Equivalent of permutation polytopes with applications

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Abstract

An open conjecture that relates to the equivalents between the permutation polytopes associated to their groups is proved and also the proof for two permutation groups which are effectively equivalent implies the permutation polytopes associated to these groups is combinatorial equivalent is given.

Keywords: polytope, lattice points, permutation groups.

Mathematics subject classification : 52B20 / 11P21 / 20Bxx

Introduction

The most famous permutation polytope is the Birkhoff polytope. It is appears naturally in various contexts like: enumerative combinatorices, optimization and statistics, [1,2] for more see, [3, 4and 5].

Lattice polytope is the convex hull of a finite sub set of the integer lattice $\mathbb{Z}^d$. Equivalently it is a polytope with all vertices are in $\mathbb{Z}^d$, [6]. Permutation polytope can be defined as a lattice polytope.

This work consists of the definitions of combinatorial equivalent and effectively equivalent together with examples and figures. The relationship between permutations polytopes together with combinatorial equivalent and effectively equivalent is also discusses.

Some remarks and theorems that relate to permutation group and permutation polytope is also given and two conjectures with its proof about permutation polytope are discuses in details.

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1. Combinatorial equivalent of permutation polytopes:

Here are some definitions and examples are given, that is benefit for defining the combinatorial equivalent and its result.

Definition 1.1 [7,p.13]:

Let G be a group, A representation of G is a homomorphism $\rho: G \to GL(V)$ for some vector space V. The dimension of V represents the degree of ρ.

Two representation $\rho: G \to GL(V)$ and $\psi: G \to GL(W)$, called equivalent if there exist an isomorphism $T: V \to W$ such that $\psi_g = T\rho_g T^{-1}$ for all $g \in G$ and denoted by $p \sim \psi$.

Definition 1.2 [8]:

A face poset define as a poset of cells of p order by inclusion where face poset denote by $\mathcal{X}(p)$.

Remark 1.1 [9]:

The set of all faces of a polytope partially ordered by inclusion is called face lattice, and Two polytopes are said to be equivalent if there face lattices are isomorphic.

Example 1.1 [10,p.12]:

Figure (1), represents the face poset of a polytope.

Definition 1.3 [1]:

For two polytopes P and Q, an equivalence of the face lattice as a posets of polytopes lattices is said to be combinatorial equivalent and denoted by $P \approx Q$.

Remark 1.2 [11,p.6]:

1. Two polytopes are combinatorially equivalent in dimensions two if and only if they have the same number of vertices.

2. If two polytopes combinatorially equivalent then they have the same number of vertices and same dimension.

Example 1.2 [12,p3-4]:

The classification of 3-dimensional 0/1-polytope $P \subseteq \mathbb{R}^3$ according to 0/1-equivalent represents by the figure below. P' is 0/1-equivalent to sub polytope of P denoted by an arrow $P \to P'$, that is P=P(V) and $P' \sim P(V')$ for subse t$V' \subseteq V$. There are 12 0/1-equivalent classes.

Definition 1.4 [12,p.7]:

Two polytopes are said to be congruent if they have the same edge lengths, volume, etc.

Definition 1.5 [11,p.196]:

Let $P \subseteq \mathbb{R}^p$ and $Q \subseteq \mathbb{R}^d$ two polytopes and let $\pi: \mathbb{R}^p \to \mathbb{R}^d$ be an affine map $\pi(x) = Ax - z$ with $A \in \mathbb{R}^{p \times d}$ and $z \in \mathbb{R}^d$ if π is injective then Q is affinely equivalent to P and denoted by $P \approx_{aff} Q$.

Example 1. 3 [11,p.155]:

For two octahedron, $P_1 = conv\{e_1, -e_1, e_2, -e_2, e_3, -e_3\}$ and P_2 which is obtained by perturbing the vertex e_1 to

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$e_1 + (1/6) \, e_2$. P_1 is a regular octahedron and

P_2 is a non-regular octahedron in R^3, that is P_1

and P_2 are combinatorial equivalent but not

affinely equivalent .

So ,P_1=conv(y_1) and P_2=conv(y_2), where

$$y_1 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \text{ and } y_2 =$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1/6 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Proposition 1.1 [12,p.8]

On the finite set of all 0/1-polytopes one

has the following hierarchy of equivalence

relation 0/1−equivalent $\Rightarrow$ congrent $\Rightarrow$

affinely equivalent $\Rightarrow$

combinatorially equivalent

For all three implications the converse is false.

**2. Effectively equivalent of permutation
polytopes:**

In this section definitions and examples

of effectively equivalent relates to a

permutation polytope are discussed.

Definition(2.1),[1]:

Let G be a permutation group of

subgroup S_n the representation $\rho: G \to S_n$ is

called a permutation representation.

Definition 2.2 [1]:

Two real representations ρ_1 and ρ_2 of G

said to be stable equivalent. If contains the

same non-trivial irreducible factors.

Definition 2.3 [1]:

The permutation representation of group

$G \, reg: G \longrightarrow S_{|G|}$ is said to be regular

representation of group G via right

multiplication.

Definition 2.4 [1]:

For finite groups G_1 and G_2 , there are

two real representation $\rho_i: G_i \to GL(V_i)$ (for

$i = 1,2$) if there exists an isomorphism

$\varphi: G_1 \to G_2$ such that ρ_1 and $\rho_2 o\varphi$ are stably

equivalent G_1-representation, then the two real

representations is said to be effectively

equivalent and denoted by $\rho_1 \approx_{eff} \rho_2$.

3. Theorems about permutation polytopes:

According to the preceding sections we

are reaching to prove the open problems which

are given in this section together with their

proofs.

Remark(3.1),[Baumeister4]:

For a symmetric group S$_n$. An injective

homomorphism $\pi: G \to S_n$ is called

permutation representation. The pair (G, π) is

called permutation group.

Theorem 3.1 [13]:

Two effectively equivalent permutation

representations relates to two permutation

polytopes are affinely equivalent.

The open conjecture for [1] which is given by

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Theorem 3.2:

The permutation group G is effectively equivalent to S_n. if $P(G)$ is combinatorial equivalent to the Birkhoff polytope B_n for some n.where $P(G)$ is a permutation polytope

Proof:

Let P(G) be a permutation polytope for a permutation group G. The n^{th}Birkhoff polytope which is defined as the convex hull of all permutations(Remark(3.1))is given by

$B_n=P(S_n)$.

Since the permutation polytope and the Birkhoff polytope has isomorphic face lattice because they are combinatorial equivalents, and the faces lattice of a permutation polytope of dimension d are the vertices which appear as a permutation matrices and the faces lattice of Birkhoff polytope of dimension d are also vertices represented by a permutation matrices. Therefore the group that represents the permutation polytope and the symmetric group S_n are effectively equivalent.

The open conjecture for [1] which is given by

Corollary 3.1

For two permutation groups G_1 and G_2, if G_1 and G_2 are effectively equivalents then $P(G_1)$ and $P(G_2)$ are combinatorial equivalents.

Proof:

Since G_1 and G_2 are effectively equivalents $P(G_1)$ and $P(G_2)$ are affinaly equivalent (theorem(3.2))

$P(G_1)$ and $P(G_2)$ are combinatorial equivalent(proposition(1.1)).

4. Conclousion

In this paper, an important theorem for the relations between permutation polytope and permutation group for any dimension is introduced. Also a corollary between two permutation polytopes is proved.

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التكافؤ بين متعدد الأضلاع والزوايا التبادلي مع تطبيقاته

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المستخلص :
تم تقديم برهان للعلاقة بين متعدد الأضلاع والزوايا التبادلي مع الزمرة المرتبطة به بالإضافة الى ذلك تم برهنه
العلاقة بين زمرتين متعلقتين بمتعددي الأضلاع والزوايا التبادلي وتم توضيح المفاهيم الأساسية لعلاقات التكافؤ.