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Additivity of Higher Multiplicative Mappings in Rings

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Abstract

We study the additivity of higher multiplicative mappings in rings.

Mathematics subject classification : .

Introduction

Let $\mathcal{R}$ and S be an arbitrary rings with associative property (not necessarily with identity elements).An injective mappings σ of $\mathcal{R}$ onto S such that $(xy)^{\sigma} = x^{\sigma}y^{\sigma}$ for all x,y $\in$ R will be said a "multiplicative isomorphism" of $\mathcal{R}$ onto S. The study of additivity of multiplicative isomorphism has been done by "Rickart[3]" ,"Wang [5]"and also by "Johnson [1]" .In both of these paper some sort of minimality conditions were imposed on the ring R. In [2] the author generalized the main theorem of Rickarts paper [3] and at the same time remove the minimality condition .In [4]Shaheen defined higher multiplicative maps and study its additivity on Triangular rings .In this work ,we introduce a study on the additivity of higher multiplicative maps in rings, we should mentioned the reader that our result in this paper is along different lines than those in shaheens results.Now , we shall give Shaheens definition see[4] of higher multiplicative maps which is basic in this paper.

Definition 1.1:-[4]

Let $\mathcal{R}$ and $\acute{\mathcal{R}}$ be two rings ,a family $\varphi = (\varphi_i)_{i\in N}$ of mappings of a ring $\mathcal{R}$ into $\acute{\mathcal{R}}$.φ is said to be higher multiplicative if $\forall$ n$\in$ N.We have

$$\varphi_n(ab) = \sum_{i=1}^{n} \varphi_i(a)\varphi_i(b) \text{ for all } a, b \in \mathcal{R}.$$

Throughout this section $\varphi_i(a)\varphi_j(b) = 0$ for all i$\neq j$and $\varphi_i(o) = 0$ for all i, suppose that $\mathcal{R}$ be a ring contain $\{e_\alpha : \alpha \in \Lambda\}$family of idempotent that satisfy:

(i)x $\mathcal{R}$ ={0} implies x=0;

(ii)If e_α $\mathcal{R}$ x={0} $\forall\alpha \in \Lambda$,then x=0 (and hence $\mathcal{R}$ x={0} $\Rightarrow$ x=0.

(iii) $\forall$ $\alpha \in \Lambda$, $e_\alpha x e_\alpha$ $\mathcal{R}$ (1-e_α)={0}$\Rightarrow$ $e_\alpha x e_\alpha = 0$.

2-Results

Now, we shall introduce some lemmas

Lemma 2.1 : For all $x_{ij} \in R_{ij}$ for all i , j

$$\varphi_k(x_{ii} + x_{jm}) = \varphi_k(x_{ii}) + \varphi_k(x_{jm}) \quad j\neq m.$$

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Proof

Let i = j =1 and m=2. Let z is an element of $\mathcal{R}$ such that

$\varphi_k(z) = \varphi_k(x_{11}) + \varphi_k(x_{12})$for all k .

For arbitrary $a_{1j} \in \mathcal{R}_{1j}$, we have

$$\varphi_k(za_{1j}) = \sum_{i=1}^{k} \varphi_i(z) \ \ \varphi_i(a_{1j})$$
$$= \sum_{i=1}^{k} (\varphi_i(x_{11}) + \varphi_i(x_{12})) \ \varphi_i(a_{1j})$$
$$= \sum_{i=1}^{k} \varphi_i(x_{11})\varphi_i(a_{1j}) + \varphi_i(x_{12}) \ \varphi_i(a_{1j})$$
$$= \sum_{i=1}^{k} \varphi_i\left((x_{11} + x_{12})(a_{1j})\right)$$

Since φ_i is injective,then $za_{1j} = (x_{11} + x_{12})a_{1j}$

In the same fashion ,for $a_{2j} \in \mathcal{R}_{2j}$,we have

$$\varphi_k(za_{2j}) = \sum_{i=1}^{k} \varphi_i(z)\varphi_i(a_{2j})$$
$$= \sum_{i=1}^{k} (\varphi_i(x_{11}) + \varphi_i(x_{12}))\varphi_i(a_{2j})$$
$$= \sum_{i=1}^{k} \varphi_i(x_{11})\varphi_i(a_{2j}) + \varphi_i(x_{12})\varphi_i(a_{2j})$$
$$= \varphi_k((x_{11} + x_{12})a_{2j})$$

Since φ_k is injective ,then

$$za_{2j} = (x_{11} + x_{12})a_{2j}$$

Thus $[z-(x_{11} + x_{12})]\mathcal{R} = 0$ and so by condition (i),we see that

$z = x_{11} + x_{12}$.

Then $\varphi_k(z) = \varphi_k(x_{11}) + \varphi_k(x_{12})$for all n .

The only essentially different choice for i,j,m is to let i=m=1 and let j=2.In this case we are led to $\mathcal{R}\,[z-(x_{11} + x_{12})] = 0$ and so once again $z = x_{11} + x_{12}$ in view of condition (ii).

Lemma 2.2:- φ_k is additive map on $\mathcal{R}_{12}$

Proof :-

Let $x_{12}, u_{12} \in \mathcal{R}_{12}$ and $z \in \mathcal{R}$ such that

$$\varphi_k(z) = \varphi_k(x_{12}) + \varphi_k(u_{12})$$

An arbitrary $a_{1j} \in \mathcal{R}_{1j}$, we have

$$\varphi_k(za_{1j}) = \sum_{i=1}^{k} \varphi_i(z)\varphi_i(a_{1j})$$
$$= \sum_{i=1}^{k} (\varphi_i(x_{12}) + \varphi_i(u_{12}))\varphi_i(a_{1j})$$
$$= \sum_{i=1}^{k} \varphi_i(x_{12})\varphi_i(a_{1j}) + \varphi_i(u_{12})\varphi_i(a_{1j})$$
$$= \sum_{i=1}^{k} \varphi_i\left((x_{12} + u_{12})(a_{1j})\right) = 0$$

Then $za_{1j}=0$

For $a_{2j} \in \mathcal{R}_{2j}$,

$$\emptyset_k(za_{2j}) = \sum_{i=1}^{k} \varphi_i(z)\varphi_i(a_{2j})$$
$$= \sum_{i=1}^{k} (\varphi_i(x_{12}) + \varphi_i(u_{12}))\varphi_i(a_{2j})$$
$$= \sum_{i=1}^{k} (\varphi_i(e_1)+\varphi_i(x_{12}))(\varphi_i(a_{2j}) + \varphi_i(u_{12}))\varphi_i(a_{2j})$$
$$= \sum_{i=1}^{k} (\varphi_i(e_1)+\varphi_i(x_{12}))(\varphi_i(a_{2j}) + \varphi_i(u_{12}a_{2j}))$$
$$= \sum_{i=1}^{k} \varphi_i(e_1 + x_{12})(a_{2j} + u_{12}a_{2j})$$

= ∑_(i = 1)^k▒〖 φ_i (〗 〖 (x〗 _12 + u_12)a_2j)

by using of [Lemma 2.1],

$za_{2j} = (x_{12} + u_{12})a_{2j}$

$[z - (x_{12} + u_{12})]a_{2j} = 0$ it follows that

$[z - (x_{12} + u_{12})]\mathcal{R} = 0$

And so by condition (i), $z = x_{12} + u_{12}$

Lemma2.3:- φ_k is additive on $\mathcal{R}_{11}$.

Proof :- Let $x_{11}, u_{11} \in \mathcal{R}_{11}$,and we write

$\varphi_k(z) = \varphi_k(x_{11}) + \varphi_k(u_{11})$ for some $z \in \mathcal{R}$.

Using [Lemma 2.2],we see that

$$\varphi_k(za_{12}) = \sum_{i=1}^{k} \varphi_i(z)\varphi_i(a_{12})$$

$= \sum_{i=1}^{n} (\emptyset_i(x_{11}) + \emptyset_i(u_{11}))\emptyset_i(a_{12})$

$= \sum_{i=1}^{k} \varphi_i(x_{11})\varphi_i(a_{12}) + \varphi_i(u_{11})\varphi_i(a_{12})$

$= \sum_{i=1}^{n} \varphi_i(x_{11}a_{12}) + \varphi_i(u_{11}a_{12}) = \varphi_k(x_{11}a_{12}) + \varphi_k(u_{11}a_{12})$

$= \varphi_k(x_{11}a_{12} + u_{11}a_{12}) = \varphi_k((x_{11} + u_{11})a_{12})$

This show that

$za_{12} = (x_{11} + u_{11})a_{12}$

In other words,

$[z - (x_{11} + u_{11})]\mathcal{R}_{12} = 0$

Next, we write $z = z_{11} + z_{12} + z_{21} + z_{22}$

And note that

$\varphi_k(z) = \varphi_k(x_{11}) + \varphi_k(u_{11})$

$= \varphi_k(e_1x_{11}) + \varphi_k(e_1u_{11})$

$= \sum_{i=1}^{k} \varphi_i(e_1)\varphi_i(x_{11}) + \varphi_i(e_1)\varphi_i(u_{11})$

$= \sum_{i=1}^{k} \varphi_i(e_1)(\varphi_i(x_{11}) + \varphi_i(u_{11}))$

$= \varphi_k(e_1z) = \varphi_k(e_1(z_{11} + z_{12} + z_{21} + z_{22}))$

$= \varphi_k(z_{11} + z_{12})$

This equation show that

$Z = z_{11} + z_{12}$,whence $z_{21} = z_{22} = 0$

By repeating the argument with e_1 multiplied in to the right ,one finds that $z_{12}=0$,thus yielding $z = z_{11} \in \mathcal{R}_{11}$.Therfore $z - (x_{11} + u_{11}) \in \mathcal{R}_{11}$ and our previous conclusion that

$(z - (x_{11} + u_{11}))\mathcal{R}_{12} = 0$

Forces $z = (x_{11} + u_{11})$]from [condition,(iii)].

Lemma2.4:-

$\varphi_k$is additive on $e_1\mathcal{R} = \mathcal{R}_{11} + \mathcal{R}_{12}$.

Proof:-

Let $x_{11}, u_{11} \in \mathcal{R}_{11}$and x_{12} and $u_{12} \in \mathcal{R}_{12}$ then by Lemma 2.2,2.3,2.4 ,we get

$$\varphi_k((x_{11} + x_{12}) + (u_{11} + u_{12})) = \varphi_k(x_{11} + u_{11}) + \varphi_k(x_{12} + u_{12})$$

$= \varphi_k(x_{11}) + \varphi_k(u_{11}) + \varphi_k(x_{12}) + \varphi_k(u_{12})$

$= \varphi_k(x_{11} + x_{12}) + \varphi_k(u_{11} + u_{12})$

Then $\varphi_k$is additive on $e_1\mathcal{R} = \mathcal{R}_{11} + \mathcal{R}_{12}$

Theorem 2.5:-

Let $\mathcal{R}$ be a ring containing $\{e_\alpha : \alpha \in \Lambda\}$ family of idempotents which satisfies:

(i)$x\mathcal{R} = \{0\} \Rightarrow x=0$;

(ii)If $e_\alpha \mathcal{R} x = \{0\}$ $\forall \alpha \in \Lambda$,then x=0(and hence $\mathcal{R} x = \{0\} \Rightarrow x=0$.

(iii)$\forall \alpha \in \Lambda$, $e_\alpha x e_\alpha \mathcal{R} (1-e_\alpha) = \{0\} \Rightarrow e_\alpha x e_\alpha = 0$.

Then any multiplicative higher isomorphism φ of $\mathcal{R}$ on to an arbitrary ring S is additive .

Proof

Let $x, y \in \mathcal{R}$,and write

$$\varphi_k(z) = \varphi_k(x) + \varphi_k(y)$$

For $\alpha \in A$,select any $t_\alpha \in e_\alpha\mathcal{R}$.Then

$$\varphi_k(t_\alpha z) = \sum_{i=1}^{k} \varphi_i(t_\alpha)\varphi_i(z)$$

$$= \sum_{i=1}^{k} \varphi_i(t_\alpha)(\varphi_i(x) + \varphi_i(y))$$

$$= \sum_{i=1}^{k} (\varphi_i(t_\alpha)\varphi_i(x) + \varphi_i(t_\alpha)\varphi_i(y))$$

$= \varphi_k(t_\alpha x + t_\alpha y)$

Since φ_k is additive on $e_\alpha\mathcal{R}$.by [Lemma 2.4]

Hence $t_\alpha z = t_\alpha(x + y)$

And so we have proved that

$e_\alpha\, \mathcal{R}[z-(x+y)] = 0$ for all $\alpha \in$A.

By [Condition ,(ii)] ⇒z=x+y. This say that

$$\varphi_k(x+y) = \varphi_k(x) + \varphi_k(y).$$

Corollary 2.6:-

Let $\mathcal{R}$ be a prime ring containing idempotent e≠0,1($\mathcal{R}$ needn't have identity) ,then any higher multiplicative isomorphism of $\mathcal{R}$ onto a ring S is additive .

Corollary 2.7:-

Let $\mathcal{R}$ satisfies the conditions of the theorem (or the preceding corollary).Then any higher multiplicative anti-isomorphism φ of $\mathcal{R}$ onto an arbitrary ring S is additive .

Proof:-Let τ be the higher anti-isomorphism of S onto the opposite ring S^* of S

By the theorem

$\sigma = \tau\varphi$ is an additive of $\mathcal{R}$ onto S^* and so φ is additive .

References

[1] R.E. Johnson, Rings with unique addition ,Proc.Amer.Math.Soc. 9 (1958) 57-61.

[2]W.S.Martindal III,When are Multiplicative mapping Additive ?,Proceeding of the American mathematical Society ,21,695-698,1969.

[3] C.E. Rickart ,one- to- one mappings of rings and lattices ,Bull .Amer . Math . Soc.54(1948),758-764.

[4] R.C. Shaheen , on Additive Mappings of Matrix Rings, Phd .Thesis, Baghdad University,(2016).

[5]Y.Wang,The additivity of Multiplicative Maps on Rings,Communication in algebra, Vol 37,(2009),2351-2356.

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