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A New Subclass of Harmonic Univalent Functions

Waggas Galib Atshan
Department of Mathematics
College of Computer Science
and Information Technology
University of Al-Qadisiyah, Diwaniya-Iraq
E-mail: waggas.galib@qu.edu.iq

Najah Ali Jiben Al-Ziadi
Department of Mathematics
College of Science
University of Baghdad, Baghdad -Iraq
E-mail: najah.ali@qu.edu.iq

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Abstract: In this paper, we define a new class of harmonic univalent functions of the form $f = h + \overline{g}$ in the open unit disk U. We obtain basic properties, like, coefficient bounds, extreme points, convex combination, distortion and growth theorems and integral operator.

Keywords: Univalent function; harmonic function; extreme point; convex combination; integral operator.
Mathematics subject classification : 30C45.

1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic function in a simply connected domain D if both u and v are real harmonic in D. In any simply connected domain $D \subset \mathbb{C}$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. Note that $f = h + \overline{g}$ reduces to h if the co-analytic part g is zero. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [1]).

Let $N_{\mathcal{H}}$ denote the class of function $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the open unit disk $U = \{z: |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in N_{\mathcal{H}}$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n ,$$

$$g(z) = \sum_{n=1}^{\infty} b_n z^n \ , |b_1| < 1. \qquad (1.1)$$

Also, Let $R_{\mathcal{H}}$ denote the subclass of $N_{\mathcal{H}}$ containing all functions $f = h + \overline{g}$, where h and g are given by

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad , \qquad g(z) = -\sum_{n=1}^{\infty} b_n z^n \ ,$$

$$(a_n \geq 0, b_n \geq 0, |b_1| < 1). \qquad (1.2)$$

We denote by $WN_{\mathcal{H}}(\lambda, \alpha, \beta)$ the class of all functions of the form (1.1) that satisfy the condition:

$$Re\left\{\frac{zf'(z) + z^2 f''(z)}{\lambda z f'(z) + (1-\lambda) f(z)}\right\} > \beta \left|\frac{zf'(z) + z^2 f''(z)}{\lambda z f'(z) + (1-\lambda) f(z)} - 1\right| + \alpha, \qquad (1.3)$$

where $0 \leq \lambda \leq 1,\ 0 \leq \alpha < 1,\ \beta \geq 0$ and $z \in U$.

Let $WR_{\mathcal{H}}(\lambda, \alpha, \beta)$ be the subclass of $WN_{\mathcal{H}}(\lambda, \alpha, \beta)$, where $WR_{\mathcal{H}}(\lambda, \alpha, \beta) = R_{\mathcal{H}} \cap WN_{\mathcal{H}}(\lambda, \alpha, \beta)$.
Note that for the case $\lambda = 1$ and $g \equiv 0$ the class $WR_{\mathcal{H}}(\lambda, \alpha, \beta)$ reduces to the class $UCT(\alpha, \beta)$ studied by Bharati et al. [2]. Also, for the case $\lambda = 0$, $\beta = 0$ and $g \equiv 0$ the class $WR_{\mathcal{H}}(\lambda, \alpha, \beta)$ reduces to the class $H(1, \beta)$ studied by Lashin [3].

Such type of study was carried out by various authors for another classes, like, Atshan and Wanas [4], El-Ashwah and Kota [5] and Ezhilarasi and Sudharsan [6].
In order to derive our main results, we have to recall here the following lemmas:

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Lemma 1[7]**.** Let $w = u + iv$ and β, α are real numbers. Then $Re(w) \geq \beta|w-1| + \alpha$ if and only if $Re\{w(1+\beta e^{i\theta}) - \beta e^{i\theta}\} > \alpha$.

Lemma 2[7]**.** Let $w = u + iv$. Then $Re(w) \geq \alpha$ if and only if $|w - (1+\alpha)| \leq |w + (1-\alpha)|$.

2. Coefficient bounds

First, we give the sufficient condition for $f = h + \overline{g}$ to be in the class $WN_{\mathcal{H}}(\lambda, \alpha, \beta)$.

Theorem 2.1. Let $f = h + \overline{g}$ with h and g are given by (1.1). If

$$\sum_{n=2}^{\infty}[n^2(\beta+1) - (\beta+\alpha)(\lambda n - \lambda + 1)]|a_n| + \sum_{n=1}^{\infty}[n^2(\beta+1) - (\beta+\alpha)(\lambda n - \lambda + 1)]|b_n| \leq 1 - \alpha, \quad (2.1)$$

where $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$, $\beta \geq 0$, then f is harmonic univalent, sense-preserving in U and $f \in WN_{\mathcal{H}}(\lambda, \alpha, \beta)$.

Proof. If $z_1 \neq z_2$, then

$$\left|\frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)}\right| \geq 1 - \left|\frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)}\right|$$

$$= 1 - \left|\frac{\sum_{n=1}^{\infty} b_n(z_1^n - z_2^n)}{(z_1 - z_2) + \sum_{n=2}^{\infty} a_n(z_1^n - z_2^n)}\right|$$

$$> 1 - \frac{\sum_{n=1}^{\infty} n|b_n|}{1 - \sum_{n=2}^{\infty} n|a_n|}$$

$$\geq 1 - \frac{\sum_{n=1}^{\infty}\frac{[n^2(\beta+1) - (\beta+\alpha)(\lambda n - \lambda + 1)]}{1-\alpha}|b_n|}{1 - \sum_{n=2}^{\infty}\frac{[n^2(\beta+1) - (\beta+\alpha)(\lambda n - \lambda + 1)]}{1-\alpha}|a_n|}$$

$$\geq 0,$$

which proves univalence. f is sense-preserving in U. This is because

$$|h'(z)| \geq 1 - \sum_{n=2}^{\infty} n\,|a_n||z|^{n-1}$$

$$> 1 - \sum_{n=2}^{\infty} n\,|a_n|$$

$$\geq 1 - \sum_{n=2}^{\infty}\frac{[n^2(\beta+1) - (\beta+\alpha)(\lambda n - \lambda + 1)]}{1-\alpha}|a_n|$$

$$\geq \sum_{n=1}^{\infty}\frac{[n^2(\beta+1) - (\beta+\alpha)(\lambda n - \lambda + 1)]}{1-\alpha}|b_n|$$

$$\geq \sum_{n=1}^{\infty} n|b_n| > \sum_{n=1}^{\infty} n\,|b_n||z|^{n-1}$$

$$\geq |g'(z)|.$$

For proving $f \in WN_{\mathcal{H}}(\lambda, \alpha, \beta)$, we must show that (1.3) holds true. By using Lemma (1), it is sufficient to show that

$$Re\left\{\frac{zf'(z) + z^2 f''(z)}{\lambda z f'(z) + (1-\lambda)f(z)}(1+\beta e^{i\theta}) - \beta e^{i\theta}\right\} > \alpha \quad (-\pi \leq \theta \leq \pi),$$

or equivalently

$$Re\left\{\frac{(1+\beta e^{i\theta})(zf'(z) + z^2 f''(z))}{\lambda z f'(z) + (1-\lambda)f(z)} - \frac{\beta e^{i\theta}(\lambda z f'(z) + (1-\lambda)f(z))}{\lambda z f'(z) + (1-\lambda)f(z)}\right\} > \alpha. \quad (2.2)$$

If we put

$$A(z) = (1+\beta e^{i\theta})(zf'(z) + z^2 f''(z)) - \beta e^{i\theta}(\lambda z f'(z) + (1-\lambda)f(z))$$

and

$$B(z) = \lambda z f'(z) + (1-\lambda)f(z).$$

In view of Lemma (2) we only need to prove that

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \geq 0, for\ \ 0 \leq \alpha < 1.$$

So, $|A(z) + (1-\alpha)B(z)|$

$$= \Bigg|(1+\beta e^{i\theta})\left(z + \sum_{n=2}^{\infty} n^2 a_n z^n + \sum_{n=1}^{\infty} n^2 b_n (\overline{z})^n\right) - \beta e^{i\theta}\left(z + \sum_{n=2}^{\infty}(\lambda n - \lambda + 1)\, a_n z^n + \sum_{n=1}^{\infty}(\lambda n - \lambda + 1)\, b_n(\overline{z})^n\right) + (1-\alpha)\left(z + \sum_{n=2}^{\infty}(\lambda n - \lambda + 1)\, a_n z^n + \sum_{n=1}^{\infty}(\lambda n - \lambda + 1)\, b_n(\overline{z})^n\right)\Bigg|$$

$$= |(2-\alpha)z$$

$$+\sum_{n=2}^{\infty}[n^2(1+\beta e^{i\theta})-(\beta e^{i\theta}+\alpha-1)(\lambda n-\lambda+1)]a_n z^n$$

$$+\sum_{n=2}^{\infty}[n^2(1+\beta e^{i\theta})-(\beta e^{i\theta}+\alpha-1)(\lambda n-\lambda+1)]b_n(\overline{z})^n\Big|.$$

Also, $|A(z)-(1+\alpha)B(z)|$

$$=\Bigg|(1+\beta e^{i\theta})\Bigg[z+\sum_{n=2}^{\infty}na_n z^n+\sum_{n=1}^{\infty}nb_n(\overline{z})^n$$

$$+\sum_{n=2}^{\infty}(n-1)a_n z^n+\sum_{n=1}^{\infty}n(n-1)b_n(\overline{z})^n\Bigg]$$

$$-\beta e^{i\theta}\Bigg[\lambda\left(z+\sum_{n=2}^{\infty}n\,a_n z^n+\sum_{n=1}^{\infty}n\,b_n(\overline{z})^n\right)$$

$$+(1-\lambda)\left(z+\sum_{n=2}^{\infty}a_n z^n+\sum_{n=1}^{\infty}b_n(\overline{z})^n\right)\Bigg]$$

$$-(1+\alpha)\Bigg[\lambda\left(z+\sum_{n=2}^{\infty}n\,a_n z^n+\sum_{n=1}^{\infty}n\,b_n(\overline{z})^n\right)$$

$$+(1-\lambda)\left(z+\sum_{n=2}^{\infty}a_n z^n+\sum_{n=1}^{\infty}b_n(\overline{z})^n\right)\Bigg]\Bigg|$$

$$=|-\alpha z$$

$$+\sum_{n=2}^{\infty}[n^2(1+\beta e^{i\theta})-(\beta e^{i\theta}+\alpha+1)(\lambda n-\lambda+1)]a_n z^n$$

$$+\sum_{n=1}^{\infty}[n^2(1+\beta e^{i\theta})-(\beta e^{i\theta}+\alpha+1)(\lambda n-\lambda+1)]b_n(\overline{z})^n\Big|.$$

Therefore,

$$|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|$$

$$\geq 2\{(1-\alpha)$$

$$-\sum_{n=2}^{\infty}[n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)]|a_n|$$

$$-\sum_{n=1}^{\infty}[n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)]|b_n|\Bigg\}\geq 0.$$

By inequality (2.1), which implies that $f\in WN_{\mathcal{H}}(\lambda,\alpha,\beta)$.

The harmonic univalent function

$$f(z)=z+\sum_{n=2}^{\infty}\frac{x_n}{n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)}z^n$$

$$+\sum_{n=1}^{\infty}\frac{\overline{y}_n}{n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)}(\overline{z})^n, \qquad (2.3)$$

where $\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1-\alpha$, show that coefficient bound given by (1.3) is sharp.

The functions of the form (2.3) are in the class $WN_{\mathcal{H}}(\lambda,\alpha,\beta)$, because

$$\sum_{n=2}^{\infty}[n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)]$$

$$\times\frac{|x_n|}{n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)}$$

$$+\sum_{n=1}^{\infty}[n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)]$$

$$\times\frac{|y_n|}{n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)}$$

$$=\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1-\alpha.$$

The restriction placed in Theorem (2.1) on the moduli of the coefficients of $f=h+\overline{g}$ enables us to conclude for arbitrary rotation of the coefficients of f that the resulting functions would still be harmonic univalent and $f\in WN_{\mathcal{H}}(\lambda,\alpha,\beta)$.

In the following theorem, it is shown that the condition (2.1) is also necessary for functions in $WR_{\mathcal{H}}(\lambda,\alpha,\beta)$.

Theorem (2.2). Let $f=h+\overline{g}$ with h and g be given by (1.2). Then $f\in WR_{\mathcal{H}}(\lambda,\alpha,\beta)$ if and only if

$$\sum_{n=2}^{\infty}[n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)]a_n$$

$$+\sum_{n=1}^{\infty}[n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)]b_n$$

$$\leq 1-\alpha, \qquad (2.4)$$

where $0\leq\lambda\leq 1, 0\leq\alpha<1$ *and* $\beta\geq 0$.

Proof. Since $WR_{\mathcal{H}}(\lambda,\alpha,\beta)\subset WN_{\mathcal{H}}(\lambda,\alpha,\beta)$, we only need to proof the " only if " part of the theorem.

Assume that $f\in WR_{\mathcal{H}}(\lambda,\alpha,\beta)$. Then by (1.3), we have

$$Re\left\{\frac{zf'(z)+z^2f''(z)}{\lambda zf'(z)+(1-\lambda)f(z)}(1+\beta e^{i\theta})-\beta e^{i\theta}\right\}\geq\alpha.$$

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This is equivalent to

$$Re\left\{\frac{\left(1+\beta e^{i\theta}\right)\left(zf'(z)+z^2f''(z)\right)}{\lambda zf'(z)+(1-\lambda)f(z)} - \frac{\left(\beta e^{i\theta}+\alpha\right)\left(\lambda zf'(z)+(1-\lambda)f(z)\right)}{\lambda zf'(z)+(1-\lambda)f(z)}\right\}$$

$$= Re\left\{\frac{(1-\alpha)-\sum_{n=2}^{\infty}[n^2+\beta e^{i\theta}n^2-\beta e^{i\theta}(\lambda n-\lambda+1)-\alpha(\lambda n-\lambda+1)]a_nz^{n-1}}{1-\sum_{n=2}^{\infty}(\lambda n-\lambda+1)a_nz^{n-1}-\sum_{n=1}^{\infty}(\lambda n-\lambda+1)b_n(\overline{z})^{n-1}} - \frac{\sum_{n=1}^{\infty}[n^2+\beta e^{i\theta}n^2-\beta e^{i\theta}(\lambda n-\lambda+1)-\alpha(\lambda n-\lambda+1)]b_n(\overline{z})^{n-1}}{1-\sum_{n=2}^{\infty}(\lambda n-\lambda+1)a_nz^{n-1}-\sum_{n=1}^{\infty}(\lambda n-\lambda+1)b_n(\overline{z})^{n-1}}\right\} \geq 0. \qquad (2.5)$$

The above required condition (2.5) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0 < |z| = r < 1$, we must have

$$Re\left\{\frac{(1-\alpha)-\sum_{n=2}^{\infty}[n^2+\beta e^{i\theta}n^2-\beta e^{i\theta}(\lambda n-\lambda+1)-\alpha(\lambda n-\lambda+1)]a_nr^{n-1}}{1-\sum_{n=2}^{\infty}(\lambda n-\lambda+1)a_nr^{n-1}-\sum_{n=1}^{\infty}(\lambda n-\lambda+1)b_nr^{n-1}} - \frac{\sum_{n=1}^{\infty}[n^2+\beta e^{i\theta}n^2-\beta e^{i\theta}(\lambda n-\lambda+1)-\alpha(\lambda n-\lambda+1)]b_nr^{n-1}}{1-\sum_{n=2}^{\infty}(\lambda n-\lambda+1)a_nr^{n-1}-\sum_{n=1}^{\infty}(\lambda n-\lambda+1)b_nr^{n-1}}\right\} \geq 0.$$

Since $Re\left(-e^{i\theta}\right) \geq -\left|e^{i\theta}\right| = -1$, and let $r \to 1^-$. This gives (2.4) and the proof is complete.

3. Extreme points

In the following theorem, we obtain the extreme points of the class $WR_{\mathcal{H}}(\lambda,\alpha,\beta)$.

Theorem 3.1. Let f be given by (1.2). Then $f \in WR_{\mathcal{H}}(\lambda,\alpha,\beta)$ if and only if f can be expressed as

$$f(z) = \sum_{n=1}^{\infty}\left(\mu_nh_n(z)+\delta_ng_n(z)\right) \quad (z \in U), \qquad (3.1)$$

where $h_1(z) = z$,

$$h_n(z) = z - \frac{1-\alpha}{n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)}z^n,$$

$(n = 2,3,\ldots)$

and

$$g_n(z) = z - \frac{1-\alpha}{n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)}(\overline{z})^n,$$

$(n = 1,2,3,\ldots)$,

$$\sum_{n=1}^{\infty}(\mu_n+\delta_n) = 1, \quad (\mu_n \geq 0, \quad \delta_n \geq 0).$$

In particular, the extreme points of $WR_{\mathcal{H}}(\lambda,\alpha,\beta)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. Assume that f can be expressed by (3.1). Then, we have

$$f(z) = \sum_{n=1}^{\infty}[\mu_nh_n(z)+\delta_ng_n(z)]$$

$$= \sum_{n=1}^{\infty}(\mu_n+\delta_n)\,z$$

$$- \sum_{n=2}^{\infty}\frac{1-\alpha}{n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)}\mu_nz^n$$

$$- \sum_{n=1}^{\infty}\frac{1-\alpha}{n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)}\delta_n(\overline{z})^n$$

$$= z - \sum_{n=2}^{\infty}\frac{1-\alpha}{n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)}\mu_nz^n$$

$$- \sum_{n=1}^{\infty}\frac{1-\alpha}{n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)}\delta_n(\overline{z})^n.$$

Therefore,

$$\sum_{n=2}^{\infty}[n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)]$$

$$\times \frac{1-\alpha}{n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)}\mu_n$$

$$+ \sum_{n=1}^{\infty}[n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)]$$

$$\times \frac{1-\alpha}{n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)}\delta_n$$

$$= (1-\alpha)\left(\sum_{n=1}^{\infty}(\mu_n+\delta_n)-\mu_1\right)$$

$$= (1-\alpha_1)(1-\mu_1) \leq 1-\alpha\,.$$

So $f \in WR_{\mathcal{H}}(\lambda,\alpha,\beta)$.

Conversely, let $f \in WR_{\mathcal{H}}(\lambda,\alpha,\beta)$, by putting

$$\mu_n = \frac{n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)}{1-\alpha}a_n,$$

$(n = 2,3,\ldots)$

and

$$\delta_n = \frac{n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)}{1-\alpha}b_n,$$

$(n = 1,2,3,\ldots)$.

We define $\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n - \sum_{n=1}^{\infty} \delta_n$.
Then, note that $0 \le \mu_n \le 1 \quad (n = 2,3, \dots)$, $0 \le \delta_n \le 1 \quad (n = 1,2, \dots)$.
Hence,

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n - \sum_{n=1}^{\infty} b_n (\bar{z})^n$$

$$= z - \sum_{n=2}^{\infty} \frac{1-\alpha}{n^2(\beta+1) - (\beta+\alpha)(\lambda n - \lambda + 1)} \mu_n z^n$$

$$- \sum_{n=1}^{\infty} \frac{1-\alpha}{n^2(\beta+1) - (\beta+\alpha)(\lambda n - \lambda + 1)} \delta_n (\bar{z})^n$$

$$= z - \sum_{n=2}^{\infty} (z - h_n(z))\mu_n - \sum_{n=1}^{\infty} (z - g_n(z))\delta_n$$

$$= \left(1 - \sum_{n=2}^{\infty} \mu_n - \sum_{n=1}^{\infty} \delta_n\right) z$$

$$+ \sum_{n=2}^{\infty} \mu_n h_n(z) + \sum_{n=1}^{\infty} \delta_n g_n(z)$$

$$= \mu_1 h_1(z) + \sum_{n=2}^{\infty} \mu_n h_n(z) + \sum_{n=1}^{\infty} \delta_n g_n(z)$$

$$= \sum_{n=1}^{\infty} [\mu_n h_n(z) + \delta_n g_n(z)],$$

that is the required representation.

4. Convex combination

Now, we show $WR_{\mathcal{H}}(\lambda, \alpha, \beta)$ is closed under convex combination of its members.

Theorem (4.1). The class $WR_{\mathcal{H}}(\lambda, \alpha, \beta)$ is closed under convex combination.

Proof. For $j = 1,2,3, \dots$, let $f_j \in WR_{\mathcal{H}}(\lambda, \alpha, \beta)$, where f_j is given by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j}\, z^n - \sum_{n=1}^{\infty} b_{n,j}\, (\bar{z})^n.$$

Then by (2.4), we have

$$\sum_{n=2}^{\infty} [n^2(\beta+1) - (\beta+\alpha)(\lambda n - \lambda + 1)]\, a_{n,j}$$

$$+ \sum_{n=1}^{\infty} [n^2(\beta+1) - (\beta+\alpha)(\lambda n - \lambda + 1)]\, b_{n,j}$$

$$\le (1-\alpha). \qquad (4.1)$$

For $\sum_{j=1}^{\infty} t_j = 1, \quad 0 \le t_j \le 1$, the convex combination of f_j may be written as

$$\sum_{j=1}^{\infty} t_j = z - \sum_{n=2}^{\infty} \left(\sum_{j=1}^{\infty} t_j a_{n,j} \right) z^n$$

$$- \sum_{n=1}^{\infty} \left(\sum_{j=1}^{\infty} t_j b_{n,j} \right) (\bar{z})^n.$$

Then by (4.1), we have

$$\sum_{n=2}^{\infty} [n^2(\beta+1) - (\beta+\alpha)(\lambda n - \lambda + 1)] \left(\sum_{j=1}^{\infty} t_j a_{n,j} \right)$$

$$+ \sum_{n=1}^{\infty} [n^2(\beta+1) - (\beta+\alpha)(\lambda n - \lambda + 1)] \left(\sum_{j=1}^{\infty} t_j b_{n,j} \right)$$

$$= \sum_{j=1}^{\infty} t_j \left\{ \sum_{n=2}^{\infty} [n^2(\beta+1) - (\beta+\alpha)(\lambda n - \lambda + 1)] a_{n,j} \right.$$

$$\left. + \sum_{n=1}^{\infty} [n^2(\beta+1) - (\beta+\alpha)(\lambda n - \lambda + 1)] b_{n,j} \right\}$$

$$\le \sum_{j=1}^{\infty} t_j\, (1-\alpha) = 1 - \alpha.$$

Therefore,

$$\sum_{j=1}^{\infty} t_j\, f_j(z) \in WR_{\mathcal{H}}(\lambda, \alpha, \beta).$$

This completes the proof.

Corollary 4.1. The class $WR_{\mathcal{H}}(\lambda, \alpha, \beta)$ is a convex set.

5. Distortion and growth theorems

We introduce the distortion theorems for the functions in the class $WR_{\mathcal{H}}(\lambda, \alpha, \beta)$.

Theorem 5.1. Let $f \in WR_{\mathcal{H}}(\lambda, \alpha, \beta)$. Then for $|z| = r < 1$, we have

$$|f(z)|$$

$$\ge (1 - b_1)r - \frac{(1-\alpha)(1-b_1)}{4(\beta+1) - (\beta+\alpha)(\lambda+1)} r^2 \qquad (5.1)$$

and

$$|f(z)|$$

$$\le (1 + b_1)r + \frac{(1-\alpha)(1-b_1)}{4(\beta+1) - (\beta+\alpha)(\lambda+1)} r^2. \qquad (5.2)$$

Proof. Assume that $f \in WR_{\mathcal{H}}(\lambda,\alpha,\beta)$. Then, by (2.4), we get

$$|f(z)| = \left| z - \sum_{n=2}^{\infty} a_n z^n - \sum_{n=1}^{\infty} b_n (\overline{z})^n \right|$$

$$\geq (1-b_1)r - \sum_{n=2}^{\infty} (a_n+b_n)\, r^n$$

$$\geq (1-b_1)r - \sum_{n=2}^{\infty} (a_n+b_n)\, r^2$$

$$= (1-b_1)r - \frac{1}{4(\beta+1)-(\beta+\alpha)(\lambda+1)}$$

$$\times \sum_{n=2}^{\infty} [4(\beta+1)-(\beta+\alpha)(\lambda+1)](a_n+b_n)r^2$$

$$\geq (1-b_1)r - \frac{1}{4(\beta+1)-(\beta+\alpha)(\lambda+1)}$$

$$\times [(1-\alpha)-(1-\alpha)b_1]r^2$$

$$= (1-b_1)r - \frac{(1-\alpha)(1-b_1)}{4(\beta+1)-(\beta+\alpha)(\lambda+1)} r^2.$$

Relation (5.2) can be proved by using similar statements. So the proof is complete.

Theorem 5.2. Let $f \in WR_{\mathcal{H}}(\lambda,\alpha,\beta)$. Then for $|z| = r < 1$, we have

$$|f'(z)| \geq (1-b_1) - \frac{2(1-\alpha)(1-b_1)}{4(\beta+1)-(\beta+\alpha)(\lambda+1)} r \qquad (5.3)$$

and

$$|f'(z)| \leq (1-b_1) + \frac{2(1-\alpha)(1-b_1)}{4(\beta+1)-(\beta+\alpha)(\lambda+1)} r. \qquad (5.4)$$

Proof. The proof is similar to that of Theorem (5.1).

6. Integral operator

Definition (6.1)[8]. The Bernardi operator is defined by

$$L_c(k(z)) = \frac{c+1}{z^c} \int_0^z \epsilon^{c-1} k(\epsilon)\, d\epsilon,$$

$$c \in \mathbb{N} = \{1,2,\dots\}. \qquad (6.1)$$

If $k(z) = z + \sum_{n=2}^{\infty} e_n z^n$, then

$$L_c(k(z)) = z + \sum_{n=2}^{\infty} \frac{c+1}{c+n} e_n z^n. \qquad (6.2)$$

Remark 6.1. If $f = h + \overline{g}$, where

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = -\sum_{n=1}^{\infty} b_n z^n,$$

$(a_n \geq 0, b_n \geq 0)$, then

$$L_c(f(z)) = L_c(h(z)) + \overline{L_c(g(z))}. \qquad (6.3)$$

Theorem 6.1. if $f \in WR_{\mathcal{H}}(\lambda,\alpha,\beta)$, then $L_c(f)(c \in \mathbb{N})$ is also in the class $WR_{\mathcal{H}}(\lambda,\alpha,\beta)$.

Proof. By (6.2) and (6.3), we get

$$L_c(f(z)) = L_c\left(z - \sum_{n=2}^{\infty} a_n z^n - \sum_{n=1}^{\infty} b_n (\overline{z})^n \right)$$

$$= z - \sum_{n=2}^{\infty} \frac{c+1}{c+n} a_n z^n - \sum_{n=1}^{\infty} \frac{c+1}{c+n} b_n (\overline{z})^n.$$

Since $f \in WR_{\mathcal{H}}(\lambda,\alpha,\beta)$, then by (2.4), we have

$$\sum_{n=2}^{\infty} \frac{[n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)]}{1-\alpha} a_n$$

$$+ \sum_{n=1}^{\infty} \frac{[n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)]}{1-\alpha} b_n \leq 1.$$

Since $c \in \mathbb{N}$, then $\frac{c+1}{c+n} \leq 1$, therefore

$$\sum_{n=2}^{\infty} \frac{[n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)]}{1-\alpha} \left(\frac{c+1}{c+n}\right) a_n$$

$$+ \sum_{n=1}^{\infty} \frac{[n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)]}{1-\alpha} \left(\frac{c+1}{c+n}\right) b_n$$

$$\leq \sum_{n=2}^{\infty} \frac{[n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)]}{1-\alpha} a_n$$

$$+ \sum_{n=1}^{\infty} \frac{[n^2(\beta+1)-(\beta+\alpha)(\lambda n-\lambda+1)]}{1-\alpha} b_n \leq 1,$$

and this gives the result.

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صنف جزئي جديد من الدوال أحادية التكافؤ التوافقية

وقاص غالب عطشان
قسم الرياضيات
كلية علوم الحاسبات وتكنولوجيا المعلومات
جامعة القادسية
E-mail: waggas.galib@qu.edu.iq

نجاح علي جبن الزيادي
قسم الرياضيات
كلية العلوم
جامعة بغداد
E-mail: najah.ali@qu.edu.iq

المستخلص :

في هذا البحث، عرفنا صنف جديد من الدوال أحادية التكافؤ التوافقية من الشكل $f = h + \overline{g}$ في قرص الوحدة المفتوح U. حصلنا على الخواص الأساسية مثل، حدود المعامل، نقاط متطرفة، التركيب المحدب، مبرهنات النمو والتشوية ومؤثر تكاملي.