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On Fuzzy Measure on Fuzzy sets

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Abstract: In this paper, we study the fuzzy measure on fuzzy sets and prove some new properties.

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Mathematics subject classification : 20CXX .

1. Introduction

The fuzzy measure, defined on σ-field, was introduced by Sugeno [4]. Ralescu and Adams [10] generalized the concepts of fuzzy measure and fuzzy integral to the case that the value of a fuzzy measure can be infinite, and to realize an approach from Subjective.

Wang [7,11]and Kruse [17] studied some structural characteristics of fuzzy measures and proved several theorem about fuzzy measure.

Wang [7, 11] introduced the concept of 'autocontinuity of a set function', used it with regard to the above-mentioned researches, and obtained a series of new results.

The notion of fuzzy measure was extended by Avallone and Barbieri, Jiang and Suzuki [14] Narukawa and Murofushi [8] , Ralescu and Adams [10] as a set function which was defined on σ-field with values in [0,∞] . After that, many authors studied the fuzzy measure and proved some results about it as Guo and Zhang [8] Kui [13], Li and Yasuda [6] Lushu and Zhaohu [15] Minghu [16].

In this paper, we mention the definition of Fuzzy Measure on Fuzzy
Set with some Properties, and prove some new relations deal with fuzzy measure.

Definition (1): [18, 19]
Let Ω be anon empty set, a fuzzy set A in Ω(or a fuzzy subset in Ω) is a function from Ω into I, i.e. $A \in I^{\Omega}$. $A(x)$ is interpreted as the degree of membership of element x in a fuzzy set A for each $x \in \Omega$. a fuzzy set A in Ω is can be represented by the set of pairs:

$$A = \{(x, A(x)) : x \in \Omega \}$$

Note that every ordinary set is fuzzy set, i.e. $P(\Omega) \subseteq I^{\Omega}$.

Definition (2): [1, 2]
A family $\mathcal{F}$ of fuzzy sets in a set Ω is called a fuzzy σ −field on a set Ω If,

1. $\phi, \Omega \in \mathcal{F}$.
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.
3. If $\{A_n\} \subset \mathcal{F}, n = 1,2,3, \dots$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Evidently, an arbitrary σ −field must be fuzzy σ −field.

A fuzzy measurable Space is a pair$(\Omega, \mathcal{F})$, where Ω is a set and $\mathcal{F}$ is a fuzzy σ −field on Ω. a fuzzy set A in Ω is called fuzzy measurable (fuzzy measurable with respect to the fuzzy σ −field) if $A \in \mathcal{F}$, i.e. any member of $\mathcal{F}$ is called a fuzzy measurable set.

Definition (3) [3]:
Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space. A set function $\mu: \mathcal{F} \to [0, \infty]$ is said to be

(1) Finite if, $\mu(A) < \infty$ for each $A \in \mathcal{F}$.

(2) Semi-finite, if for each $A \in \mathcal{F}$ with $\mu(A) = \infty$, there exists $B \in \mathcal{F}$ with $B \subseteq A$ and $0 < \mu(B) < \infty$.

(3) Bounded, if
$$\sup\{|\mu(A): A \in \mathcal{F}|\} < \infty$$

(4) σ-finite, if for each $A \in \mathcal{F}$, there is a sequence $\{A_n\}$ of sets in $\mathcal{F}$ such that
$$A \subset \bigcup_{n=1}^{\infty} A_n$$
And $\mu(A_n) < \infty$ for all n.

(5) Additive if,
$\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathcal{F}$ and $\mathbf{A} \cap \mathbf{B} = \emptyset$.

(6) Finitely additive if,
$$\mu(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} \mu(A_k)$$
whenever $A_1, A_{2,\dots,}A_n$ are disjoint sets in $\mathcal{F}$.

(7) σ-additive (sometimes called Completely additive, or A counably additive) if,
$$\mu(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)$$
, whenever $\{A_n\}$is a sequence of disjoint sets in $\mathcal{F}$.

(8) Measure, if μ is σ - additive and $\mu(A) \geq 0$ for all $A \in \mathcal{F}$.

(9) Probability, if μ is a measure and $\mu(\Omega)$ =1.

(10) Continuous from below at $A \in \mathcal{F}$, if $\lim_{n\to\infty} \boldsymbol{\mu}(\boldsymbol{A_n}) = \boldsymbol{\mu}(\boldsymbol{A})$,whenever $\{A_n\}$ is a sequence of sets in $\mathcal{F}$, and $A_n \uparrow A$.

(11) Continuous from above at $A \in \mathcal{F}$, if $\lim_{n\to\infty} \boldsymbol{\mu}(\boldsymbol{A_n}) = \boldsymbol{\mu}(\boldsymbol{A})$,whenever $\{A_n\}$ is a sequence of sets in $\mathcal{F}$, and $A_n \downarrow A$.

(12) Continuous at $A \in \mathcal{F}$, if it is continuous both from below and from above at A.

Definition (4): [4]

Let$(\Omega, \mathcal{F})$ be a fuzzy measurable space. A set function $\mu: \mathcal{F} \longrightarrow [0, \infty]$ is said to be a fuzzy measure on $(\Omega, \mathcal{F})$ if it satisfies the following properties:

(1) $\mu(\emptyset)$=0

(2) If $A, B \in \mathcal{F}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$

Definition (5): [5]

Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space. A set function $\mu: \mathcal{F} \longrightarrow [0, \infty]$ is called:

(1) Upper semi continuous fuzzy measure if and only if
$$\lim_{n\to\infty} \mu(\mathrm{A_n}) = \mu(\bigcup_{n=1}^{\infty} \mathrm{A}_n)$$
whenever $\{\mathrm{A}_n\}$ is increasing sequence.

(2) Lower semi continuous fuzzy measure if and only if
$$\lim_{n\to\infty} \mu(\mathrm{A_n}) = \mu(\bigcap_{n=1}^{\infty} \mathrm{A}_n)$$
whenever $\{\mathrm{A}_n\}$ is decreasing sequence.

(3) Semi continuous fuzzy measure if it is both upper and lower semi continuous fuzzy measure.

(4) Regular if and only if
$\Omega \in \mathcal{F}$ and $\mu(\Omega) = 1$.

Definition (6): [5]

Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space. A set function $\mu: \mathcal{F} \longrightarrow [0, \infty)$ is said to be

1. Exhaustive if
$\mu(A_n) \longrightarrow 0$ whenever $\{A_n\}$ is infinite sequence of disjoint sets in $\mathcal{F}$
2. Order-continuous if $\mu(A_n) \longrightarrow 0$, whenever $A_n \in \mathcal{F}$,$n = 1,2, \dots$ and $A_n \downarrow \emptyset$.

Definition (7): [6]

Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space. A set function $\mu: \mathcal{F} \longrightarrow [0, \infty)$ is said to be additive,if $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathcal{F}$ and $A \cap B = \emptyset$.

Definition (8): [6, 7]

Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space. A set function $\mu: \mathcal{F} \longrightarrow [0, \infty)$ is said to be Null-additive,if $\mu(A \cup B) = \mu(A)$ whenever $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$, and $\mu(B) = 0$.

Definition (9): [8]

Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space. A set function $\mu: \mathcal{F} \longrightarrow [0, \infty)$ is said to be weakly null-additive, if for any $A, B \in \mathcal{F}$,
$$\boldsymbol{\mu}(\boldsymbol{A}) = \boldsymbol{\mu}(\boldsymbol{B}) = \mathbf{0} \Longrightarrow \boldsymbol{\mu}(\boldsymbol{A} \cup \boldsymbol{B}) = \mathbf{0}$$

Remark (10):

The concept of null-null additive stems from a wings textbook which the book[8] derived from , in which it is said to be weak null additive. But we consider that it is more precise and vivid to call it "null- null additive".

Definition (11):

Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space. A set function $\mu: \mathcal{F} \longrightarrow [0, \infty)$ is said to be finitely weakly null-additive, if for any $\{\mathrm{A_i}\} \subset \mathcal{F}, \mu(A_i) = 0$
$$\text{, for all } \mathrm{i} = 1, \dots \dots, \mathrm{n} \Longrightarrow \mu\left(\bigcup_{i=1}^{n} A_i\right) = 0$$

Definition (12): [6]
Let $(\Omega,\mathcal{F})$ be a fuzzy measurable space. A set function $\mu:\mathcal{F}\longrightarrow[0,\infty)$ is said to be Countably weakly null-additive, if for any $\{A_n\}\subset\mathcal{F},\mu(A_n)=0$

$$,\text{for all } n\geq 1 \Rightarrow \mu\left(\bigcup_{n=1}^{\infty}A_n\right)=0$$

Definition (13): [6]
Let $(\Omega,\mathcal{F})$ be a fuzzy measurable space. A set function $\mu:\mathcal{F}\longrightarrow[0,\infty)$ is said to be null-continuous, if $\mu(\bigcup_{n=1}^{\infty}A_n)=0$ for every increasing sequence $\{A_n\}$ in $\mathcal{F}$ such that $\mu(A_n)=0$, for all $n\geq 1$.

Definition (14):[9]
Let $(\Omega,\mathcal{F})$ be a fuzzy measurable space. A set function $\mu:\mathcal{F}\longrightarrow[0,\infty)$ is said to be null-subtractive, if we have $\mu(A\cap B^c)=\mu(A)$, whenever $A,B\in\mathcal{F}$ and $\mu(B)=0$.

Definition (15):[9]
Let $A\in\mathcal{F},\mu(A)<\infty$. μ is called pseudo-null-subtractive with respect to A, if for any $B\in A\cap\mathcal{F}$, we have $\mu(B\cap C)=\mu(B)$, whenever $C\in\mathcal{F}$, $\mu(A\cap C)=\mu(A)$. here $A\cap\mathcal{F}=\{A\cap D: D\in\mathcal{F}\}$.

Definition (16):[9]
Let $(\Omega,\mathcal{F})$ be a fuzzy measurable space. A set function $\mu:\mathcal{F}\longrightarrow[0,\infty)$ is said to be auto continuous from above (resp. autocontinuous from below), if $\mu(B_n)\to 0$ implies $\mu(A\cup B_n)\to\mu(A)$ (resp. $\mu(A\cap B^c{}_n)\to\mu(A)$), whenever $A\in\mathcal{F},\{B_n\}\subset\mathcal{F}$, μ is called autocontinuous if it is both autocontinuous from above and autocontinuous from below.

Definition (17): [9]
Let $A\in\mathcal{F}$, $\mu(A)<\infty,\mu$ is called pseudo-autocontinuous from above with respect to A (resp. from below with respect to A), if for any $\{B_n\}\subset\mathcal{F}$, when $\mu(B_n\cap A)\to\mu(A)$, then $\mu((B_n^c\cap A)\cup C)\to\mu(C)$, (resp. $\mu(B_n\cap C)\to\mu(C)$ whenever $C\in A\cap\mathcal{F}$.

μ is called pseudo-autocontinuous with respect to A, if it is both pseudo-autocontinuous from above with respect to A and pseudo-autocontinuous from below with respect to A.

2. Main results

Theorem (1):
Let $(\Omega,\mathcal{F},\mu)$ be a fuzzy measure space, if μ is $\sigma-additive$ then $\mathcal{F}^*=\{A\Delta B, A\in\mathcal{F}, B\subseteq\Omega \text{ and } \mu(B)=0\}$ is fuzzy σ −field on Ω.

Proof:

(1) Since $\Omega\Delta\emptyset=\Omega,\Omega\in\mathcal{F}$, $\emptyset\subseteq\Omega$ and $\mu(\emptyset)=0$, we have $\Omega\in\mathcal{F}^*$

(2) Let $V\in\mathcal{F}^*$, we have

$V=A\Delta B, A\in\mathcal{F},B\subseteq\Omega$ with $\mu(B)=0$

$$\Rightarrow V^c=(A\Delta B)^c=[(A/B)\cup(B/A)]^c$$
$$=(A^c\cap B^c)\cup(A\cap B)$$
$$=(A^c/B)\cup(B/A^c)$$
$$=A^c\Delta B, \text{where } B\subseteq\Omega \text{ with } \mu(B)=0$$

Since $A\in\mathcal{F}$ and $\mathcal{F}$ is a fuzzy σ −field, we have

$A^c\in\mathcal{F}$ and $B\subseteq\Omega$ with $\mu(B)=0$

$\Rightarrow V^c=A^c\Delta B, A^c\in\mathcal{F}$ and

$B\subseteq\Omega$ with $\mu(B)=0$

$\therefore V^c\in\mathcal{F}^*$

(3) Let $\{V_n\}$ be a sequence of sets in $\mathcal{F}^*$ with

$V_n=A_n\Delta B_n$, $A_n\in\mathcal{F},B_n\subseteq\Omega$ and $\mu(B_n)=0$ for all n.

We have

$$\bigcup_{n=1}^{\infty}A_n\in\mathcal{F}$$

$\mathcal{F}$ is fuzzy σ −field

$$\bigcup_{n=1}^{\infty}B_n\subseteq\Omega \text{ and } \mu\left(\bigcup_{n=1}^{\infty}B_n\right)=\sum_{n=1}^{\infty}\mu(B_n)=\mathbf{0}$$

So

$$\bigcup_{n=1}^{\infty}V_n=\bigcup_{n=1}^{\infty}(A_n\Delta B_n)$$
$$=\bigcup_{n=1}^{\infty}[(A_n/B_n)\cup(B_n/A_n)]$$
$$=\bigcup_{n=1}^{\infty}A_n\Delta\bigcup_{n=1}^{\infty}B_n$$
$$\Rightarrow\bigcup_{n=1}^{\infty}V_n\in\mathcal{F}^*$$

Consequently $\mathcal{F}^*$ is fuzzy σ −field on Ω.

Remark (2):
The union of a collection of fuzzy σ −field need not be fuzzy σ −field as in the following example.

Example (3):
Let A,B,C,D are fuzzy sets and $\Omega=\{A(x),B(x),C(x),D(x)\}$, such that

$$A(x)=\begin{cases}2x & 0\leq x\leq 1/2\\ 0 & 1/2<x\leq 1\end{cases}$$

$$B(x)=\begin{cases}0 & 0\leq x\leq 1/4\\ 2x & 1/4<x\leq 1/2\\ 1 & 1/2<x\leq 1\end{cases}$$

$$C(x)=\begin{cases}1-2x & 0\leq x\leq 1/2\\ 1 & 1/2<x\leq 1\end{cases}$$

$$D(x)=\begin{cases}1 & 0\le x\le 1/4\\ 1-2x & 1/4<x\le 1/2\\ 0 & 1/2<x\le 1\end{cases}\quad , \quad B(x)=\begin{cases}0 & 0\le x\le 1/4\\ 2x & 1/4<x\le 1/2\\ 1 & 1/2<x\le 1\end{cases}$$

Let $\mathcal{F}_1=\{\emptyset, A(x), C(x), \Omega\}$, $\mathcal{F}_2=\{\emptyset, B(x), D(x), \Omega\}$ are two fuzzy σ −fields, but $\mathcal{F}_1\cup\mathcal{F}_2$ is not fuzzy σ −field.

Solution:

First we must prove that $\mathcal{F}_1$ and $\mathcal{F}_2$ is fuzzy σ −field.

$\mathcal{F}_1$ Is fuzzy σ −field

(1) $\emptyset, \Omega\in\mathcal{F}_1$.

(2) (i) Let $A(x)\in\mathcal{F}_1$, to prove $A^c(x)\in\mathcal{F}_1$

From Definition (1.1.6) we get on

$$A^c(x)=1-A(x)$$

$$=1-\begin{cases}2x & 0\le x\le 1/2\\ 0 & 1/2<x\le 1\end{cases}$$

$$=\begin{cases}1-2x & 0\le x\le 1/2\\ 1 & 1/2<x\le 1\end{cases}$$

$$=C(x)$$

But $C(x)\in\mathcal{F}_1$

$\Longrightarrow A^c(x)\in\mathcal{F}_1$.

(ii) Let $C(x)\in\mathcal{F}_1$, to prove $C^c(x)\in\mathcal{F}_1$

$C^c(x)=1-C(x)$

$$=1-\begin{cases}1-2x & 0\le x\le 1/2\\ 1 & 1/2<x\le 1\end{cases}$$

$$=\begin{cases}2x & 0\le x\le 1/2\\ 0 & 1/2<x\le 1\end{cases}$$

$$=A(x)$$

But $A(x)\in\mathcal{F}_1$

$$\Longrightarrow C^c(x)\in\mathcal{F}_1$$

(iii) It is clear that $\emptyset^c=\Omega\in\mathcal{F}_1$

And $\Omega^c=\emptyset\in\mathcal{F}_1$

(3) (i) if $0\le x\le 1/2$

$\Longrightarrow (A\cup C)(x)=\max\{A(x), C(x)\}$

$=\max\{2x, 1-2x\}=2x$

(a) If $x=0$

$\Longrightarrow (A\cup C)(x)=0$

$=\emptyset(x)\in\mathcal{F}_1$.

(b) If $x=\frac{1}{2}$ $\Longrightarrow$ $(A\cup C)(x)=1=\Omega(x)\in\mathcal{F}_1$.

(ii) $1/2<x\le 1$

$\Longrightarrow (A\cup C)(x)=\max\{A(x), C(x)\}$

$=\max\{0,1\}=1$

$\therefore A\cup C(x)=1=\Omega(x)\in\mathcal{F}_1$.

$\therefore \mathcal{F}_1$ Is fuzzy σ −field

In the same way we can prove that $\mathcal{F}_2$ is fuzzy σ −field.

Now to prove that $\mathcal{F}_1\cup\mathcal{F}_2$ is not fuzzy σ −field

$\longrightarrow \mathcal{F}_1\cup\mathcal{F}_2=\{\emptyset, A(x), B(x), C(x), D(x), \Omega\}$

$$A(x)=\begin{cases}2x & 0\le x\le 1/4\\ 2x & 1/4<x\le 1/2\\ 0 & 1/2<x\le 1\end{cases}$$

(i) if $0\le x\le 1/4$

$\Longrightarrow (A\cup B)(x)=\max\{A(x), B(x)\}=\max\{2x, 0\}=2x$

(a) If $x=0 \Longrightarrow A\cup B(x)=0=\emptyset(x)\in\mathcal{F}_1$.

(b) If $x=\frac{1}{4}\Longrightarrow (A\cup B)(x)=1/2\notin\mathcal{F}_1$.

$\therefore \mathcal{F}_1\cup\mathcal{F}_2$ is not fuzzy σ −field

Theorem (4):

Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space, suppose that $\mathcal{F}^*$ is σ −field and μ^* is a measure on $(\Omega, \mathcal{F}^*)$, for any $A\in\mathcal{F}$ such that $\mu(B)=\mu^*(A\cap B)$ For any $B\in\mathcal{F}^*$ is fuzzy measure on $(\Omega, \mathcal{F})$.

Proof:

(1) Since $\mathcal{F}^*$ is σ −field $\Longrightarrow \emptyset\in\mathcal{F}^*$

$\therefore \mu(\emptyset)=\mu^*(A\cap\emptyset)=\mu^*(\emptyset)=\emptyset$.

(2) Let $A_1, A_2\in\mathcal{F}$, if $A_1\subseteq A_2$, then

$\mu(A_1)=\mu^*(A_1\cap B)\le\mu^*(A_2\cap B)=\mu(A_2)$

$\therefore$ μ is fuzzy measure on $(\Omega, \mathcal{F})$.

Theorem (5):

Let $(\Omega, \mathcal{F}, \mu)$ be a fuzzy measure space such that there is $B\in\mathcal{F}$ with $0<\mu(B)<\infty$,define $\mu^*:\mathcal{F}\longrightarrow[0,\infty]$ by $\mu^*(A)=\mu(A\cap B)\,/\,\mu(B)$,then$(\Omega, \mathcal{F}, \mu^*)$ is fuzzy measure space.

Proof:

(1) $\mu^*(\emptyset)=\mu(\emptyset\cap B)/\mu(B)=0$.

(2) let $A, B\in\mathcal{F}$, if $A\subseteq B$, we have $\mu(A)\le\mu(B)$

Since $A\subseteq B$, hence $A\cap B=A$

$$\Longrightarrow \mu(A\cap B)=\mu(A)$$

$$\Longrightarrow \mu(A\cap B)=\mu(A)\le\mu(B)$$

$$\Longrightarrow \mu(A\cap B)/\mu(B)\le\mu(B\cap B)/\mu(B)$$

$$\Longrightarrow \mu^*(A)\le\mu^*(B)\,.$$

Consequently μ^* is a fuzzy measure.

Theorem (6):

Let $(\Omega, \mathcal{F})$ be a fuzzy measurable space, μ, ν be a fuzzy measures on Ω , then $\mu+\nu$ which denoted by

$$(\mu+\nu)(A)=\mu(A)+\nu(A)$$

is fuzzy measure on Ω.

Proof:

(1) Since μ, ν be two fuzzy measures

$\Longrightarrow (\mu+\nu)(\emptyset)=0$.

(2) let $A, B\in\mathcal{F}$, if $A\subseteq B$, we have

$(\mu+\nu)(A)=\mu(A)+\nu(A)\le\mu(B)+\nu(B)=(\mu+\nu)(B)$.

So $\mu+\nu$ is fuzzy measure.

Corollary (1):
Let $(\Omega,\mathcal{F})$ be a fuzzy measurable space, μ be a fuzzy measure on Ω, and $\alpha > 0$, define a set function$(\alpha\mu)(A) = \alpha\mu(A)$, then $\alpha\mu$ is fuzzy measure on Ω.
Proof:
(1) Since μ be a fuzzy measure, we have
$$(\alpha\mu)(\emptyset) = \alpha\mu(\emptyset) = 0 .$$
(2) let $A, B \in \mathcal{F}$, if $A \subseteq B$, we have
$$\mu(A) \le \mu(B)$$
$$\Longrightarrow (\alpha\mu)(A) = \alpha\mu(A) \le \alpha\mu(B)$$
$$= (\alpha\mu)(B) .$$
So $\alpha\mu$ is fuzzy measure.
Remark (7)
The points (1) and (2) from Definition (5) explain fuzzy measure is upper semi continuous and lower semi continuous; the following results take us to the converse direction.
Theorem (8)
Let $(\Omega,\mathcal{F})$ be a fuzzy measurable space and let μ be a function$\mu: \mathcal{F} \longrightarrow \mathbb{R}_+$, if μ is additive, non-decreasing and upper semi continuous, then μ is fuzzy measure.
Proof:
(1) Since $A = A \cup \emptyset$
Also μ is additive we have
$$\mu(A) = \mu(A \cup \emptyset) = \mu(A) + \mu(\emptyset)$$
$$\Longrightarrow \mu(\emptyset) = 0$$
(1) Let $A, B \in \mu$, if $A \subseteq B$, we have
$$\mathrm{B} = A \cup (B \setminus A)$$
$$\text{and } \mathrm{A} \cap (B \setminus A) = \emptyset$$
Since μ is additive we have, we obtain
$$\mu(B) = \mu(A) + \mu(B \setminus A)$$
Consequently
$$\mu(B \setminus A) = \mu(B) - \mu(A)$$
In addition, $\mu(B \setminus A) \ge 0$
Hence
$$\mu(A) \le \mu(B)$$
Then μ is fuzzy measure.
Theorem (9):
Let $(\Omega,\mathcal{F})$ be a fuzzy measurable space, let $\{\mathrm{A}_n\}$ be a sequence of disjoint fuzzy set in $\mathcal{F}$ and it is decreasing, if $\mu(\mathrm{A}_n) < \infty$ and μ is lower semi continuous fuzzy measure at $\emptyset$, then $\lim_{n\to\infty} \mu(\mathrm{A}_n) = 0$.
Proof:
Since $\{\mathrm{A}_n\}$ is lower continuous fuzzy measure at $\emptyset$, we have
$$\lim_{n\longrightarrow\infty} \mu(\mathrm{A_n}) = \mu(\emptyset)$$
But
$$\mu(\emptyset) = 0$$
Consequently, we have
$$\lim_{n\to\infty} \mu(\mathrm{A}_n) = 0.$$
Theorem (10):
Let $(\Omega,\mathcal{F})$ be a fuzzy measurable space, and for any$A \in \mathcal{F}$, $\mu(A) \neq 0, then\ \mu$ is null additive.

Proof:
If there exists some set $B \in \mathcal{F}$ such that $\mu(B) = 0$, then $B = \emptyset$.
Consequently, for any $A \in \mathcal{F}$, we have $\mu(A \cup B) = \mu(A)$.
Theorem (11):
Let $(\Omega,\mathcal{F})$ be a fuzzy measurable space, if μ is autocontinuous from below, then it is null-subtractive.
Proof:
Let $A, B_n \in \mathcal{F}$
Since if μ is autocontinuous from below, we have
$$\lim_{n\to\infty} \mu(B_n) = 0$$
Also we have
$$\mu(A \cap B_n^c) \to \mu(A)$$
Consequently μ is null-subtractive.
Theorem (12):
Let $(\Omega,\mathcal{F})$ be a fuzzy measurable space, if μ is pseudo-autocontinuous from below with respect to A, then it is pseudo-null-subtractive with respect to A.
Proof:
Let $A, B_n \in \mathcal{F}$
Since if μ is pseudo-autocontinuous from below, we have
$\mu(A) < \infty$. And $C \in A \cap \mathcal{F}$
$$\mu(B_n \cap C) \to \mu(C)$$
Consequently μ is pseudo-null-subtractice with respect to A.
Theorem (13):
Let $(\Omega,\mathcal{F})$ be a fuzzy measurable space, if μ is upper semi continuous fuzzy measure and ccountably weakly null additive then μ is exhaustive.
Proof:
Let $\{A_n\}$ be a disjoint of sequence of sets in $\mathcal{F}$
Since μ is countably weakly null additive
$\therefore$ $\mu(A_n) = 0$, for all $n \ge 1$
$$\Longrightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0$$
Also μ is upper semi continuous
$$\Longrightarrow \lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$$
$$\Longrightarrow \lim_{n\to\infty} \mu(A_n) = 0$$
$\therefore$ μ is exhaustive.

Theorem (14):
Let $(\Omega,\mathcal{F})$ be a fuzzy measurable space, if μ is ccountably weakly null additive then μ is null-continuous.

Proof:

Let $\{A_n\}$ be a increasing sequence of sets in $\mathcal{F}$, such that

$\mu(A_n) = 0$, for all $n \geq 1$

Since μ is ccountably weakly null additive

$$\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0$$

$\therefore$ μ is null-continuous .

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حول القياس الضبابي على المجموعات الضبابية

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المستخلص:

في هذا البحث، سندرس القياس الضبابي على مجموعات ضبابية ونبرهن بعض الخصائص الجديدة.