamma+1}_{\lambda,p}\mathfrak{L}(z)}{S^{\gamma}_{\lambda,p}\mathfrak{L}(z)} - 1\right)\right] + \left[z^{p}\ S^{\gamma}_{\lambda,p}\mathfrak{L}(z)\right]^{\tau}, \tag{3.3}$$

then

$$\left[z^{p}\ S^{\gamma}_{\lambda,p}\mathfrak{L}(z)\right]^{\tau} \prec q(z), \tag{3.4}$$

the best dominance is attained by $q(z)$.

Proof. Letting $\wp(z)$ known for:

$$\wp(z) = \left[z^{\wp}\, S^{\gamma}_{\lambda,\wp}\mathfrak{L}(z)\right]^{\tau}, \tag{3.5}$$

therefore in $\mathfrak{U}$, $\wp(z)$ is analytic and $\wp(0) = 1$ as a result of taking the derivative of (3.5) involving z then applying the identity of (1.8) in the provided equation.

$$\psi(z) = \varrho(\gamma+1)\left[z^{\wp}\ S^{\gamma}_{\lambda,\wp}\mathfrak{L}(z)\right]^{\tau}\left[\left(\frac{S^{\gamma+1}_{\lambda,\wp}\mathfrak{L}(z)}{S^{\gamma}_{\lambda,\wp}\mathfrak{L}(z)} - 1\right)\right] + \left[z^{\wp}\ S^{\gamma}_{\lambda,\wp}\mathfrak{L}(z)\right]^{\tau} = \wp(z) + \frac{\varrho}{\tau} z\wp''(z).$$

Therefore, the subordnation (3.2) be the same as

$$\wp(z) + \frac{\varrho}{\tau} zp''(z) \prec q(z) + \frac{\varrho}{\tau} zq''(z).$$

Lemma (2.1) is used in this context, with $\mathfrak{P} = \frac{\varrho}{\tau}$, $\alpha = 1$, we get (3.4).

Applying $q(z) = \left(\frac{1+Az}{1+\mathfrak{B}z}\right)$, and $(-1 \le \mathfrak{B} < A \le 1)$ from theorem 3.1, the subsequent outcome is calculated:

Corollary 3.1. Given $\tau, \varrho \in \mathbb{C}\backslash\{0\}$ with $(-1 \le \mathfrak{B} < A \le 1)$. Assume as

$$\mathcal{R}\left\{\frac{1-\mathcal{B}z}{1+\mathcal{B}z}\right\} > \max\left\{0, -\mathcal{R}\left(\frac{\tau}{\varrho}\right)\right\}.$$

considering that $\mathfrak{L} \in \sum_{\wp}$ fulfils the subordnation condition:

$$\psi(z) \prec \left(\frac{1+Az}{1+\mathfrak{B}z}\right) + \left(\frac{\varrho}{\tau}\right)\frac{z(A-\mathfrak{B})}{(1+\mathfrak{B}z)^2},$$

when $\psi(z)$ as defined in equation (3.3), then

$$\left[z^{\wp}\ S^{\gamma}_{\lambda,\wp}\mathfrak{L}(z)\right]^{\tau} \prec \left(\frac{1+\mathcal{A}z}{1+\mathcal{B}z}\right),$$

where the bested domineting is $\left(\frac{1+Az}{1+\mathfrak{B}z}\right)$.

By use corollary (3.1) for $A = 1$, $\mathfrak{B} = -1$, we obtain our next conclusion.

Corollary 3.2. Given $\tau, \varrho \in \mathbb{C}\backslash\{0\}$, assuming that

$$\mathcal{R}\left\{\frac{1+z}{1-z}\right\} > \max\left\{0, -\mathcal{R}\left(\frac{\tau}{\varrho}\right)\right\}$$

considrring that $\mathfrak{L} \in \sum_{\wp}$ fulfils the subordnation condition:

$$\psi(z) \prec \left(\frac{1+z}{1-z}\right) + \left(\frac{\tau}{\varrho}\right)\frac{2z}{(1-z)^2},$$

when $\psi(z)$ expressed as equation (3.3), then

$$\left[z^{\wp}\ S^{\gamma}_{\lambda,\wp}\mathfrak{L}(z)\right]^{\tau} \prec \left(\frac{1+z}{1-z}\right),$$

where the best dominating is $\left(\frac{z+1}{1-z}\right)$.

Theorem 3.2 : Consider a function $q(z)$, which is both convax and univalent within $\mathfrak{U}$ and $q(0) = 1$, where $q'(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ is star like and univalent in $\mathfrak{U}$. Letting $j, v, \eta, \tau \in \mathbb{C}^*$, $\mathrm{Y}, t \in \mathbb{C}$ with $v + \eta \neq 0$, $\frac{vz^{\wp}\, S^{\gamma+1}_{\lambda,\wp}\mathfrak{L}(z) + \eta z^{\wp}\, S^{\gamma}_{\lambda,\wp}\mathfrak{L}(z)}{v+\eta} \neq 0$, $z \in \mathfrak{U}$, assuming that q fulfil the next condition

$$\mathcal{R}\left\{1 + \frac{2\mathrm{Y}}{j}(q(z))^2 + \frac{z\, q''(z)}{q'(z)} - \frac{z\, q'(z)}{q(z)}\right\} > 0, \tag{3.6}$$

if $\mathfrak{L} \in \sum_{\wp}$ fulfil:

$$\Delta(z) \prec \mathrm{Y}(q(z))^2 - \mathrm{t} + \mathrm{j}\frac{z\, q'(z)}{q(z)}, \tag{3.7}$$

where

$$\Delta(z) = \mathrm{Y}\left[\frac{vz^{\wp}\, S^{\gamma+1}_{\lambda,\wp}\mathfrak{L}(z) + \eta z^{\wp}\, S^{\gamma}_{\lambda,\wp}\mathfrak{L}(z)}{v+\eta}\right]^{\frac{2}{\tau}} - t + j\left(\frac{1}{\tau}\right)(1+\gamma)\left[\left(\frac{vz^{\wp}\, S^{\gamma+2}_{\lambda,\wp}\mathfrak{L}(z) + \eta z^{\wp}\, S^{\gamma+1}_{\lambda,\wp}\mathfrak{L}(z)}{vz^{\wp}\, S^{\gamma+1}_{\lambda,\wp}\mathfrak{L}(z) + \eta z^{\wp}\, S^{\gamma}_{\lambda,\wp}\mathfrak{L}(z)} - 1\right)\right], \tag{3.8}$$

thus

$$\left[\frac{vz^{\wp}\, S_{\lambda,\wp}^{\gamma+1}\mathfrak{L}(z)+\eta z^{\wp}\, S_{\lambda,\wp}^{\gamma}\mathfrak{L}(z)}{v+\eta}\right]^{\frac{1}{\tau}} \prec q(z),$$

when the best dominant is denoted as $q(z)$.

Proof. Write $p(z)$ by the following manner:

$$p(z)=\left[\frac{vz^{\wp}\, S_{\lambda,\wp}^{\gamma+1}\mathfrak{L}(z)+\eta z^{\wp}\, S_{\lambda,\wp}^{\gamma}\mathfrak{L}(z)}{v+\eta}\right]^{\frac{1}{\tau}}, \tag{3.9}$$

then p is analytic in $\mathfrak{U}$. A computing the derivetive of (3.9) with regard to z, then substituting the idenetity of (1.8) into the resultant solution, we get

$$\frac{z\,p'(z)}{p(z)}=\left(\frac{1}{\tau}\right)(1+\gamma)\left[\left(\frac{vz^{\wp}\, S_{\lambda,\wp}^{\gamma+2}\mathfrak{L}(z)+\eta z^{\wp}\, S_{\lambda,\wp}^{\gamma+1}\mathfrak{L}(z)}{vz^{\wp}\, S_{\lambda,\wp}^{\gamma+1}\mathfrak{L}(z)+\eta z^{\wp}\, S_{\lambda,\wp}^{\gamma}\mathfrak{L}(z)}-1\right)\right]. \tag{3.10}$$

Setting $\wp(\mathfrak{w})=\text{Ɣ}\mathfrak{w}^2-\mathrm{t}$ with $\Phi(\mathfrak{w})=\frac{j}{\mathfrak{w}},\mathfrak{w}\neq 0$, reveals the $\wp(\mathfrak{w})$ is analytic function in $\mathbb{C}$, also $\mathfrak{G}(\mathfrak{w})$ is analytic in $\mathbb{C}\backslash\{0\}$ and $\Phi(\mathfrak{w})\neq 0, \mathfrak{w}\in\mathbb{C}\backslash\{0\}$. Furthermore, there is

$$\mathfrak{S}(z)=zq'(z)\Phi(q(z))=\mathrm{j}\,\frac{z\,q'(z)}{q(z)},$$

and

$$\mathfrak{Y}(z)=\wp(q(z))+\mathfrak{S}(z)=\text{Ɣ}(q(z))^2-\mathrm{t}+\mathrm{j}\,\frac{z\,q'(z)}{q(z)},$$

$\mathfrak{S}(z)$ is found to be a starlika univalent functions in $\mathfrak{U}$, we have

$$\mathfrak{Y}'(z)=2\text{Ɣ}\,q(z)q'(z)+j\,\frac{z\,q''(z)}{q'(z)}-jz\left(\frac{q'(z)}{q(z)}\right)^2+j\,\frac{q'(z)}{q(z)},$$

hence that

$$\mathcal{R}\left\{\frac{z\,\mathfrak{Y}'(z)}{\mathfrak{S}(z)}\right\}=\mathcal{R}\left\{1+\frac{2\text{Ɣ}}{j}(q(z))^2+\frac{z\,q''(z)}{q'(z)}-\frac{z\,q'(z)}{q(z)}\right\}>0.$$

Applying equation (3.10), getting

$$\text{Ɣ}(p(z))^2-\mathrm{t}+j\,\frac{zp'(z)}{p(z)}=\text{Ɣ}\left[\frac{vz^{\wp}\, S_{\lambda,\wp}^{\gamma+1}\mathfrak{L}(z)+\eta z^{\wp}\, S_{\lambda,\wp}^{\gamma}\mathfrak{L}(z)}{v+\eta}\right]^{\frac{2}{\tau}}-t+j\left(\frac{1}{\tau}\right)(1+\gamma)\left[\left(\frac{vz^{\wp}\, S_{\lambda,\wp}^{\gamma+2}\mathfrak{L}(z)+\eta z^{\wp}\, S_{\lambda,\wp}^{\gamma+1}\mathfrak{L}(z)}{vz^{\wp}\, S_{\lambda,\wp}^{\gamma+1}\mathfrak{L}(z)+\eta z^{\wp}\, S_{\lambda,\wp}^{\gamma}\mathfrak{L}(z)}-1\right)\right].$$

By using (3.7), we have

$$\text{Ɣ}(p(z))^2-\mathrm{t}+\mathrm{j}\,\frac{zp'(z)}{p(z)}=\text{Ɣ}(q(z))^2-\mathrm{t}+\mathrm{j}\,\frac{z\,q''(z)}{q'(z)},$$

it may be deduced that subordination (3.7) means that $p(z)\prec q(z)$, furthermore, it can be deduced from Lemma (2.2), the fanction $q(z)$ is best for the domain.

Putting $q(z)=\left(\frac{1+\mathrm{A}z}{1+\mathfrak{B}z}\right)$,with $(-1\leq\mathfrak{B}<\mathrm{A}\leq 1)$ from Theorem3.2, equation (3.6) is transformed into

$$\mathcal{R}\left\{1+\frac{2\text{Ɣ}}{j}\left(\frac{1+\mathrm{A}z}{1+\mathfrak{B}z}\right)^2+\frac{z(A-\mathcal{B})}{(1+\mathrm{A}z)(1+\mathfrak{B}z)}-\frac{2\mathcal{B}z}{1+z\mathcal{B}}\right\}>0, \tag{3.11}$$

therefore, we can infer the consequent conclusion.

Corollary 3.3. Letting $(-1\leq\mathfrak{B}<\mathrm{A}\leq 1)$, , with $j,v,\eta,\tau\in\mathbb{C}^*$, $\text{Ɣ},t\in\mathbb{C}$, consider that (3.11) fulfills. If $\mathfrak{L}\in\sum_{\wp}$ and

$$\Delta(z)\prec\text{Ɣ}\left(\frac{1+\mathrm{A}z}{1+z\mathfrak{B}}\right)^2-\mathrm{t}+j\,\frac{z(A-\mathcal{B})}{(1+\mathrm{A}z)(1+z\mathfrak{B})},$$

the function $\Delta(z)$ is define in equation (3.8), then

$$\left[\frac{vz^{\wp}\,S_{\lambda,\wp}^{\gamma+1}\mathfrak{L}(z)+\eta z^{\wp}\,S_{\lambda,\wp}^{\gamma}\mathfrak{L}(z)}{v+\eta}\right]^{\frac{1}{\tau}} \prec \left(\frac{1+Az}{1+\mathcal{B}z}\right),$$

where the bested domineting is $\left(\frac{1+Az}{1+\mathfrak{B}}\right)$.

Putting $q(z)=\left(\frac{1+z}{1+z}\right)$, from theorem 3.2, equation (3.6) is transformed into

$$\mathcal{R}\left\{1+\frac{2\text{ɣ}}{j}\left(\frac{1+z}{1+z}\right)^2+\frac{2z}{1-z^2}+\frac{2z}{1-z}\right\}>0, \tag{3.12}$$

therefore, we can infer the consequent conclusion.

Corollary 3.4. Letting $j, v, \eta, \tau \in \mathbb{C}^*$, $\text{ɣ}, t \in \mathbb{C}$. Let's suppose that (3.12) fulfills. If $\mathfrak{L} \in \sum_{\wp}$ and

$$\Delta(z) \prec \text{ɣ}\left(\frac{1+z}{1+z}\right)^2 - \mathrm{t} + j\,\frac{2z}{1-z^2},$$

the function $\Delta(z)$ is defined in equation (3.8), then

$$\left[\frac{vz^{\wp}\,S_{\lambda,\wp}^{\gamma+1}\mathfrak{L}(z)+\eta z^{\wp}\,S_{\lambda,\wp}^{\gamma}\mathfrak{L}(z)}{v+\eta}\right]^{\frac{1}{\tau}} \prec \left(\frac{z+1}{1+z}\right),$$

where the best domineting is $\left(\frac{z+1}{1+z}\right)$.

4. Results of Differantial Superordnations:

Theorem 4.1: Let us consider a function $q(z)$ to be convax univalent in $\mathfrak{U}$ also $q(0)=1, \tau \in \mathbb{C}\backslash\{0\}$, $\mathcal{R}\{\varrho\}>0$, if $\mathfrak{L} \in \sum_{\mathcal{P}}$, where

$$\left[z^{\wp}\;S_{\lambda,\wp}^{\gamma}\mathfrak{L}(z)\right]^{\tau} \in \mathfrak{R}[q(0),1] \cap \mathfrak{S}. \tag{4.1}$$

If $\psi(z)$ function in (3.3), be univalent and the superordnation criteria will be obtained:

$$q(z)+\frac{\varrho}{\tau}zq'(z) \prec \psi(z), \tag{4.2}$$

thus

$$q(z) \prec \left[z^{\wp}\;S_{\lambda,\wp}^{\gamma}\mathfrak{L}(z)\right]^{\tau}, \tag{4.3}$$

where the best subordnant is $q(z)$.

Proof. Letting $\wp(z)$ be a function specified by

$$\wp(z)=\left[z^{\wp}\;S_{\lambda,\wp}^{\gamma}\mathfrak{L}(z)\right]^{\tau}. \tag{4.4}$$

Taking the derivative of (4.4) with regard to z, it has

$$\frac{z\,\wp'(z)}{\wp(z)}=\tau\left[\frac{z\left(S_{\lambda,\wp}^{\gamma}\mathfrak{L}(z)\right)'+\mathcal{P}\,S_{\lambda,\wp}^{\gamma}\mathfrak{L}(z)}{S_{\lambda,\wp}^{\gamma}\mathfrak{L}(z)}\right]. \tag{4.5}$$

By a simple calculation and applying the equation (1.8) to the value (4.5), we can obtain:

$$\psi(z)=\varrho(\gamma+1)\left[z^{\wp}\;S_{\lambda,\wp}^{\gamma}\mathfrak{L}(z)\right]^{\tau}\left[\left(\frac{S_{\lambda,\wp}^{\gamma+1}\mathfrak{L}(z)}{S_{\lambda,\wp}^{\gamma}\mathfrak{L}(z)}-1\right)\right]+\left[z^{\wp}\;S_{\lambda,\wp}^{\gamma}\mathfrak{L}(z)\right]^{\tau}=\wp(z)+\frac{\varrho}{\tau}z\wp''(z).$$

Applying Lemma 2.3 yields the desired outcome.

Setting $q(z)=\left(\frac{1+Az}{1+\mathfrak{B}z}\right)$, with $(-1 \le \mathfrak{B} < A \le 1)$, the next conclusion can be derived of theorem 4.1.

Corollary 4.1: Given $\tau \in \mathbb{C}\backslash\{0\}$, $\mathcal{R}\{\varrho\} > 0$, with $(-1 \leq \mathcal{B} < \mathcal{A} \leq 1)$, where

$$\left[z^{\wp}\ S^{\gamma}_{\lambda,\wp}\mathfrak{L}(z)\right]^{\tau} \in \mathfrak{R}[q(0),1] \cap \mathfrak{S}.$$

When $\psi(z)$ in (3.3) is univalnet in $\mathfrak{U}$, and $\mathfrak{L} \in \sum_{\mathcal{P}}$ satisfying the superordnation condetion,

$$\left(\frac{1+Az}{1+\mathcal{B}z}\right) + \left(\frac{\varrho}{\tau}\right)\frac{z(A-\mathcal{B})}{(1+\mathcal{B}z)^2} \prec \psi(z),$$

thus

$$\left(\frac{1+Az}{1+\mathfrak{B}z}\right) \prec \left[z^{\wp}\ S^{\gamma}_{\lambda,\wp}\mathfrak{L}(z)\right]^{\tau}.$$

Were the best subordnant is $\left(\frac{1+Az}{1+z\mathfrak{B}}\right)$.

Theorem 4.2: Consider $q(z)$ as a convax univalent within $\mathfrak{U}$ and $q(0) = 1$, also $q'(z) \neq 0$, when $\frac{z\,q'(z)}{q(z)}$ is starlikes univalnet in $\mathfrak{U}$. Letting $j, v, \eta, \tau \in \mathbb{C}^*, \text{Ɣ}, t \in \mathbb{C}$ with $v + \eta \neq 0$, $\frac{vz^{\wp}\ S^{\gamma+1}_{\lambda,\wp}\mathfrak{L}(z)+\eta z^{\wp}\ S^{\gamma}_{\lambda,\wp}\mathfrak{L}(z)}{v+\eta} \neq 0, z \in U$. Assume that q fulfill the next condition:

$$\mathcal{R}\left\{\frac{2\text{Ɣ}}{j}(q(z))^2 q'(z)\right\} > 0.$$

Let $\mathfrak{L} \in \sum_{\wp}$ and satisfy the condition:

$$\left[\frac{vz^{\wp}\ S^{\gamma+1}_{\lambda,\wp}\mathfrak{L}(z)+\eta z^{\wp}\ S^{\gamma}_{\lambda,\wp}\mathfrak{L}(z)}{v+\eta}\right]^{\frac{1}{\tau}} \in \mathfrak{R}[q(0),1] \cap \mathfrak{S}. \tag{4.6}$$

We have $\Delta(z)$ function provided by (3.8), is univalnet in $\mathfrak{U}$,

$$\text{Ɣ}(q(z))^2 - t + j\frac{z\,q'(z)}{q(z)}\Delta(z), \tag{4.7}$$

then

$$q(z) \prec \left[\frac{vz^{\wp}\ S^{\gamma+1}_{\lambda,\wp}\mathfrak{L}(z)+\eta z^{\wp}\ S^{\gamma}_{\lambda,\wp}\mathfrak{L}(z)}{v+\eta}\right]^{\frac{1}{\tau}},$$

when the best subordnant is $q(z)$.

Proof. Letting $\wp(z)$ define on $\mathfrak{U}$ by (3.9). Subsequently, the calculation revealed as:

$$\frac{z\,\wp'(z)}{\wp(z)} = \left(\frac{1}{\tau}\right)(1+\gamma)\left[\left(\frac{vz^{\wp}\ S^{\gamma+2}_{\lambda,\wp}\mathfrak{L}(z)+\eta z^{\wp}\ S^{\gamma+1}_{\lambda,\wp}\mathfrak{L}(z)}{vz^{\wp}\ S^{\gamma+1}_{\lambda,\wp}\mathfrak{L}(z)+\eta z^{\wp}\ S^{\gamma}_{\lambda,\wp}\mathfrak{L}(z)} - 1\right)\right]. \tag{4.8}$$

Choosing $\wp(\mathfrak{w}) = \text{Ɣ}\mathfrak{w}^2 - t$ with $\Phi(\mathfrak{w}) = \frac{j}{\mathfrak{w}}, \mathfrak{w} \neq 0$, it's clearly the $\wp(\mathfrak{w})$ in $\mathbb{C}$ and $\Phi(\mathfrak{w})$ in $\mathbb{C}\backslash\{0\}$ are analytics functions, that $\Phi(\mathfrak{w}) \neq 0, (\mathfrak{w} \in \mathbb{C}\backslash\{0\})$. Furthermore, getting

$$\mathfrak{S}(z) = zq'(z)\phi(q(z)) = j\frac{z\,q'(z)}{q(z)},$$

it was discovered $\mathfrak{S}(z)$ is a function that is both starlike and univalant within $\mathfrak{U}$. Since $q(z)$ is convex, we may deduce that

$$\mathcal{R}\left\{\frac{\wp'(q(z))}{\Phi(q(z))}\right\} = \mathcal{R}\left\{\frac{2\text{Ɣ}}{j}(q(z))^2\ q'(z)\right\} > 0.$$

The hypothesis (4.7) can be equivalently utilised by employing (4.8)

$$\wp(q(z)) + zq'(z)\Phi(q(z)) \prec \wp(\wp(z)) + z\wp'(z)\Phi(\wp(z)).$$

A conclusion is thus achieved application of the lemma 2.4.

5. Sandwich Results:

Theorem 5.1: Consider q_1 , q_2 as convax univalant function in $\mathfrak{U}$, employing $q_1(0) = q_2(0) = 1$ and q_2 fulfils (3.1). Assume $\tau \in \mathbb{C}\backslash\{0\}$, $\mathcal{R}\{\varrho\} > 0$. If $\mathfrak{L} \in \sum_{p}$, where

$$\left[z^{p}\ S^{\gamma}_{\lambda,p}\mathfrak{L}(z)\right]^{\tau} \in \mathfrak{R}[q(0),1] \cap \mathfrak{S},$$

the function $\psi(z)$, specified in equation (3.3), is univalent and fulfills the specified criteria:

$$q_1(z) + \frac{\varrho}{\tau} z q_1'(z) \prec \psi(z) \prec q_2(z) + \frac{\varrho}{\tau} z q_2'(z), \tag{5.1}$$

thus

$$q_1(z) \prec \left[z^{p}\ S^{\gamma}_{\lambda,p}\mathfrak{L}(z)\right]^{\tau} \prec q_2(z),$$

when q_1, q_2 represent the best subordnant and dominent respactively.

Theorem 5.2: Letting q_i denote a pair of univalent convax functions in $\mathfrak{U}$, through $q_i(0) = 1, q_i{}'(z) \neq 0, (i = 1,2)$. Say it q_1 and q_2 fulfill all the conditions specified in calculations (3.7) and (4.7), respectively. If $\mathfrak{L} \in \sum_{p}$, assume that function $\mathfrak{L}$ fulfills the following condition:

$$\left[\frac{v z^{p}\ S^{\gamma+1}_{\lambda,p}\mathfrak{L}(z) + \eta z^{p}\ S^{\gamma}_{\lambda,p}\mathfrak{L}(z)}{v+\eta}\right]^{\frac{1}{\tau}} \in \mathfrak{R}[q(0),1] \cap \mathfrak{S},$$

where $\frac{v z^{p}\ S^{\gamma+1}_{\lambda,p}\mathfrak{L}(z) + \eta z^{p}\ S^{\gamma}_{\lambda,p}\mathfrak{L}(z)}{v+\eta} \neq 0$, and $\Delta(z)$ is univalent in $\mathfrak{U}$, as indicated by equation (3.8),

$$\mathrm{Y}(q_1(z))^2 - \mathrm{t} + \mathrm{j}\frac{z\, q_1{}'(z)}{q_1(z)} \prec \Delta(z) \prec \mathrm{Y}(q_2(z))^2 - \mathrm{t} + \mathrm{j}\frac{z\, q_2{}'(z)}{q_2(z)}, \tag{5.2}$$

implies

$$q_1(z) \prec \left[\frac{v z^{p}\ S^{\gamma+1}_{\lambda,p}\mathfrak{L}(z) + \eta z^{p}\ S^{\gamma}_{\lambda,p}\mathfrak{L}(z)}{v+\eta}\right]^{\frac{1}{\tau}} \prec q_2(z),$$

where the best subordnant and dominant q_1 and q_2, respectively.

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