Journal of Al-Qadisiyah for Computer Science and Mathematics Vol. 16(1) 2024, pp Math 1-15

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)

Applications of Gegenbauer Polynomials for Two Families of Bi-univalent Functions Associating λ-Pseudo-Starlike and Convex Functions with Sakaguchi Type Functions

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ARTICLE INFO

Article history:
Received: 26 /12/2023
Rrevised form: 18 /2/2024
Accepted : 3 /3/2024
Available online: 30 /3/2024

Keywords:

Bi-univalent function,
Holomorphic function, function λ-pseudo,
Estimation of coefficients,
polynomials Gegenbauer,
Fekete-Szegö problem .

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ABSTRACT

In the present article, we introduce two families $B_{\Sigma}(\delta,\alpha,\lambda,m,n,t)$ and $F_{\Sigma}(\delta,\lambda,\gamma,m,n,t)$ of holomorphic and bi-univalent functions associating λ-pseudo-starlike and convex functions with Sakaguchi type functions defined by Gegenbauer polynomials. We derive the initial Maclaurin coefficients estimates and determinate the Fekete-Szegö problem of functions in these families.

https://doi.org/10.29304/jqcsm.2024.16.11443

1. Introduction

Legendre [18] studied orthogonal polynomials Comprehensively. The importance of orthogonal polynomials for contemporary mathematics as well as for a wide range of their applications in physics and engineering, is beyond any doubt. It is well-known that these polynomials play an essential role in problems of the approximation theory. They occur in the theory of differential and integral equations as well as in mathematical statistics. Their applications in

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Communicated by ‘sub etitor’

quantum mechanics, scattering theory, automatic control, signal analysis, and axially symmetric potential theory are also known [6,11]. In practical, the Gegenbauer polynomials is special case of orthogonal polynomials. They are representatively related with typically real functions T_R as discovered in [17]. Typically, real functions play an important role in the geometric function theory because of the relation $T_R = \overline{co}S_R$ and its role of estimating coefficient bounds, where S_R indicates the family of univalent functions in the unit disk with real coefficients and $\overline{co}S_R$ denotes the closed convex hull of S_R.

Denote by $\mathcal{A}$ the family of all functions f which are holomorphic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and have the type:

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n , \tag{1.1}$$

Symbolized by S the subfamily of $\mathcal{A}$ consisting the functions that are univalent in U. According to the "Koebe One-Quarter Theorem" [12] each function f from S has an inverse f^{-1}, which fulfills

$$f^{-1}\big(f(z)\big)=z,\ (z\in U)$$

and

$$f\big(f^{-1}(w)\big)=w,\ \left(|w|<r_0(f),\ r_0(f)\geq\frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is named bi-univalent in U if together f and f^{-1} are univalent in U. Let Σ indicate the family of bi-univalent functions in U satisfying (1.1). Beginning with Srivastava et al. pioneering work [32] on the subject, the large number of works associated with the subject have been presented (see, for example [1,2,8,9,10,13,16,19,22,26,27,28,30,33]). We see that the set Σ is not empty. We see that the functions

$$\frac{z}{1-z}\ ,\ \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\ \text{ and }\ -\log(1-z)$$

Are in the set Σ with the corresponding inverse functions

$$\frac{w}{1+w}\ ,\ \frac{e^{2w}-1}{e^{2w}+1}\ \text{ and }\ \frac{e^{w}-1}{e^{w}},$$

respectively. But the functions

$$z-\frac{z^2}{2}\ \text{ and }\ \frac{z}{1-z^2}.$$

are not a member of the set Σ.

We say that a function $f \in S$ is starlike with respect to symmetrical points if (see [24])

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\}>0,\quad z\in U.$$

The set of all such functions is denoted by S_S^* .

The family of starlike functions with respect to symmetric points obviously includes the family of convex functions with respect to symmetric points C_s, satisfying the following condition:

$$\mathrm{Re}\left\{\frac{(zf'(z))'}{(f(z)-f(-z))'}\right\}>0,\quad z\in U.$$

Recently, Frasin [15] introduced and studied the family $S(\gamma,m,n)$ consisting of functions $f \in \mathcal{A}$ which satisfy the condition

$$\mathrm{Re}\left\{\frac{(m-n)zf'(z)}{f(mz)-f(nz)}\right\}>\gamma ,$$

for some $0 \leq \gamma < 1, m, n \in \mathbb{C}$ with $m \neq n$, $|n| \leq 1$ and for all $z \in U$. We note that the family S(γ,1,n) was given by Owa et al. [25] while the family $S(\gamma,1,-1) \equiv S_s(\gamma)$ was considered by Sakaguchi [24] and is called Sakaguchi function of order γ. Also, $S(0,1,-1) \equiv S_s$ is the family of starlike functions with respect to symmetrical points in U and $S(\gamma,1,0) \equiv S^*(\gamma)$ which is the family of starlike functions of order γ $(0 \leq \gamma < 1)$.In [7] Babalola defined the family $L_\lambda(\gamma)$ of λ-pseudostarlike functions of order γ which are the function $f \in \mathcal{A}$ such that

$$\mathrm{Re}\left\{\frac{z(f'(z))^\lambda}{f(z)}\right\} > \gamma,$$

where $0 \leq \gamma < 1$, $\lambda \geq 1$, and $z \in U$. In particular, Babalola [7] showed that all λ- pseudo-starlike functions are Bazilevic of type $1 - \frac{1}{\lambda}$ and order $\gamma^{\frac{1}{\lambda}}$ and are univalent in U. It is observed that for $\lambda = 1$, we have the family of starlike functions.

For the functions f and g be holomorphic in U. We say that the function f is said to be subordinate to g, if there exists a Schwarz function w holomorphic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. This subordination is indicated by $f \prec g$ or $f(z) \prec g(z) (z \in U)$. It is well known that (see [21]), if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Recently, Amourah et al. [5] studied the generating function of Gegenbauer polynomials $H_\delta(z,t)$ that are given by the following recurrence relation:

$$H_\delta(z,t) = \frac{1}{(1-2tz+z^2)^\delta},$$

where $\delta \in \mathbb{R} \setminus \{0\}$, $t \in [-1,1]$ and $z \in U$. For fixed t, the function H_δ is holomorphic in U, so it may be expanded in a Taylor-Maclaurin series as note that if $t = \cos\beta$, where $\beta \in (-\frac{\pi}{3}, \frac{\pi}{3})$, then

$$H_\delta(z,t) = \frac{1}{(1-2tz+z^2)^\delta} = \sum_{n=0}^{\infty} G_n^\delta(t) z^n,$$

where $\mathcal{G}_n^\delta(t)$ is Gegenbauer polynomial of degree n.

Clearly, H_δ generates nothing when $\delta = 0$. Thus, the generating function of the Gegenbauer polynomial is

$$H_0(z,t) = 1 - \log(1-2tz+z^2) = \sum_{n=0}^{\infty} G_n^0(t) z^n.$$

Furthermore, it is worth to mention that a normalization of δ to be greater than $-\frac{1}{2}$ is desirable [11,23]. Also, Gegenbauer polynomials can be introduced by the following recurrence relations:

$$G_n^\delta(t) = \frac{1}{2}\left[2t(n+\delta-1)G_{n-1}^\delta(t) - (n+2\delta-2)G_{n-1}^\delta(t)\right],$$

with the initial values

$$G_0^\delta(t) = 1, \quad G_1^\delta(t) = 2\delta t \quad \text{and} \quad G_2^\delta(t) = 2\delta(\delta+1)t^2 - \delta. \tag{1.3}$$

Remark 1.1. Choosing the special values of δ, the Gegenbauer polynomial $\mathcal{G}_n^\delta(t)$ reduces to the following well-known polynomials:

1) Taking $\delta = 1$, we have the Chebyshev Polynomials.
2) Taking $\delta = \frac{1}{2}$, we obtain the Legendre Polynomials.

In geometric function theory, the so-called Fekete-Szegö type inequalities (or problems) which estimate some upper bounds for $|a_3-\mu a_2^2|$ for holomorphic univalent functions. Its origin was in the disproof by Fekete and Szegö [14] conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity.
The results related to initial coefficient estimates and Fekete- Szegö functional problem for special subcalsses of Σ associated with special polynomials appeared like the ones in [4,20,29,31,36].

2. Main Results

This section start with defining the families $B_\Sigma(\delta,\alpha,\lambda,m,n,t)$ and $F_\Sigma(\delta,\lambda,\gamma,m,n,t)$ as follows:

Definition 2.1. A function $f\in\Sigma$ is called in the family $B_\Sigma(\delta,\alpha,\lambda,m,n,t)$ if it fulfills the subordinations:

$$(1-\alpha)\frac{(m-n)z\left(f'(z)\right)^\lambda}{f(mz)-f(nz)}+\alpha\frac{(m-n)\left(\left(zf'(z)\right)'\right)^\lambda}{\left(f(mz)-f(nz)\right)'}\prec\frac{1}{(1-2tz+z^2)^\delta}$$

and

$$(1-\alpha)\frac{(m-n)w\left(g'(w)\right)^\lambda}{g(mw)-g(nw)}+\alpha\frac{(m-n)\left(\left(wg'(w)\right)'\right)^\lambda}{\left(g(mw)-g(nw)\right)'}\prec\frac{1}{(1-2tw+w^2)^\delta},$$

where $0\le\alpha\le1,\lambda\ge1,t\in\left(\frac{1}{2},1\right]$, $m,n\in C$, $m\neq n$, $|n|\le1$, δ is a nonzero real constant and $g=f^{-1}$ is given by (1.2).

Definition 2.2. A function $f\in\Sigma$ is called in the family $F_\Sigma(\delta,\lambda,\gamma,m,n,t)$ if it fulfills the subordinations:

$$\left(\frac{(m-n)z\left(f'(z)\right)^\lambda}{f(mz)-f(nz)}\right)^\gamma\left(\frac{(m-n)\left(\left(zf'(z)\right)'\right)^\lambda}{\left(f(mz)-f(nz)\right)'}\right)^{1-\gamma}\prec\frac{1}{(1-2tz+z^2)^\delta}$$

and

$$\left(\frac{(m-n)w(g'(w))^\lambda}{g(mw)-g(nw)}\right)^\gamma\left(\frac{(m-n)((wg'(w))')^\lambda}{(g(mw)-g(nw))'}\right)^{1-\gamma}\prec\frac{1}{(1-2tw+w^2)^\delta},$$

where $\lambda\ge1,0\le\gamma\le1,t\in\left(\frac{1}{2},1\right]$, $m,n\in C$, $m\neq n$, $|n|\le1$, δ is a nonzero real constant and $g=f^{-1}$ is given by (1.2).

Remark 2.1. If we choose

1) $\delta=1$ in Definition 2.1, the family $B_\Sigma(\delta,\alpha,\lambda,m,n,t)$ reduces to the family $R_\Sigma(\alpha,\lambda,m,n,t)$, which Wanas studied in [37].
2) $m=1$ and $n=-1$ in Definition 2.2, the family $F_\Sigma(\delta,\lambda,\gamma,m,n,t)$ becomes family $S_\Sigma(\delta,\lambda,\gamma,t)$, which Al-Shbeil et al. defined in [3].
3) $\delta=m=1$ and $n=-1$ in Definition 2.2, the family $F_\Sigma(\delta,\lambda,\gamma,m,n,t)$ reduces to the family $T_\Sigma^s(\lambda,\gamma,t)$, which Wanas gave in [34] .
4) $\delta=\lambda=m=1$ and $n=-1$ in Definition 2.2, the family $F_\Sigma(\delta,\lambda,\gamma,m,n,t)$ becomes family $D_\Sigma^s(\gamma,t)$, which Wanas and Swamy introduced in [35].

If we choose $\alpha=1$ in the Definition 2.1, the family $B_\Sigma(\delta,\alpha,\lambda,m,n,t)$ reduces to the family $M_\Sigma(\delta,\lambda,m,n,t)$ of bi-convex functions which fulfills the conditions:

$$\frac{(m-n)((zf'(z))')^{\lambda}}{(f(mz)-f(nz))'} \prec \frac{1}{(1-2tz+z^2)^{\delta}}$$

and

$$\frac{(m-n)((wg'(w))')^{\lambda}}{(g(mw)-g(nw))'} \prec \frac{1}{(1-2tw+w^2)^{\delta}},$$

where the function $g = f^{-1}$ is given by (1.2).

If we choose $\gamma = 1$ in Definition 2.2, the family $F_{\Sigma}(\delta,\lambda,\gamma,m,n,t)$ reduces to the family $H_{\Sigma}(\delta,\lambda,m,n,t)$ of bi-starlike functions which fulfills the conditions:

$$\frac{(m-n)z(f'(z))^{\lambda}}{f(mz)-f(nz)} \prec \frac{1}{(1-2tz+z^2)^{\delta}}$$

and

$$\frac{(m-n)w(g'(w))^{\lambda}}{g(mw)-g(nw)} \prec \frac{1}{(1-2tw+w^2)^{\delta}},$$

where the function $g = f^{-1}$ is given by (1.2).

Theorem 2.1. For $0\leq\alpha\leq1, \lambda\geq1, t\in\left(\frac{1}{2},1\right]$, $m,n\in C$, $m\neq n$, $|n|\leq1$ and δ is nonzero real constant, let $f \in \mathcal{A}$ be in the family $B_{\Sigma}(\delta,\alpha,\lambda,m,n,t)$. Then

$$|a_2| \leq \frac{2|\delta|t\sqrt{2|\delta|t}}{\sqrt{\left|2\delta^2t^2[2(\Omega(\alpha,\lambda,m,n)-mn)-(\alpha+1)^2(2\lambda-m-n)^2]-\delta(\alpha+1)^2(2\lambda-m-n)^2(2t^2-1)\right|}}$$

and

$$|a_3| \leq \frac{4\delta^2t^2}{(\alpha+1)^2(2\lambda-m-n)^2} + \frac{2|\delta|t}{(2\alpha+1)(3\lambda-m^2-n^2-mn)},$$

where

$$\Omega(\alpha,\lambda,m,n) = \alpha[(m^2+n^2+4mn)-6\lambda(m+n-\lambda)] + \lambda\big(1-2(m+n-\lambda)\big). \quad (2.1)$$

Proof. Let $f \in B_{\Sigma}(\delta,\alpha,\lambda,m,n,t)$. Then there exists two holomorphic functions $u,v:U\longrightarrow U$ given by

$$u(z) = u_1z + u_2z^2 + u_3z^3 + \cdots \qquad (z \in U) \quad (2.2)$$

and

$$v(w) = v_1w + v_2w^2 + v_3w^3 + \cdots \qquad (w \in U), \quad (2.3)$$

with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$, $z,w \in U$ such that

$$(1-\alpha)\frac{(m-n)z\left(f'(z)\right)^{\lambda}}{f(mz)-f(nz)}+\alpha\frac{(m-n)\left(\left(zf'(z)\right)'\right)^{\lambda}}{\left(f(mz)-f(nz)\right)'}=1+\mathcal{G}_1^{\delta}(t)u(z)+\mathcal{G}_2^{\delta}(t)u^2(z)+\cdots \quad (2.4)$$

and

$$(1-\alpha)\frac{(m-n)w\left(g'(w)\right)^{\lambda}}{g(mw)-g(nw)}+\alpha\frac{(m-n)\left(\left(wg'(w)\right)'\right)^{\lambda}}{\left(g(mw)-g(nw)\right)'}$$

$$=1+\mathcal{G}_1^{\delta}(t)v(w)+\mathcal{G}_2^{\delta}(t)v^2(w)+\cdots \quad (2.5)$$

Combining (2.2), (2.3), (2.4) and (2.5), we obtain

$$(1-\alpha)\frac{(m-n)z\left(f'(z)\right)^{\lambda}}{f(mz)-f(nz)}+\alpha\frac{(m-n)\left(\left(zf'(z)\right)'\right)^{\lambda}}{\left(f(mz)-f(nz)\right)'}$$

$$=1+\mathcal{G}_1^{\delta}(t)u_1z+\left[\mathcal{G}_1^{\delta}(t)u_2+\mathcal{G}_2^{\delta}(t)u_1^2\right]z^2+\cdots \quad (2.6)$$

and

$$(1-\alpha)\frac{(m-n)w\left(g'(w)\right)^{\lambda}}{g(mw)-g(nw)}+\alpha\frac{(m-n)\left(\left(wg'(w)\right)'\right)^{\lambda}}{\left(g(mw)-g(nw)\right)'}$$

$$=1+\mathcal{G}_1^{\delta}(t)v_1w+\left[\mathcal{G}_1^{\delta}(t)v_2+\mathcal{G}_2^{\delta}(t)v_1^2\right]w^2+\cdots\ . \quad (2.7)$$

It is quite well-known that if $|u(z)|<1$ and $|v(w)|<1$, z,w∈U, then

$$|u_i|\le 1 \quad \text{and} \quad |v_i|\le 1 \quad \text{for all } i\in\mathbb{N}. \quad (2.8)$$

Equating the coefficients in (2.6) and (2.7), we deduce that

$$(\alpha+1)(2\lambda-m-n)a_2=G_1^{\delta}(t)u_1, \quad (2.9)$$

$$(2\alpha+1)(3\lambda-m^2-n^2-mn)a_3+(3\alpha+1)[(m+n)^2-2\lambda(m+n-\alpha+1)]a_2=G_1^{\delta}(t)u_2+G_2^{\delta}(t)u_1^2, \quad (2.10)$$

$$-(\alpha+1)(2\lambda-m-n)a_2=\mathcal{G}_1^{\delta}(t)v_1 \quad (2.11)$$

and

$$[(6\lambda-m^2-n^2)-2\lambda(m+n-\lambda+1)-\alpha(6\lambda(m+n-\lambda-1)+(m-n)^2)]\,a_2^2 - (2\alpha+1)(3\lambda-m^2-n^2-mn)a_3=\mathcal{G}_1^{\delta}(t)v_2+\mathcal{G}_2^{\delta}(t)v_1^2. \quad (2.12)$$

From (2.9) and (2.11), we conclude that

$$u_1=-v_1 \quad (2.13)$$

and

$$2(\alpha+1)^2(2\lambda-m-n)^2a_2^2=\left(\mathcal{G}_1^\delta(t)\right)^2(u_1^2+v_1^2). \quad (2.14)$$

Adding (2.10) to (2.12), yields

$$2\{\alpha[(m^2+n^2+4mn)-6\lambda(m+n-\lambda)]+\lambda\big(1-2(m+n-\lambda)\big)-mn\}a_2^2=\mathcal{G}_1^\delta(t)(u_2+v_2)+\mathcal{G}_2^\delta(t)(u_1^2+v_1^2).$$

where($\Omega(\alpha,\lambda,m,n)$ is given by (2.1). Consequently, we have

$$2\left[\Omega(\alpha,\lambda,m,n)-mn-\frac{\mathcal{G}_2^\delta(t)}{\left(\mathcal{G}_1^\delta(t)\right)^2}(\alpha+1)^2(2\lambda-m-n)^2\right]a_2^2=\mathcal{G}_1^\delta(t)(u_2+v_2). \quad (2.15)$$

Further computations using (1.3), (2.8) and (2.15), we obtain

$$|a_2|\le\frac{2|\delta|t\sqrt{2|\delta|t}}{\sqrt{\left|2\delta^2t^2[2(\Omega(\alpha,\lambda,m,n)-mn)-(\alpha+1)^2(2\lambda-m-n)^2]-\delta(\alpha+1)^2(2\lambda-m-n)^2(2t^2-1)\right|}}.$$

Next, if we subtract (2.12) from (2.10), we deduce that

$$2(2\alpha+1)(3\lambda-m^2-n^2-mn)(a_3-a_2^2)=\mathcal{G}_1^\delta(t)(u_2-v_2)+\mathcal{G}_2^\delta(t)(u_1^2-v_1^2). \quad (2.16)$$

In view of (2.13) and (2.14), we get from (2.16)

$$a_3=\frac{\left(\mathcal{G}_1^\delta(t)\right)^2(u_1^2+v_1^2)}{2(\alpha+1)^2(2\lambda-m-n)^2}+\frac{\mathcal{G}_1^\delta(t)(u_2-v_2)}{2(2\alpha+1)(3\lambda-m^2-n^2-mn)}$$

Thus applying (1.3), we obtain

$$|a_3|\le\frac{4\delta^2t^2}{(\alpha+1)^2(2\lambda-m-n)^2}+\frac{2|\delta|t}{(2\alpha+1)(3\lambda-m^2-n^2-mn)}.$$

Putting $\alpha=1$ Theorem 2.1, we obtain the next result:

Corollary 2.1. For $\lambda\geq1, t\in\left(\frac{1}{2},1\right]$, $m,n\in C$, $m\neq n$, $|n|\leq1$ and δ is nonzero real constant, let $f\in\mathcal{A}$ be in the family $M_\Sigma(\delta,\lambda,m,n,t)$ Then

$$|a_2|\le\frac{|\delta|t\sqrt{2|\delta|t}}{\sqrt{\left|\delta^2t^2[(\Psi(\lambda,m,n)-mn)-2(2\lambda-m-n)^2]-\delta(2\lambda-m-n)^2(2t^2-1)\right|}}$$

and

$$|a_3|\le\frac{\delta^2t^2}{(2\lambda-m-n)^2}+\frac{2|\delta|t}{3(3\lambda-m^2-n^2-mn)},$$

where

$$\Psi(\lambda,m,n)=(m^2+n^2+4mn)-6\lambda(m+n-\lambda)+\lambda\big(1-2(m+n-\lambda)\big). \quad (2.17)$$

Theorem 2.2. For $\lambda\geq 1$, $0\leq\gamma\leq 1$, $t\in\left(\frac{1}{2},1\right]$, $m,n\in C$, $m\neq n$, $|n|\leq 1$ and δ is nonzero real constant, let $f\in\mathcal{A}$ be in the family $F_\Sigma(\delta,\lambda,\gamma,m,n,t)$. Then

$$|a_2| \leq \frac{2|\delta|t\sqrt{2|\delta|t}}{\sqrt{\left|\begin{array}{c}\delta^2t^2\left[4\left((3-2\gamma)(3\lambda-m^2-n^2-mn)+\frac{1}{2}\gamma(\gamma-1)(2\lambda-m-n)^2+(4-3\gamma)\left((m+n)^2-2\lambda(m+n-\lambda+1)\right)\right)\right. \\ \left.-2(2-\gamma)^2(2\lambda-m-n)^2\right]-\delta(2-\gamma)^2(2\lambda-m-n)^2(2t^2-1)\end{array}\right|}}$$

and

$$|a_3| \leq \frac{4\delta^2t^2}{(2-\gamma)^2(2\lambda-m-n)^2}+\frac{2|\delta|t}{(3-2\gamma)(3\lambda-m^2-n^2-mn)}.$$

Proof. Let $f\in F_\Sigma(\delta,\lambda,\gamma,m,n,t)$. Then there exists two holomorphic functions $u,v: U\longrightarrow U$

$$\left(\frac{(m-n)z(f'(z))^\lambda}{f(mz)-f(nz)}\right)^\gamma\left(\frac{(m-n)((zf'(z))')^\lambda}{(f(mz)-f(nz))'}\right)^{1-\gamma} = 1+\mathcal{G}_1^\delta(t)u(z)+\mathcal{G}_2^\delta(t)u^2(z)+\cdots \quad (2.18)$$

and

$$\left(\frac{(m-n)w(g'(z))^\lambda}{g(mz)-g(nz)}\right)^\gamma\left(\frac{(m-n)((wg'(z))')^\lambda}{(g(mz)-g(nz))'}\right)^{1-\gamma} = 1+\mathcal{G}_1^\delta(t)v(w)+\mathcal{G}_2^\delta(t)v^2(w)+\cdots. \quad (2.19)$$

where u and v have the forms (2.2) and (2.3). Combining (2.18) and (2.19), yield

$$\left(\frac{(m-n)z(f'(z))^\lambda}{f(mz)-f(nz)}\right)^\gamma\left(\frac{(m-n)((zf'(z))')^\lambda}{(f(mz)-f(nz))'}\right)^{1-\gamma}$$

$$=1+\mathcal{G}_1^\delta(t)u_1z+\left[\mathcal{G}_1^\delta(t)u_2+\mathcal{G}_2^\delta(t)u_1^2\right]z^2+\cdots \quad (2.20)$$

and

$$\left(\frac{(m-n)w(g'(z))^\lambda}{g(mz)-g(nz)}\right)^\gamma\left(\frac{(m-n)((wg'(z))')^\lambda}{(g(mz)-g(nz))'}\right)^{1-\gamma}$$

$$=1+\mathcal{G}_1^\delta(t)v_1w+\left[\mathcal{G}_1^\delta(t)v_2+\mathcal{G}_2^\delta(t)v_1^2\right]w^2+\cdots. \quad (2.21)$$

Equating the coefficients in (2.20) and (2.21), we deduce that

$$(2-\gamma)(2\lambda-m-n)a_2=G_1^\delta(t)u_1, \quad (2.22)$$

$$(3\text{-}2\gamma)(3\lambda\text{-}m^2\text{-}n^2\text{-}mn)a_3+\left[\frac{1}{2}\gamma(\gamma\text{-}1)(2\lambda\text{-}m\text{-}n)^2+(4\text{-}3\gamma)\left((m+n)^2\text{-}2\lambda(m+n\text{-}\lambda+1)\right)\right]a_2^2=G_1^\delta(t)u_2+G_2^\delta(t)u_1^2\,, \quad (2.23)$$

$$-(2-\gamma)(2\lambda-m-n)a_2=\mathcal{G}_1^\delta(t)v_1 \quad (2.24)$$

and

$$(3-2\gamma)(3\lambda-m^2-n^2-mn)(2a_2^2-a_3)+\left[\frac{1}{2}\gamma(\gamma-1)(2\lambda-m-n)^2+(4-3\gamma)\left((m+n)^2-2\lambda(m+n-\lambda+1)\right)\right]a_2^2=\mathcal{G}_1^\delta(t)v_2+\mathcal{G}_2^\delta(t)v_1^2. \quad (2.25)$$

In view of (2.22) and (2.24), we have

$$u_1=-v_1 \quad (2.26)$$

and

$$2(2-\gamma)^2(2\lambda-m-n)^2a_2^2=\left(\mathcal{G}_1^\delta(t)\right)^2(u_1^2+v_1^2). \quad (2.27)$$

If we add (2.23) to (2.25), we conclude that

$$2\left[(3-2\gamma)(3\lambda-m^2-n^2-mn)+\frac{1}{2}\gamma(\gamma-1)(2\lambda-m-n)^2+(4-3\gamma)\left((m+n)^2-2\lambda(m+n-\lambda+1)\right)\right]a_2^2=\mathcal{G}_1^\delta(t)(u_2+v_2)+\mathcal{G}_2^\delta(t)(u_1^2+v_1^2), \quad (2.28)$$

By substitute the value of $u_1^2+v_1^2$ from (2.27) in (2.28), yields

$$\left[2\left((3\text{-}2\gamma)(3\lambda\text{-}m^2\text{-}n^2\text{-}mn)+\frac{1}{2}\gamma(\gamma\text{-}1)(2\lambda\text{-}m\text{-}n)^2+(4\text{-}3\gamma)\left((m+n)^2\text{-}2\lambda(m+n\text{-}\lambda+1)\right)\right)\text{-}\frac{2(2\text{-}\gamma)^2(2\lambda\text{-}m\text{-}n)^2G_2^\delta(t)}{\left(G_1^\delta(t)\right)^2}\right]a_2^2=G_1^\delta(t)(u_2+v_2),$$

or equivalently

$$a_2^2=\frac{\left(G_1^\delta(t)\right)^3(u_2+v_2)}{2\left[\left((3\text{-}2\gamma)(3\lambda\text{-}m^2\text{-}n^2\text{-}mn)+\frac{1}{2}\gamma(\gamma\text{-}1)(2\lambda\text{-}m\text{-}n)^2+(4\text{-}3\gamma)\left((m+n)^2\text{-}2\lambda(m+n\text{-}\lambda+1)\right)\right)\left(G_1^\delta(t)\right)^2\text{-}(2\text{-}\gamma)^2(2\lambda\text{-}m\text{-}n)^2G_2^\delta(t)\right]}\,, \quad (2.29)$$

Further computations using (1.3), (2.7) and (2.29), we obtain

$$|a_2| \leq \frac{2|\delta|t\sqrt{2|\delta|t}}{\sqrt{\left|\begin{array}{c}\delta^2t^2\left[4\left((3-2\gamma)(3\lambda-m^2-n^2-mn)+\frac{1}{2}\gamma(\gamma-1)(2\lambda-m-n)^2+(4-3\gamma)\left((m+n)^2-2\lambda(m+n-\lambda+1)\right)\right)\right. \\ \left.-2(2-\gamma)^2(2\lambda-m-n)^2\right]-\delta(2-\gamma)^2(2\lambda-m-n)^2(2t^2-1)\end{array}\right|}}.$$

Next, if we subtract (2.25) from (2.23), we deduce that

$$2(3-2\gamma)(3\lambda-m^2-n^2-mn)(a_3-a_2^2) = \mathcal{G}_1^\delta(t)(u_2-v_2)+\mathcal{G}_2^\delta(t)(u_1^2-v_1^2) \quad (2.30)$$

In view of (2.26) and (2.27), we get from (2.30)

$$a_3 = \frac{\left(\mathcal{G}_1^\delta(t)\right)^2(u_1^2+v_1^2)}{2(2-\gamma)^2(2\lambda-m-n)^2}+\frac{\mathcal{G}_1^\delta(t)(u_2-v_2)}{2(3-2\gamma)(3\lambda-m^2-n^2-mn)}.$$

Thus applying (1.3), we obtain

$$|a_3| \leq \frac{4\delta^2t^2}{(2-\gamma)^2(2\lambda-m-n)^2}+\frac{2|\delta|t}{(3-2\gamma)(3\lambda-m^2-n^2-mn)}.$$

Putting $\gamma=1$, in Theorem 2.2, we obtain the next result:

Corollary 2.2. For $\lambda\geq1$, $t\in\left(\frac{1}{2},1\right]$, $m,n\in C$, $m\neq n$, $|n|\leq1$ and δ is nonzero real constant, let $f\in\mathcal{A}$ be in the family $H_\Sigma(\delta,\lambda,m,n,t)$. Then

$$|a_2| \leq \frac{2|\delta|t\sqrt{2|\delta|t}}{\sqrt{\left|\begin{array}{c}\delta^2t^2\left[4\left((3\lambda-m^2-n^2-mn)+(m+n)^2-2\lambda(m+n-\lambda+1)\right)-2(2\lambda-m-n)^2\right] \\ -\delta(2\lambda-m-n)^2(2t^2-1)\end{array}\right|}}$$

and

$$|a_3| \leq \frac{4\delta^2t^2}{(2\lambda-m-n)^2}+\frac{2|\delta|t}{3\lambda-m^2-n^2-mn}.$$

In the next theorems, we discus "Fekete-Szegö problem" of the families $B_\Sigma(\delta,\alpha,\lambda,m,n,t)$ and $F_\Sigma(\delta,\lambda,\gamma,m,n,t)$.

Theorem 2.3. For $0\leq\alpha\leq1$, $\lambda\geq1$, $t\in\left(\frac{1}{2},1\right]$, $m,n\in C$, $m\neq n$, $|n|\leq1$, $\xi\in R$ and δ is nonzero real constant, let $f\in\mathcal{A}$ be in the family $B_\Sigma(\delta,\alpha,\lambda,m,n,t)$. Then

$$|a_3-\xi a_2^2|\leq\begin{cases}\dfrac{2t|\delta|}{(2\alpha+1)(3\lambda-m^2-n^2-mn)}\ ; \\ \qquad \text{for } |\xi-1|\leq\dfrac{|2\delta^2t^2[2(\Omega(\alpha,\lambda,m,n)-mn)-(\alpha+1)^2(2\lambda-m-n)^2]-\delta(\alpha+1)^2(2\lambda-m-n)^2(2t^2-1)|}{4\delta^2t^2(2\alpha+1)(3\lambda-m^2-n^2-mn)}, \\ \dfrac{8t^3|\delta^3||\xi-1|}{|2\delta^2t^2[2(\Omega(\alpha,\lambda,m,n)-mn)-(\alpha+1)^2(2\lambda-m-n)^2]-\delta(\alpha+1)^2(2\lambda-m-n)^2(2t^2-1)|}\ ; \\ \qquad \text{for } |\xi-1|\geq\dfrac{|2\delta^2t^2[2(\Omega(\alpha,\lambda,m,n)-mn)-(\alpha+1)^2(2\lambda-m-n)^2]-\delta(\alpha+1)^2(2\lambda-m-n)^2(2t^2-1)|}{4\delta^2t^2(2\alpha+1)(3\lambda-m^2-n^2-mn)}\ .\end{cases}$$

Proof. In the light of (2.15) and (2.16), we deduce that

$$a_3-\xi a_2^2=(1-\xi)a_2^2+\frac{\mathcal{G}_1^\delta(t)(u_2-v_2)}{2(2\alpha+1)(3\lambda-m^2-n^2-mn)}$$

$$=(1-\xi)\frac{\left(G_1^\delta(t)\right)^3(u_2+v_2)}{2\left[(\Omega(\alpha,\lambda,m,n)-mn)\left(G_1^\delta(t)\right)^2-G_2^\delta(t)(\alpha+1)^2(2\lambda-m-n)^2\right]}+\frac{G_1^\delta(t)(u_2-v_2)}{2(2\alpha+1)(3\lambda-m^2-n^2-mn)}$$

$$=\frac{G_1^\delta(t)}{2}\left[\left(\psi(\xi)+\frac{1}{(2\alpha+1)(3\lambda-m^2-n^2-mn)}\right)u_2+\left(\psi(\xi)-\frac{1}{(2\alpha+1)(3\lambda-m^2-n^2-mn)}\right)v_2\right],$$

where

$$\psi(\xi)=\frac{\left(G_1^\delta(t)\right)^2(1-\xi)}{(\Omega(\alpha,\lambda,m,n)-mn)\left(G_1^\delta(t)\right)^2-G_2^\delta(t)(\alpha+1)^2(2\lambda-m-n)^2}.$$

According to (1.3), we deduce that

$$|a_3-\xi a_2^2|\leq\begin{cases}\dfrac{2t|\delta|}{(2\alpha+1)(3\lambda-m^2-n^2-mn)}, \\ \qquad 0\leq|\psi(\xi)|\leq\dfrac{1}{(2\alpha+1)(3\lambda-m^2-n^2-mn)} \\ 2t|\delta||\psi(\xi)|, \\ \qquad |\psi(\xi)|\geq\dfrac{1}{(2\alpha+1)(3\lambda-m^2-n^2-mn)}\end{cases}.$$

After some computations, we obtain

$$|a_3-\xi a_2^2|\leq\begin{cases}\dfrac{2t|\delta|}{(2\alpha+1)(3\lambda-m^2-n^2-mn)}\,; \\ \qquad \text{for } |\xi-1|\leq\dfrac{\left|2\delta^2t^2[2(\Omega(\alpha,\lambda,m,n)-mn)-(\alpha+1)^2(2\lambda-m-n)^2]-\delta(\alpha+1)^2(2\lambda-m-n)^2(2t^2-1)\right|}{4\delta^2t^2(2\alpha+1)(3\lambda-m^2-n^2-mn)}, \\ \dfrac{8t^3|\delta^3||\xi-1|}{\left|2\delta^2t^2[2(\Omega(\alpha,\lambda,m,n)-mn)-(\alpha+1)^2(2\lambda-m-n)^2]-\delta(\alpha+1)^2(2\lambda-m-n)^2(2t^2-1)\right|}\,; \\ \qquad \text{for } |\xi-1|\geq\dfrac{\left|2\delta^2t^2[2(\Omega(\alpha,\lambda,m,n)-mn)-(\alpha+1)^2(2\lambda-m-n)^2]-\delta(\alpha+1)^2(2\lambda-m-n)^2(2t^2-1)\right|}{4\delta^2t^2(2\alpha+1)(3\lambda-m^2-n^2-mn)}\,.\end{cases}$$

Putting $\xi = 1$ in Theorem 2.3, we obtain the next result:

Corollary 2.3. For $0\leq\alpha\leq1, \lambda\geq1, t\in\left(\frac{1}{2},1\right]$, $m,n\in C$, $m\neq n$, $|n|\leq1$ and δ is nonzero real constant, let $f \in \mathcal{A}$ be in the family $B_\Sigma(\delta,\alpha,\lambda,m,n,t)$. Then

$$|a_3 - a_2^2| \leq \frac{2t|\delta|}{(2\alpha+1)(3\lambda-m^2-n^2-mn)}.$$

Putting $\alpha = 1$ in Theorem 2.3, we obtain the next result:

Corollary 2.4. For $\lambda\geq1, t\in\left(\frac{1}{2},1\right]$, $m,n\in C$, $m\neq n$, $|n|\leq1$, $\xi\in R$ and δ is nonzero real constant, let $f \in \mathcal{A}$ be in the family $M_\Sigma(\delta,\lambda,m,n,t)$. Then

$$|a_3-\xi a_2^2|\leq\begin{cases}\dfrac{2t|\delta|}{(2\alpha+1)(3\lambda-m^2-n^2-mn)} \\ \qquad \text{for } |\xi-1|\leq\dfrac{\left|\delta^2t^2[(\Psi(\lambda,m,n)-mn)-2(2\lambda-m-n)^2]-\delta(2\lambda-m-n)^2(2t^2-1)\right|}{\delta^2t^2(2\alpha+1)(3\lambda-m^2-n^2-mn)}, \\ \dfrac{2t^3|\delta^3||\xi-1|}{\left|\delta^2t^2[(\Psi(\lambda,m,n)-mn)-2(2\lambda-m-n)^2]-\delta(2\lambda-m-n)^2(2t^2-1)\right|}\,; \\ \qquad \text{for } |\xi-1|\geq\dfrac{\left|\delta^2t^2[(\Psi(\lambda,m,n)-mn)-2(2\lambda-m-n)^2]-\delta(2\lambda-m-n)^2(2t^2-1)\right|}{\delta^2t^2(2\alpha+1)(3\lambda-m^2-n^2-mn)}\,.\end{cases}$$

Theorem 2.4. For $\lambda\geq1$, $0\leq\gamma\leq1$, $t\in\left(\frac{1}{2},1\right]$, $m,n\in C$, $m\neq n$, $|n|\leq1$, $\xi\in R$ and δ is nonzero real constant, let $f \in \mathcal{A}$ be in the family $F_\Sigma(\delta,\lambda,\gamma,m,n,t)$. Then

$$|a_3-\xi a_2^2|\le\begin{cases}\dfrac{2t|\delta|}{(3-2\gamma)(3\lambda-m^2-n^2-mn)}\,; \\ \quad for\ |\xi-1|\le\dfrac{\left|\begin{array}{c}\delta^2t^2\left[4\left((3-2\gamma)(3\lambda-m^2-n^2-mn)+\frac{1}{2}\gamma(\gamma-1)(2\lambda-m-n)^2+(4-3\gamma)\left((m+n)^2-2\lambda(m+n-\lambda+1)\right)\right)\right. \\ \left.-2(2-\gamma)^2(2\lambda-m-n)^2\right]-\delta(2-\gamma)^2(2\lambda-m-n)^2(2t^2-1)\end{array}\right|}{4\delta^2t^2(3-2\gamma)(3\lambda-m^2-n^2-mn)}, \\ \dfrac{8t^3|\delta^3||\xi-1|}{\left|\begin{array}{c}\delta^2t^2\left[4\left((3-2\gamma)(3\lambda-m^2-n^2-mn)+\frac{1}{2}\gamma(\gamma-1)(2\lambda-m-n)^2+(4-3\gamma)\left((m+n)^2-2\lambda(m+n-\lambda+1)\right)\right)\right. \\ \left.-2(2-\gamma)^2(2\lambda-m-n)^2\right]-\delta(2-\gamma)^2(2\lambda-m-n)^2(2t^2-1)\end{array}\right|}\,; \\ \quad for\ |\xi-1|\ge\dfrac{\left|\begin{array}{c}\delta^2t^2\left[4\left((3-2\gamma)(3\lambda-m^2-n^2-mn)+\frac{1}{2}\gamma(\gamma-1)(2\lambda-m-n)^2+(4-3\gamma)\left((m+n)^2-2\lambda(m+n-\lambda+1)\right)\right)\right. \\ \left.-2(2-\gamma)^2(2\lambda-m-n)^2\right]-\delta(2-\gamma)^2(2\lambda-m-n)^2(2t^2-1)\end{array}\right|}{4\delta^2t^2(3-2\gamma)(3\lambda-m^2-n^2-mn)}.\end{cases}$$

Proof. In view of (2.29) and (2.30), we deduce that

$$a_3-\xi a_2^2=(1-\xi)a_2^2+\frac{\mathcal{G}_1^\delta(t)(u_2-v_2)}{2(3-2\gamma)(3\lambda-m^2-n^2-mn)}$$

$$=\frac{\left(\mathcal{G}_1^\delta(t)\right)^3(u_2+v_2)(1-\xi)}{\begin{array}{c}2\left[\left((3-2\gamma)(3\lambda-m^2-n^2-mn)+\frac{1}{2}\gamma(\gamma-1)(2\lambda-m-n)^2+(4-3\gamma)\left((m+n)^2-2\lambda(m+n-\lambda+1)\right)\right)\left(\mathcal{G}_1^\delta(t)\right)^2\right. \\ \left.-(2-\gamma)^2(2\lambda-m-n)^2\mathcal{G}_2^\delta(t)\right]\end{array}}$$

$$+\frac{\mathcal{G}_1^\delta(t)(u_2-v_2)}{2(3-2\gamma)(3\lambda-m^2-n^2-mn)}$$

$$=\frac{G_1^\delta(t)}{2}\left[\left(\varphi(\xi)+\frac{1}{(3-2\gamma)(3\lambda-m^2-n^2-mn)}\right)u_2+\left(\varphi(\xi)-\frac{1}{(3-2\gamma)(3\lambda-m^2-n^2-mn)}\right)v_2\right],$$

where

$$\varphi(\xi)$$

$$=\frac{\left(\mathcal{G}_1^\delta(t)\right)^2(1-\xi)}{\begin{array}{c}\left((3-2\gamma)(3\lambda-m^2-n^2-mn)+\frac{1}{2}\gamma(\gamma-1)(2\lambda-m-n)^2+(4-3\gamma)\left((m+n)^2-2\lambda(m+n-\lambda+1)\right)\right)\left(\mathcal{G}_1^\delta(t)\right)^2 \\ -(2-\gamma)^2(2\lambda-m-n)^2\mathcal{G}_2^\delta(t)\end{array}}.$$

According to (1.3), we deduce that

$$|a_3-\xi a_2^2|\le\begin{cases}\dfrac{2t|\delta|}{(3-2\gamma)(3\lambda-m^2-n^2-mn)}, & 0\le|\varphi(\xi)|\le\dfrac{1}{(3-2\gamma)(3\lambda-m^2-n^2-mn)} \\ 2t|\delta||\varphi(\xi)|, & |\varphi(\xi)|\ge\dfrac{1}{(3-2\gamma)(3\lambda-m^2-n^2-mn)}\end{cases}.$$

After some computations, we obtain

$$|a_3-\xi a_2^2| \le \begin{cases} \dfrac{2t|\delta|}{(3-2\gamma)(3\lambda-m^2-n^2-mn)}\,; \\ \quad for\ |\xi-1| \le \dfrac{\left|\delta^2t^2\left[4\left((3-2\gamma)(3\lambda-m^2-n^2-mn)+\frac{1}{2}\gamma(\gamma-1)(2\lambda-m-n)^2+(4-3\gamma)\left((m+n)^2-2\lambda(m+n-\lambda+1)\right)\right)-2(2-\gamma)^2(2\lambda-m-n)^2\right]-\delta(2-\gamma)^2(2\lambda-m-n)^2(2t^2-1)\right|}{4\delta^2t^2(3-2\gamma)(3\lambda-m^2-n^2-mn)}, \\ \dfrac{8t^3|\delta^3||\xi-1|}{\left|\delta^2t^2\left[4\left((3-2\gamma)(3\lambda-m^2-n^2-mn)+\frac{1}{2}\gamma(\gamma-1)(2\lambda-m-n)^2+(4-3\gamma)\left((m+n)^2-2\lambda(m+n-\lambda+1)\right)\right)-2(2-\gamma)^2(2\lambda-m-n)^2\right]-\delta(2-\gamma)^2(2\lambda-m-n)^2(2t^2-1)\right|}\,; \\ \quad for\ |\xi-1| \ge \dfrac{\left|\delta^2t^2\left[4\left((3-2\gamma)(3\lambda-m^2-n^2-mn)+\frac{1}{2}\gamma(\gamma-1)(2\lambda-m-n)^2+(4-3\gamma)\left((m+n)^2-2\lambda(m+n-\lambda+1)\right)\right)-2(2-\gamma)^2(2\lambda-m-n)^2\right]-\delta(2-\gamma)^2(2\lambda-m-n)^2(2t^2-1)\right|}{4\delta^2t^2(3-2\gamma)(3\lambda-m^2-n^2-mn)}. \end{cases}$$

Putting $\xi = 1$ in Theorem 2.4, we obtain the next result:

Corollary 2.5. For $\lambda\ge1$, $0\le\gamma\le1$, $t\in\left(\frac{1}{2},1\right]$, $m,n\in C$, $m\neq n$, $|n|\le1$, and δ is nonzero real constant, let $f \in \mathcal{A}$ be in the family $F_\Sigma(\delta,\lambda,\gamma,m,n,t)$. Then

$$|a_3-a_2^2| \le \frac{2t|\delta|}{(3-2\gamma)(3\lambda-m^2-n^2-mn)}.$$

Putting $\gamma = 1$, in Theorem 2.4, we obtain the next result:

Corollary 2.6. For $\lambda\ge1$, $t\in\left(\frac{1}{2},1\right]$, $m,n\in C$, $m\neq n$, $|n|\le1$, $\xi\in R$ and δ is nonzero real constant, let $f \in \mathcal{A}$ be in the family $H_\Sigma(\delta,\lambda,m,n,t)$Then

$$|a_3-\xi a_2^2| \le \begin{cases} \dfrac{2t|\delta|}{3\lambda-m^2-n^2-mn}\,; \\ \quad for\ |\xi-1| \le \dfrac{\left|\delta^2t^2\left[4\left(3\lambda-m^2-n^2-mn+(m+n)^2-2\lambda(m+n-\lambda+1)\right)-2(2\lambda-m-n)^2\right]-\delta(2\lambda-m-n)^2(2t^2-1)\right|}{4\delta^2t^2(3\lambda-m^2-n^2-mn)}, \\ \dfrac{8t^3|\delta^3||\xi-1|}{\left|\delta^2t^2\left[4\left(3\lambda-m^2-n^2-mn+(m+n)^2-2\lambda(m+n-\lambda+1)\right)-2(2\lambda-m-n)^2\right]-\delta(2\lambda-m-n)^2(2t^2-1)\right|}\,; \\ \quad for\ |\xi-1| \ge \dfrac{\left|\delta^2t^2\left[4\left(3\lambda-m^2-n^2-mn+(m+n)^2-2\lambda(m+n-\lambda+1)\right)-2(2\lambda-m-n)^2\right]-\delta(2\lambda-m-n)^2(2t^2-1)\right|}{4\delta^2t^2(3\lambda-m^2-n^2-mn)}. \end{cases}$$

Conclusion

The fact that we can obtain many unique and effective uses of a large variety of interesting functions and specific polynomial in Geometric Function Theory provided the primary inspiration for our analysis in this article. The primary objective was to define the families $B_\Sigma(\delta,\alpha,\lambda,m,n,t)$ and $F_\Sigma(\delta,\lambda,\gamma,m,n,t)$ of bi-univalent functions associated with λ-pseudo-starlike and convex functions defined by Sakaguchi type functions which governed by Gegenbauer polynomials. We generated Taylor coefficient inequalities for functions in the families $B_\Sigma(\delta,\alpha,\lambda,m,n,t)$ and $F_\Sigma(\delta,\lambda,\gamma,m,n,t)$ and viewed the famous Fekete-Szegӧ problem.

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