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Theorems of Strong Differential Sandwich Results for Analytic Functions Associated with Wanas Fractional Integral Operator

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ABSTRACT

The objective of this research is to produce robust differential subordination and differential superordination results using the fractional integral of the Wanas differential operator. These results apply to Analytic functions defined on $U \times \overline{U}$, with Coefficient functions that are holomorphic in U. Furthermore, for each instance of strong differential subordination and strong differential superordination, we provide the most superior dominant and the most subordinate subordinant. These findings are used to achieve strong sandwich results.

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1. Introduction

The Concept of Strong Differential Subordination was first proposed in Antonino and Romaguera's study [6], where they examined the Briot-Bouquet strong differential subordination. This notion was suggested as an expansion of the conventional differential subordination, as stated by Miller and Mocanu [17,19]. In 2009, Oros and Oros [23] expanded upon the theory of differential subordination [18] by developing the theory of strong differential subordination. Oros [22] expanded upon the dual concept of differential subordination and differential superordination [20] by introducing the notion of strong differential superordination. The theories of strong differential subordination and superordination underwent significant advancement. The optimal dominant for a strong differential subordination, as well as its counterpart, the optimal subordinant for a powerful differential superordination, were introduced in [21]. Additionally, the initial instances of strong differential subordinations and superordinations of analytic functions were shown in [11]. The text mentions several well-known operators that exhibit strong differential subordination and superordination,

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including the Liu-Srivastava operator [10], Salagean differential operator [31], Ruscheweyh derivative [25], multiplier transformation [14,30], general differential operators [4,24], and various other operators [1,2,27,33].

We intend to investigate the fractional integral of the Wanas operator in relation to differential strong subordinations and strong superordinations, building on the findings of [36], which investigated this topic using differential subordination theory, We used the theories of strong differential subordination and strong differential superordination in this study. In addition, we extended the Wanas operator to include the $U \times \bar{U}$ family of analytic functions. Furthermore, we investigated the previously unexplored application of fractional integral to this extended operator., which has not been previously examined.

Taking into account U ={$\mathfrak{z}\in\mathbb{C}:|\mathfrak{z}|<1$}and with its closure $\bar{U}$={$\mathfrak{z}\in\mathbb{C}:|\mathfrak{z}|<1$}Let $\mathcal{H}(U \times \bar{U})$ represents the family of analytic functions in $U \times \bar{U}$. For n a positive integer and $a \in \mathbb{C}$, let $\mathcal{H}^*[a, n, \varsigma] = \{f \in \mathcal{H}(U \times \bar{U}): f(\mathfrak{z}, \varsigma) = a + a_n(\varsigma)\mathfrak{z}^n + a_{n+1}(\varsigma)\mathfrak{z}^{n+1} + \cdots, \mathfrak{z} \in U, \varsigma \in \bar{U}\}$, where $a_k(\varsigma)$ are holomorphic functions in $\bar{U}$ for $k \geq n$.

Let $\mathcal{A}^*_{n\,\varsigma} = \{f \in \mathcal{H}(U \times \bar{U}): f(\mathfrak{z}, \varsigma) = \mathfrak{z} + a_{n+1}(\varsigma)\mathfrak{z}^{n+1} + \cdots, \mathfrak{z} \in U, \varsigma \in \bar{U}\}$, with $\mathcal{A}^*_{1\,\varsigma} = \mathcal{A}^*_{\varsigma}$, where $a_k(\varsigma)$ are holomorphic functions in $\bar{U}$ for $k \geq n + 1$.
The concept of strong differential subordination is precisely defined as follows:

Definition 1.1 [23]. Let $f(\mathfrak{z}, \varsigma), g(\mathfrak{z}, \varsigma)$ Analytic in $U \times \bar{U}$. The function $f(\mathfrak{z}, \varsigma)$ is purported to be strongly differential subordinate to $g(\mathfrak{z}, \varsigma)$, written $f(\mathfrak{z}, \varsigma) \prec\prec g(\mathfrak{z}, \varsigma), \mathfrak{z} \in U, \varsigma \in \bar{U}$, if there exists an analytic function w in U with $w\varsigma 0) = 0$ and $|w(\mathfrak{z})| < 1, \mathfrak{z} \in U$ such that $f(\mathfrak{z}, \varsigma) = g(w(\mathfrak{z}), \varsigma)$ for all $\varsigma \in \bar{U}$.

Remark 1.1 [23].

(i) Since $f(\mathfrak{z}, \varsigma)$ is Analytic in $U \times \bar{U}$, for all $\varsigma \in \bar{U}$ and univalent in U, for all $\varsigma \in \bar{U}$, Definition1.1 is equivalent to $f(0, \varsigma) = g(0, \varsigma)$ for all $\varsigma \in \bar{U}$ and $f(U \times \bar{U}) \subset g(U \times \bar{U})$.

(ii) If $f(\mathfrak{z}, \varsigma) = f(\mathfrak{z})$ and $g(\mathfrak{z}, \varsigma) = g(\mathfrak{z})$, the strong differential subordination reduce to the classical differential subordination.

In order to examine the concept of strong differential subordination, it is necessary to Utilize the following lemma..

Lemma 1.1 [16]. If the holomorphic function

$$p(\mathfrak{z}, \varsigma) = q(0, \varsigma) + p_n(\varsigma)t^n + p_{n+1}(\varsigma)t^{n+1} + \cdots,$$

in $U \times \bar{U}$ Implementing the strong differential subordination

$$p(\mathfrak{z}, \varsigma) + \eta\mathfrak{z}p'_{\mathfrak{z}}(\mathfrak{z}, \varsigma) \prec\prec h(\mathfrak{z}, \varsigma), \quad (\mathfrak{z} \in U, \varsigma \in \bar{U}),$$

where $q(\mathfrak{z}, \varsigma)$ is a convex function and

$$h(\mathfrak{z}, \varsigma) = q(\mathfrak{z}, \varsigma) + n\eta\mathfrak{z}q'_{\mathfrak{z}}(\mathfrak{z}, \varsigma),$$

for n a Whole number greater than zero and $\eta > 0$, after that, we get the sharp object. strong differential subordination

$$p(\mathfrak{z}, \varsigma) \prec\prec q(\mathfrak{z}, \varsigma).$$

The concept of strong differential superordination is precisely defined in the following manner:

Definition 1.2 [22]. Let $f(\mathfrak{z}, \varsigma), g(\mathfrak{z}, \varsigma)$ analytic in $U \times \bar{U}$. The function $g(\mathfrak{z}, \varsigma)$ is said to be strongly differential superordination to $f(\mathfrak{z}, \varsigma)$, written $g(\mathfrak{z}, \varsigma) \prec\prec f(\mathfrak{z}, \varsigma), \mathfrak{z} \in U, \varsigma \in \bar{U}$, if there exists an analytic function w in U with $w(0) = 0$ and $|w(\mathfrak{z})| < 1, \mathfrak{z} \in U$ such that $g(\mathfrak{z}, \varsigma) = f(w(\mathfrak{z}), \varsigma)$ for all $\varsigma \in \bar{U}$.

Remark 1.1 [22].

(i) Since $f(z,\varsigma)$ is analytic in $U \times \overline{U}$, for all $\varsigma \in \overline{U}$ and univalent in U, for all $\varsigma \in \overline{U}$, Definition1.1 is equivalent to $g(0,\varsigma) = f(0,\varsigma)$ for all $\varsigma \in \overline{U}$ and $g(U \times \overline{U}) \subset f(U \times \overline{U})$.

(ii) If $g(z,\varsigma) = g(z)$ and $f(z,\varsigma) = f(z)$, the strong differential superordination reduce to the classical differential superordination.

Definition 1.3 [13]. We are denote by Q_ς the set of functions that are analytic and injective on $\overline{U} \times \overline{U} \backslash E(f,\varsigma)$, where

$$E(f,\varsigma) = \left\{\xi \in \partial U: \lim_{z\to\xi} f(z,\varsigma) = \infty\right\},$$

and $f_z^{'}(\xi,\varsigma) \neq 0$ for $\xi \in \partial U \times \overline{U} \backslash E\varsigma f,\varsigma)$. The subclass of Q_ς with $f(0,\varsigma) = a$ is denoted by $Q_\varsigma(a)$.

To explore strong differential superordination, the following lemma is needed.

Lemma 1.2 [15]. If the function $p \in \mathcal{H}^*[a,n,\varsigma] \cap Q_\varsigma$ satisfying the strong differential superordination

$$q(z,\varsigma) + \frac{1}{\eta} z q_z'(z,\varsigma) \prec\prec p(z,\varsigma) + \frac{1}{\eta} z p_z'(z,\varsigma), \quad (z \in U, \varsigma \in \overline{U}),$$

and $p(z,\varsigma) + \frac{1}{\eta} z p_z'(z,\varsigma)$ is univalent in $U \times \overline{U}$, $q(z,\varsigma)$ is a convex function and

$$h(z,\varsigma) = q(z,\varsigma) + \frac{1}{\eta} z q_z'(z,\varsigma)$$

for $\eta \in \mathbb{C}^*$ with $Re\varsigma\eta) \geq 0$, Subsequently, that we acquire the strong differential superordination

$$q(z,\varsigma) \prec\prec p(z,\varsigma), \qquad (z \in U, \varsigma \in \overline{U})$$

and with the convex function $q(z,\varsigma) = \frac{\eta}{n\, z^{\frac{\eta}{n}}} \int_0^z h(x,\varsigma)\, x^{\frac{\eta}{n}-1} dx$ $z \in U, \varsigma \in \overline{U}$ is the best subordinant.

Definition 1.4 [35]. For $f \in \mathcal{A}$. The Wanas differential operator is defined by

$$W_{\alpha,\beta}^{k,\eta} f(z,\varsigma) = z + \sum_{n=2}^{\infty} \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m}\right)\right]^{\eta} a_n z^n, \tag{1.1}$$

where $\alpha \in \mathbb{R}, \beta \geq 0$ with $\alpha + \beta > 0$, $k \in \mathbb{N}, \eta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 1.5 [9]. The proportional integral of order where λ ($\lambda > 0$) is defined for

the function

$$D_z^{-\lambda} f(z,\varsigma) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{\lambda-1}} dt,$$

Let f be an analytic function in an area of the z-plane that is simply-connected and contains the origin. The multiplicity of f $(z - t)^{\lambda-1}$ is removed by requiring log $(z - t)$ to be real, at what for $(z - t) > 0$.

From Definition 1.3 also Definition 1.4, we conclude that

$$D_z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma) = \frac{1}{\Gamma(2+\lambda)} z^{1+\lambda} + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\lambda)} \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m}\right)\right]^{\eta} a_n(\varsigma) z^{n+\lambda}.$$

Form [36] we need this result

$$z\left(D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)_z' =$$

$$\left[1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right] D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+1} f(z,\varsigma)+\left[\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right] D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z,\varsigma). \quad (1.2)$$

This operator has special cases in [3,4,7,8,12,26,28,29,32]. For more details see [34].

2-Main Results

Theorem 2.1. Suppose that $q(z,\varsigma)$ be convex function with the property $q(\underline{0},\varsigma)=0$, we take the function $h(z,\varsigma)=q(z,\varsigma)+\lambda z q_z'(z,\varsigma), z\in U, \varsigma\in\overline{U}$, with λ a positive integer. If the strong differential subordination

$$\left(D_z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)_z' \prec\prec h(z,\varsigma) \quad (2.1)$$

is content when $f\in\mathcal{A}_\varsigma^*$, Subsequently, we gain the ensuing distinct strong differential subordination:

$$\frac{D_z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z} \prec\prec q(z,\varsigma).$$

Proof. Let

$$p(z,\varsigma)=\frac{D_z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z}, \quad z\in U, \varsigma\in\overline{U}, \quad (2.2)$$

Then $p\in\mathcal{H}^*[0,\lambda,\varsigma]$.

We have

$$D_z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)=z p(z,\varsigma).$$

Taking the derivative of the above equation with respect to z, we have

$$\left(D_z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)_z' = p(z,\varsigma)+z p_z'(z,\varsigma).$$

Then, strong subordination (2.1) is in the following format:

$$p(z,\varsigma)+z p_z'(z,\varsigma) \prec\prec h(z,\varsigma)= q(z,\varsigma)+\gamma z q_z'(z,\varsigma),$$

and, applying Lemma 1.1, we get $p(z,\varsigma)\prec\prec q(z,\varsigma)$. By (2.2), We are acquire

$$\frac{D_z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z} \prec\prec q(z,\varsigma).$$

Corollary 2.1. Applying the an convexity function $h(z,\varsigma)=\frac{\varsigma+2(\delta-\varsigma)z}{1+z}$, with $0\le\delta<1$ satisfying the strong subordination (2.1) for $f\in\mathcal{A}_\varsigma^*$, then we get the strong subordination

$$\frac{D_z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z} \prec\prec q(z,\varsigma),$$

and the function that is convex outward $q(z,\varsigma)=2(\delta-\varsigma)+2(\varsigma-\delta)\frac{Ln(1+z)}{z}$, $z\in U, \varsigma\in\overline{U}$ is the most dominant and notable.

Proof. Reiterating the procedures employed in the demonstration of Theorem 2.1, taking

$$p(z,\varsigma)=\frac{D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z},$$

the strong subordination (2.1) adopts the shape

$$p(z,\varsigma)+zp_z'(z,\varsigma) \prec\prec h(z,\varsigma)=\frac{\varsigma+2(\delta-\varsigma)z}{1+z},$$

for which, applying Lemma 1.2, we get $p(z,\varsigma) \prec\prec q(z,\varsigma)$, and thus

$$\frac{D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z} \prec\prec q(z,\varsigma)=2(\delta-\varsigma)+2(\varsigma-\delta)\frac{Ln(1+z)}{z},\quad z\in U,\varsigma\in\bar{U}.$$

Theorem 2.2. Let $q(z,\varsigma)$ Let f be a convex its function. $q(0,\varsigma)=0$, and the a function

$h(z,\varsigma)= q(z,\varsigma) + \frac{1}{\mu}zq_z'(z,\varsigma)$, with μ a Whole number greater than zero, $z\in U,\varsigma\in\bar{U}$.

If the strong subordination is accomplished

$$\left(\frac{D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}\right)^{\mu-1}\left(D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f\varsigma\right)_z' \prec\prec h(z,\varsigma), \tag{2.3}$$

for $f\in\mathcal{A}_\varsigma^*$, Next, we have the following distinct or clearly defined. strong subordination:

$$\left(\frac{D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}\right)^{\mu} \prec\prec q(z,\varsigma).$$

Proof. Define the function p by

$$p(z,\varsigma)=\left(\frac{D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}\right)^{\mu},\quad z\in U,\varsigma\in\bar{U}\,. \tag{2.4}$$

It is clear that $p\in\mathcal{H}^*[0,\lambda\mu,\varsigma]$. By applying differentiation with respect to z, we have

$$\begin{aligned}zp_z'(z,\varsigma)&=\mu\left(\frac{D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}\right)^{\mu-1}\left(D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)_z' - \mu\left(\frac{D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}\right)^{\mu}\\&=\mu\left(\frac{D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}\right)^{\mu-1}\left(D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)_z'-\mu p(z,\varsigma).\end{aligned}$$

Therefore

$$p(z,\varsigma)+\frac{1}{\mu}zp_z'(z,\varsigma)=\left(\frac{D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}\right)^{\mu-1}\left(D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)_z'.$$

Under such conditions, strong subordination(2.3) assumes the form of

$p(z,\varsigma) + \frac{1}{\mu}zp_z'(z,\varsigma) \prec\prec h(z,\varsigma) = q(z,\varsigma) +\frac{1}{\mu}zq_z'(z,\varsigma)$,

Applying the lemma 1.1, we are. conclude that

$p(z,\varsigma) \prec\prec q(z,\varsigma)$.

It follows from (2.4) that

$$\left(\frac{D_{z}^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z}\right)^{\mu} \prec\prec q(z,\varsigma).$$

Theorem 2.3. Let $q(z,\varsigma)$ be an convex function with $q(0,\varsigma) = \frac{1}{1+\lambda}$, let the function $h(z,\varsigma) = q(z,\varsigma) + zq_z'(z,\varsigma)$, $z \in U, \varsigma \in \overline{U}$. If the strong subordination

$$\frac{\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2}\left(D_{z}^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)^{2} - \left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2} D_{z}^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z,\varsigma).D_{z}^{-\lambda} W_{\alpha,\beta}^{k,\eta+2} f(z,\varsigma)}{\left(\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right) D_{z}^{-\lambda} W_{\alpha,\beta}^{k,\eta+1} f(z,\varsigma) + \left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right) D_{z}^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)^{2}}$$

$$-\frac{\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right) D_{z}^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z,\varsigma).D_{z}^{-\lambda} W_{\alpha,\beta}^{k,\eta+1} f(z,\varsigma) - \left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)\left(D_{z}^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)^{2}}{\left(\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right) D_{z}^{-\lambda} W_{\alpha,\beta}^{k,\eta+1} f(z,\varsigma) + \left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right) D_{z}^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)^{2}}$$

$$\prec\prec h(z,\varsigma) \tag{2.5}$$

Applies to f $\in \mathcal{A}_{\varsigma}^{*}$, Then we possess the keen or acute. strong subordination

$$\frac{D_{z}^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z\left(D_{z}^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)_{z}'} \prec\prec q(z,\varsigma).$$

Proof. Considering the function

$$p(z,\varsigma) = \frac{D_{z}^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z\left(D_{z}^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)_{z}'}. \tag{2.6}$$

Differentiating (2.6) with respect to z, yields

$$1 - \frac{D_{z}^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma).\left(D_{z}^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)_{z}''}{z\left(D_{z}^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)_{z}'} = p(z,\varsigma) + zp_{z}'(z,\varsigma).$$

Following a brief calculation and utilizing the given relationship (1.2), noticed that

$$1-\frac{D_{z}^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma).\left(D_{z}^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)''_{z}}{z\left(D_{z}^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)'_{z}}$$

$$=\frac{\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2}\left(D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)^{2}-\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2}D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma).D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta+2}f(z,\varsigma)}{\left(\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma)+\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)^{2}}$$

$$-\frac{\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma).D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma)-\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)\left(D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)^{2}}{\left(\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma)+\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)^{2}}.$$

In these conditions, the strong subordination (2.5) transforms

$$p(z,\varsigma)+zp'_{z}(z,\varsigma)=h(z,\varsigma)=q(z,\varsigma)+zq'_{z}(z,\varsigma),$$

and applying Lemma 1.1 , we get the sharp strong subordination $p(z,\varsigma)\prec\prec q(z,\varsigma)$. Using (2.6), to obtain

$$\frac{D_{z}^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z\left(D_{z}^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)'_{zz}}\prec\prec q(z,\varsigma).$$

Theorem 2.4. Allow $q(z,\varsigma)$ be an convex function with $q(0,\varsigma)=0$, let the function $h(z,\varsigma)=q(z,\varsigma)+\lambda\, zq'_{z}(z,\varsigma)$, where λ is a value that is an integer greater than zero, $z\in U,\varsigma\in\overline{U}$. If the strong sub ordination

$$\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2}\frac{D_{z}^{-\lambda}w_{\alpha,\beta}^{k,\eta+2}f(z,\varsigma)}{z}+2\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)\frac{D_{z}^{-\lambda}w_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma)}{z}+\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2}\frac{D_{z}^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}\prec\prec h(z,\varsigma) \quad (2.7)$$

is content with $f\in\mathcal{A}_{\varsigma}^{*}$, Next, we acquire the sharp object. strong sub ordination

$$\left(D_{z}^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)'_{z}\prec\prec q(z,\varsigma).$$

Proof. We define the function $p(z,\varsigma)$ by

$$p(z,\varsigma)=\left(D_{z}^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)'_{z}. \quad (2.8)$$

It is clear that $p(z,\varsigma)\in\mathcal{H}^{*}[0,\lambda,\varsigma]$. Using the relation (1.2), we conclude that

$$zp(z,\varsigma)=\left[1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right]D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma)+\left[\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right]D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma).$$

Differentiating the last equation with respect to$z$, yields

$$p(z,\varsigma)+zp'_z(z,\varsigma)$$
$$=\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2\frac{D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta+2}f(z,\varsigma)}{z}$$
$$+2\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)\frac{D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma)}{z}$$
$$+\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2\frac{D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}.$$

Under such conditions, the strong sub ordination (2.7) assumes the shape

$$p(z,\varsigma)\ +zp'_z(z,\varsigma)\prec\prec\ h(z,\varsigma)=\ q(z,\varsigma)\ +\lambda\, zq'_z(z,\varsigma),$$

and making use of Lemma 1.1, we are get the sharp strong sub ordination $p(z,\varsigma)\ \prec\prec\ h(z,\varsigma)$. It follows from(2.8) that

$$\left(D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)'_z\ \prec\prec q(z,\varsigma).$$

Theorem 2.5. Let $q(z,\varsigma)$ be an convex function with $q(0,\varsigma)=0$, let the function $h(z,\varsigma)=q(z,\varsigma)+\lambda zq'_z(z,\varsigma)$, for λ a positive integer $z\in U,\varsigma\in\overline{U}$. Assume that

$$\frac{D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}\in Q_\varsigma\cap\ \mathcal{H}^*[0,\lambda,\varsigma]$$

and $\left(D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)'_z$ is univalent and the strong differential superordination is satisfied

$$h(z,\varsigma)\prec\prec\ \left(D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)'_z \tag{2.9}$$

for $f\ \in\mathcal{A}^*_\varsigma$, then we get the strong superordination

$$q(z,\varsigma)\prec\prec\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z},$$

and the an convex function $q(z,\varsigma)=\frac{1}{z}\int_0^z h(x,\varsigma)dx$ is the best dominant.

Proof. Let us define the function $p(z,\varsigma)$ be defined by (2.2). Then $p\in\mathcal{H}^*[0,\lambda,\varsigma]$. Differentiating relation $D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\ =\ z\,p(z,\varsigma)$, with respect to z, we have

$$\left(D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)'_z=\ p(z,\varsigma)+zp'_z(z,\varsigma),$$

The strong superordination (2.9)) uses the following type:

$$h(z,\varsigma)\prec\prec q(z,\varsigma)+\gamma zq'_z(z,\varsigma)\prec\prec p(z,\varsigma)+zp'_z(z,\varsigma)\ ,$$

and using Lemma 1.2, we acquire the strong superordination $q(z,\varsigma)\prec\prec p(z,\varsigma)$. It follows from (2.2) that

$$q(z,\varsigma)=\frac{1}{z}\int_0^z h(x,\varsigma)dx\prec\prec\frac{D_z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z},$$

The function q is the most superior subordinate.

Corollary 2.2. Considering the convex function $h(z,\varsigma) = \frac{\varsigma+2(\delta-\varsigma)z}{1+z}$, with $0 \le \delta < 1$ and $f \in \mathcal{A}^*_\varsigma$, we assume that

$$\frac{D_z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z} \in Q_\varsigma \cap \mathcal{H}^*[0,\lambda,\varsigma]$$

and $\left(D_z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)'_z$ is univalent and satisfies the strong superordination (2.9), Next, we acquire the strong next

$$q(z,\varsigma) \prec\prec \frac{D_z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z}$$

and the convex function $q(z,\varsigma) = 2(\delta-\varsigma) + 2(\varsigma-\delta)\frac{Ln(1+z)}{z}$, $z \in U, \varsigma \in \overline{U}$ is the best subordinant.

Proof. Repeating the steps made in the proof of Theorem 2.5 considering

$p(z,\varsigma) = \frac{D_z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z}$, and the strong superordination (2.9) takes the form

$$h\varsigma z,\varsigma) = \frac{\varsigma+2(\delta-\varsigma)z}{1+z} \prec\prec z p'_z(z,\varsigma) + p(z,\varsigma).$$

Using Lemma 1.2, it yields $q(z,\varsigma) \prec\prec p(z,\varsigma)$ and so

$$q(z,\varsigma) = \frac{1}{z}\int_0^z h(x,\varsigma)dx \prec\prec \frac{D_z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z},$$

and with the best subordinant is the function

$$q(z,\varsigma) = \frac{1}{z}\int_0^z h(x,\varsigma)dx = 2(\delta-\varsigma) + 2(\varsigma-\delta)\frac{1}{z}Ln(z+1), z \in U, \varsigma \in \overline{U}$$

Theorem 2.6. Let $q(z,\varsigma)$ be convex function with $q(0,\varsigma) = 0$, let the function $h(z,\varsigma) = q(z,\varsigma) + \frac{1}{\mu} z q'_z(z,\varsigma)$, with μ a positive integer, $z \in U, \varsigma \in \overline{U}$. Assume that for $f \in \mathcal{A}^*_\varsigma$,

$$\left(\frac{D_z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z}\right)^\mu \in Q_\varsigma \cap \mathcal{H}^*[0,\lambda\mu,\varsigma],$$

and $\left(\frac{D_z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z}\right)^{\mu-1} \left(D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)'_z$, is univalent and with the strong superordination

$$h(z,\varsigma) \prec\prec \left(\frac{D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z}\right)^{\mu-1} \left(D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)'_z \qquad (2.10)$$

is verified, then we are have the strong superordination

$$q(z,\varsigma) \prec\prec \left(\frac{D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z}\right)^\mu,$$

and the convex function $q(z,\varsigma) = \frac{\mu}{z^\mu}\int_0^z h(x,\varsigma)\, x^{\mu-1} dx$ is the best subordinant.

Proof. Let the function $p(z,\varsigma)$ be defined by (2.4). Then $p \in \mathcal{H}^*[0,\lambda\mu,\varsigma]$.

By applying differentiation and with respect to z, we are have

$$zp'_z(z,\varsigma) = \mu\left(\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}\right)^{\mu-1}\left(D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)'_z - \mu\left(\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}\right)^{\mu}$$
$$= \mu\left(\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}\right)^{\mu-1}\left(D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)'_z - \mu p(z,\varsigma),$$

Therefore

$$p(z,\varsigma) + \frac{1}{\mu}zp'_z(z,\varsigma) = \left(\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}\right)^{\mu-1}\left(D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)'_z.$$

In these conditions, the strong superordination (2.10) An optimistic integer

$$h(z,\varsigma) = q\varsigma z,\varsigma) + \frac{1}{\mu}zq'_z(z,\varsigma) \prec\prec p\varsigma z,\varsigma) + \frac{1}{\mu}zp'_z(z,\varsigma),$$

and by Lemma 1.2, we are get the strong superordination $q(z,\varsigma) \prec\prec p(z,\varsigma)$. It follows from (2.4) that

$$q(z,\varsigma) \prec\prec \left(\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}\right)^{\mu},$$

and the best subordinant is the function $q(z,\varsigma) = \frac{\mu}{z^{\mu}}\int_0^z h(x,\varsigma)\,x^{\mu-1}dx$.

Theorem 2.7. Let $q(x,\varsigma)$ be convex function with $q(0,\varsigma) = \frac{1}{1+\lambda}$, let the function $h(x,\varsigma) = q(z,\varsigma) + zq'_z(z,\varsigma)$, $f \in \mathcal{A}^*_\varsigma$ and

$$\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z\left(D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)'_z} \in Q_\varsigma \cap \mathcal{H}^*\left[\frac{1}{1+\lambda},1,\varsigma\right].$$

If

$$\frac{\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2\left(D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)^2 - \left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2 D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma).D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta+2}f(z,\varsigma)}{\left(\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma)+\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)^2}$$
$$-\frac{\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma).D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma) - \left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)\left(D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)^2}{\left(\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma)+\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)D_z^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)^2}$$

is univalent and verifies the strong differential superordination

$h(z,\varsigma) \prec\prec$

$$\frac{\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2}\left(D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)^{2}-\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2}D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma).D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta+2}f(z,\varsigma)}{\left(\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma)+\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)^{2}}$$

$$-\frac{\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma).D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma)-\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)\left(D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)^{2}}{\left(\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma)+\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)^{2}}, \quad (2.11)$$

then we acquire the strong differential superordination

$$q(z,\varsigma) \prec\prec \frac{D_{Z}^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z\left(D_{Z}^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)_{z}'}$$

with and the best subordinant is the convex function

$$q(z,\varsigma) = \frac{1}{z}\int_{0}^{z} h(x,\varsigma)dx.$$

Proof. Differentiating the relation $p(z,\varsigma) = \frac{D_{Z}^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z\left(D_{Z}^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)_{z}'}$, with respect to z, yields

$$1-\frac{D_{Z}^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma).\left(D_{Z}^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)_{z}''}{z\left(D_{Z}^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)_{z}'} = p(z,\varsigma) + zp_{z}'(z,\varsigma)$$

Performing a brief calculation with and applying relation (1.2), we are get

$$1-\frac{D_{Z}^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma).\left(D_{Z}^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)_{z}''}{z\left(D_{Z}^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)_{z}'}$$

$$=\frac{\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2}\left(D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)^{2}-\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2}D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma).D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta+2}f(z,\varsigma)}{\left(\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma)+\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)^{2}}$$

$$-\frac{\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma).D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma)-\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)\left(D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)^{2}}{\left(\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma)+\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)^{2}}.$$

Under these circumstances, the strong superordination (2.11) adopts the shape

$$h(z,\varsigma) = q(z,\varsigma) + zq'_z(z,\varsigma) \prec\prec p(z,\varsigma) + zp'_z(z,\varsigma),$$

and with applying Lemma 1.2, we are get the strong superordination $q(z,\varsigma) \prec\prec p(z,\varsigma)$. Using (2.6), we have

$$q(z,\varsigma) \prec\prec \frac{D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z\left(D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)'_z}.$$

And with the best subordinant represents the function

$$q(z,\varsigma) = \frac{1}{z}\int_0^z h(x,\varsigma)dx.$$

Theorem 2.8. Let $q(z,\varsigma)$ be convex function with $q(0,\varsigma) = 0$, let the function $h(z,\varsigma) = q(z,\varsigma) + \lambda\, zq'_z(z,\varsigma)$, with λ a positive integer, $z \in U, \varsigma \in \overline{U}, f \in \mathcal{A}^*_\varsigma$. Assume that

$$\left(D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)'_z \in Q_\varsigma \cap \mathcal{H}^*[0,\lambda,\varsigma]$$

and

$$\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2 \frac{D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta+2} f(z,\varsigma)}{z} + 2\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)\frac{D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta+1} f(z,\varsigma)}{z} + \left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2 \frac{D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z}$$

is univalent and verifies the strong differential superordination

$$h(z,\varsigma) \prec\prec \left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2 \frac{D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta+2} f(z,\varsigma)}{z} + 2\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)\frac{D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta+1} f(z,\varsigma)}{z} + \left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2 \frac{D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z}, \tag{2.12}$$

then we are having the sharp strong superordination

$$q\varsigma z,\varsigma) \prec\prec \left(D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)'_z$$

And with the best subordinant is the convex function $q(z,\varsigma) = \frac{1}{z}\int_0^z h(x,\varsigma)dx$.

Proof. Let $p(z,\varsigma)$ be defined by (2.8). Then $p(z,\varsigma) \in \mathcal{H}^*[0,\lambda,\varsigma]$.

Utilizing a connection or association (1.2), we are have

$$zp(z,\varsigma) = \left[1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right] D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+1} f(z,\varsigma) + \left[\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right] D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z,\varsigma)$$

and with differentiating it with respect to z, it yields

$$p(z,\varsigma)+zp_z'(z,\varsigma)$$
$$=\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta+2}f(z,\varsigma)}{z}$$
$$+2\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma)}{z}$$
$$+\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}$$

Under these circumstances, the strong superordination (2.12) can be expressed as

$$h(z,\varsigma)=\ q(z,\varsigma)+\gamma\ zq_z'(z,\varsigma)\prec\prec p(z,\varsigma)+zp_z'(z,\varsigma),$$

generating Lemma 1.2. Therefore, we are obtain the strong superordination $q(z,\varsigma)\ \prec\prec p(z,\varsigma)$. It follows from (2.8) that

$$q(z,\varsigma)\prec\prec\left(D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)_z'.$$

and the best subordinant is the convex function $q(z,\varsigma)=\frac{1}{z}\int_0^z h(x,\varsigma)dx$.

Concluding the results of strong differential subordination and superordination, we arrive at the following " strong sandwich results".

Theorem 2.9. Let $q_1(z,\varsigma)$ and $q_2(z,\varsigma)$ be convex functions with $q_1(0,\varsigma)=q_2(0,\varsigma)=0$, let the functions $h_1(z,\varsigma)=q_1(z,\varsigma)+\lambda z{q_1}_z'(z,\varsigma)$ and $h_2(z,\varsigma)=q_2(z,\varsigma)+\lambda z{q_2}_z'(z,\varsigma)$, for λ a positive integer, $z\in\mathrm{U},\varsigma\in\overline{\mathrm{U}}$. Assume that h_2 satisfies (2.1) and h_1 satisfies (2.10). Let $f\in\mathcal{A}_\varsigma^*$,

$$\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}\in Q_\varsigma\cap\ \mathcal{H}^*[0,\lambda,\varsigma]$$

and $\left(D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)_z'$ is univalent. If

$$h_1(z,\varsigma)\prec\prec\left(D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)_z'\prec\prec h_2(z,\varsigma)\,.$$

Next, that we obtain the strong differential sandwich

$$q_1(z,\varsigma)\prec\prec\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}\prec\prec q_2(z,\varsigma)\,,$$

and the convex functions $q_1(z,\varsigma)=\frac{1}{z}\int_0^z h_1(x,\varsigma)dx$ and $q_2(z,\varsigma)=\frac{1}{z}\int_0^z h_2(x,\varsigma)dx$ are, respectively, the best subordinant and the most superior.

Theorem 2.10. Let $q_1(z,\varsigma)$ and $q_2(z,\varsigma)$ be convex functions with $q_1(0,\varsigma)=q_2(0,\varsigma)=0$, let the functions $h_1(z,\varsigma)=q_1(z,\varsigma)+\frac{1}{\mu}z{q_1}_z'(z,\varsigma)$ and $h_2(z,\varsigma)=q_2(z,\varsigma)+\frac{1}{\mu}z{q_2}_z'(z,\varsigma)$, with μ a positive integer, $z\in U,\varsigma\in\overline{U}$. Assume that h_2 satisfies (2.4) and h_1 satisfies (2.12). Let $f\in\mathcal{A}_\varsigma^*$,

$$\left(\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}\right)^\mu\in Q_\varsigma\cap\ \mathcal{H}^*[0,\lambda\mu,\varsigma],$$

and $\left(\frac{D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z}\right)^{\mu-1} \left(D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)'_z$ is univalent. If

$$h_1(z,\varsigma) \prec\prec \left(\frac{D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z}\right)^{\mu-1} \left(D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)'_z \prec\prec h_2(z,\varsigma),$$

Subsequently, that we acquire the strong differential sandwich

$$q_1(z,\varsigma) \prec\prec \left(\frac{D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z}\right)^{\mu} \prec\prec q_2(z,\varsigma),$$

and the convex functions $q_1(z,\varsigma) = \frac{\mu}{z^\mu}\int_0^z h_1(x,\varsigma)\, x^{\mu-1} dx$ and $q_2(z,\varsigma) = \frac{\mu}{z^\mu}\int_0^z h_2(x,\varsigma)\, x^{\mu-1} dx$ are, respectively, the best subordinant and the best dominant.

Theorem 2.11. Let $q_1(z,\varsigma)$ and $q_2(z,\varsigma)$ be convex functions with $q_1(0,\varsigma) = q_2(0,\varsigma) = \frac{1}{1+\lambda}$, let the functions $h_1(z,\varsigma) = q_1(z,\varsigma) + z q_{1}{}'_z(z,\varsigma)$ and $h_2(z,\varsigma) = q_2(z,\varsigma) + z q_{2}{}'_z(z,\varsigma)$, $z \in U, \varsigma \in \overline{U}$. Assume that h_2 satisfies ς2.6) and h_1 satisfies (2.13). Let $f \in \mathcal{A}^*_\varsigma$,

$$\frac{D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)}{z\left(D_Z^{-\lambda} w_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)'_z} \in Q_\varsigma \cap \mathcal{H}^*\left[\frac{1}{1+\lambda}, 1, \varsigma\right],$$

and

$$\frac{\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2 \left(D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)^2 - \left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2 D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z,\varsigma).D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+2} f(z,\varsigma)}{\left(\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right) D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+1} f(z,\varsigma) + \left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right) D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)^2}$$

$$-\frac{\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right) D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z,\varsigma).D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+1} f(z,\varsigma) - \left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)\left(D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)^2}{\left(\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right) D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+1} f(z,\varsigma) + \left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right) D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)^2}$$

is univalent. If

$h_1(z,\varsigma) \prec\prec$

$$\frac{\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2 \left(D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)^2 - \left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right)^2 D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z,\varsigma).D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+2} f(z,\varsigma)}{\left(\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right) D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+1} f(z,\varsigma) + \left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^m\right) D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z,\varsigma)\right)^2}$$

$$-\frac{\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma).D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma)-\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)\left(D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)^{2}}{\left(\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma)+\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)D_{z}^{-\lambda}W_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)^{2}}$$

$\prec\prec h_2(z,\varsigma)$,

then we acquire the strong differential sandwich

$$q_1(z,\varsigma)\prec\prec\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z\left(D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)_z'}\prec\prec q_2(z,\varsigma)$$

and the convex functions $q_1(z,\varsigma)=\frac{1}{z}\int_0^z h_1(x,\varsigma)dx$ and $q_2(z,\varsigma)=\frac{1}{z}\int_0^z h_2(x,\varsigma)dx$ are, respectively, the best subordinant and the best dominant.

Theorem 2.12. Let $q_1(z,\varsigma)$ and $q_2(z,\varsigma)$ be convex functions with $q_1(0,\varsigma)=q_2(0,\varsigma)=0$, let the functions $h_1(z,\varsigma)=q_1(z,\varsigma)+\lambda z q_{1}{}_z'(z,\varsigma)$ and $h_2(z,\varsigma)=q_2(z,\varsigma)+\lambda z q_{2}{}_z'(z,\varsigma)$, for λ a positive integer, $z\in U,\varsigma\in\overline{U}$. Assume that h_2 satisfies (2.8) and h_1 satisfies (2.15). Let $f\in\mathcal{A}_\varsigma^*$,

$$\left(D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)_z'\in Q_\varsigma\cap\mathcal{H}^*[0,\lambda,\varsigma],$$

and

$$\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2}\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta+2}f(z,\varsigma)}{z}+2\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma)}{z}+\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2}\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}$$

is univalent. If

$$h_1(z,\varsigma)\prec\prec\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2}\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta+2}f(z,\varsigma)}{z}+2\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)\left(1+\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta+1}f(z,\varsigma)}{z}+\left(\lambda-\sum_{m=1}^{k}\binom{k}{m}(-1)^{m+1}\left(\frac{\alpha}{\beta}\right)^{m}\right)^{2}\frac{D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)}{z}\prec\prec h_1(z,\varsigma),$$

then we acquire the strong differential sandwich

$$q_1(z,\varsigma)\prec\prec\left(D_Z^{-\lambda}w_{\alpha,\beta}^{k,\eta}f(z,\varsigma)\right)_z'\prec\prec q_2(z,\varsigma)$$

and the convex functions $q_1(z,\varsigma)=\frac{1}{z}\int_0^z h_1(x,\varsigma)dx$ and $q_2(z,\varsigma)=\frac{1}{z}\int_0^z h_2(x,\varsigma)dx$ are, respectively, with best subordinant and the best dominant.

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