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Some Separation Axioms Via D-Set in Bitopological Space

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ABSTRACT

In this paper, notions of some separation axioms by using D-set in bitopological space. We studied some of the fundamental properties and relations among types of D_k-bitopological spaces where k =0,1.

MSC..

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1. Introduction:

Kelly In 1963 [3] defined bitopological spac . In 1966[4] Murdeshwar studied the concepts of pairwise- T_o and weak pairwise T_o spaces. In [5] 1982 Tong introduced definition of D-set . Tallafha In [1] studied continuous and pairwise continuous functions of bitopological spaces. In [6] O.Ravi investigated open set in bitopological space . Rupaya in [7] presented defined T_o-bitopological space . In [2] Khadiga investigated defined subbitopological spaces. In this paper we introduce and study the definition of D-set in bitopological space and some types of D_k –bitopological spaces for k = 0,1.

2. D-set in bitopological space

Definition (2.1)[5]

Let (X , τ) is topological space .The subset H of X is said to be difference set (D-set) if there exist two open sets M and N in X such that M $\neq$ **X** , H= M $-$ N .

Definition (2.2)[3]

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If X non-empty set and σ_1 , σ_2 are two Topologies on X . A space (X , σ_1 , σ_2) is called bitopological space .

Definition (2.3)[6]
In bitopological space (X , σ_1, σ_2) the subset K is said to be $\sigma_1\sigma_2$-open set if K= U∪V where U∈σ_1 and V∈σ_2 . The complement of $\sigma_1\sigma_2$-open is called $\sigma_1\sigma_2$ -closed .

Definition (2.4)
A subset W of bitopological space (X , σ_1, σ_2) is called $\sigma_1\sigma_2$-difference set ($\sigma_1\sigma_2$ -D -set) if W= $U_1 \cup U_2$ where U_1 is D-set in (X , σ_1) and U_2 is D-set in (X , σ_2) .

Remark (2.1)
Every $\sigma_1\sigma_2$-open (is not equal to X) is $\sigma_1\sigma_2$-D-set .

But the converse is not true for example

Example (2.1)
Let X={h_1, h_2 , h_3, h_4} and σ_1={ ∅, X,{h_1, h_2, h_3} , {h_2, h_3 }} , σ_2={∅, X,{h_2,h_4}, {h_4}} are two topologies on X . Since A={h_1, h_2 }=$\{h_1\}\cup\{h_2\}$. Since $\{h_1\}\notin \sigma_1$ and $\{h_2\}\notin \sigma_2$ hence $\{h_1, h_2\}$ is not $\sigma_1\sigma_2$ -open . But $\{h_1\}$ is σ_1-D- set and $\{h_2\}$ is σ_2-D- set then A is $\sigma_1\sigma_2$-D- set

Definition (2.5)[1]
Afunction f:(X ,σ_1,σ_2)→ (Y, ρ_1, ρ_2) is called continuous if f:(X,σ_1)→(Y,ρ_1) and f:(X,σ_2)→(Y, ρ_2) are continuous where (X , σ_1,σ_2) and (Y, ρ_1, ρ_2) be two bitopological spaces

Theorem (2.1)
If f:(X ,σ_1,σ_2)→ (Y, ρ_1, ρ_2) is continuous therefore the inverse image of $\rho_1\rho_2$-D-set is $\sigma_1\sigma_2$-D-set .
Proof
Suppose G is $\rho_1\rho_2$-D-set in Y .Then G =A∪B suchthat A is ρ_1-D-set and B is ρ_2-D-sets in Y thus A=U−V and B=O−S where U,V ≠ Y and (U,V ∈ ρ_1) and (O,S ∈ ρ_2) .Since f:(X ,σ_1,σ_2)→ (Y, ρ_1, ρ_2) is continuous hence f:(X, σ_1)→(Y,ρ_1) and f:(X, σ_2)→(Y, ρ_2) are continuous . Thus f^{-1}(U) ,f^{-1} (O)∈σ_1 and f^{-1}(U) , f^{-1}(V) ∈ σ_2. And f^{-1}(U) , f^{-1}(O) ≠ X. Therefore f^{-1}(U) $-f^{-1}$(V)= $f^{-1}(U-V)$ = $f^{-1}(A)$ hence $f^{-1}(A)$ is σ_1 -D-set and f^{-1}(O) − f^{-1} (S) = $f^{-1}(O-S)$ = $f^{-1}(B)$ is σ_2-D-set in X . Thus the inverse image of $\rho_1\rho_2$-D-set is $\sigma_1\sigma_2$-D-set

3. D_k -bitopological spaces

Definition (3.1)[7]

Bitopological space (X ,$\boldsymbol{\sigma_1}$, $\boldsymbol{\sigma_2}$) is said to be $\boldsymbol{T_o}$ if and only if for each different points in X there exists U is $\boldsymbol{\sigma_1\sigma_2}$-open set containing one not containing other*.*

Definition (3.2)
A bitopological space (X ,σ_1, σ_2) is called D_o if and only if for each different points in X there exists $\sigma_1\sigma_2$ -D- set containing one but not containing other *.*

Theorem (3.1)
Every T_o bitopological space is D_o .
Proof :
Let (X ,σ_1, σ_2) is T_o and x, y in X such that $x \neq y$ .Then there exist A is $\sigma_1\sigma_2$-open set suchthat $x \in A$, y∉A . Thus A is $\sigma_1\sigma_2$-D-set we have (X ,σ_1, σ_2) is D_o

Theorem (3.2)
If (X ,σ_1) and (X , σ_2) is D_o then (X , σ_1,σ_2) is D_o .
Proof :

Suppose $x \neq y$ in X . Thus ∃ G_1={ $x \in G_1$, $y \notin G_1$ } is σ_1-D-set in (X , σ_1) and G_2={$x \notin G_2$, $y \in G_2$ } is σ_2-D-set in (X , σ_2) because (X , σ_1) and (X , σ_2) are D_o . Let G = $G_1 \cup G_2$ then x ∈G and y∉G then (X , σ_1, σ_2) is D_o .

The following example shows the opposite of Theorem (3.2) is not true

Example (3.1)

Suppose X={ m , n , o , p } , σ_1 ={ ∅ , X ,{m , o },{o}} and σ_2 = { ∅ , X , {n , p},{ p}} . It is clear that (X , σ_1, σ_2) is D_o . But $n \neq p$ and there is no σ_1-D-set containing n not containing p thus (X,σ_1) is not D_o . And m ≠ o and there is no σ_2-D-set containing 1 not 3 thus (X , σ_2) is not D_o .

Theorem (3.3)
If f:(M ,σ_1,σ_2)→ (N , ρ_1, ρ_2) is continuous and one to one and N is D_o -space then M D_o-space .
Proof :
Let N is D_o and $x \neq y$ in M . Then there exist a, b in Y and $a \neq b$, (f(x)=a , f(y)=b) . Since f is one to one hence $f(x) \neq f(y)$. Since Y is D_o then there exist U is ρ_1 ρ_2 -D-set in N such that $a \in U$ and b∉ U. we have the inverse image of U is $\sigma_1\sigma_2$-D-set in M containing x not y. Therefore (M , σ_1, σ_2) is D_o .

Definition(3.3)[2]
For a bitopological space (Y,σ_1,σ_2) and A ⊆ X . (A ,σ_{1_A},σ_{2_A}) is said to be subspace of (Y,σ_1,σ_2) when σ_{1_A}={ U_1∩A : $U_1 \in \sigma_1$} and σ_{2_A}={ U_2∩A : $U_2 \in \sigma_2$} .

Theorem(3.4)
If (Y,σ_1,σ_2) is D_o and A subset of X then (A , σ_{1_A} , σ_{2_A}) is D_o
Proof :
Suppose a and b are two distance points in A thus a and b in Y . Since Y is D_o then there exist G is $\sigma_1\sigma_2$-D-set in Y and G containing a not b . Hence $G = V_1 \cup V_2$ where V_1 is σ_1-D-set and V_2 is σ_2-D-set in Y . Then $V_1 = O_1 - S_1$ and $V_2 = O_2 - S_2$ such that (O_1, $S_1 \in \sigma_1$) , (O_2, $S_2 \in \sigma_2$) and O_1, $O_2 \neq$Y .
$A \cap G = A \cap [\, (O_1 - S_1) \cup (\, O_2 - S_2)]$
$= [\, (A \cap O_1) - (A \cap S_1)\,] \cup [\, (A \cap O_2) - (A \cap S_2)\,]$
Since $(A \cap O_1)$, $(A \cap S_1) \in \sigma_{1_A}$ and $(A \cap O_1) \neq Y$
$(A \cap O_2)$, $(A \cap S_2) \in \sigma_{2_A}$ and $(A \cap O_2) \neq Y$
Let $O_1^* - S_1^* = (A \cap O_1) - (A \cap S_1)$ and $O_2^* - S_2^* = (A \cap O_2) - (A \cap S_2)$
Hence $O_1^* - S_1^*$ is σ_{1_A}-D-set and $A_2^* = O_2^* - S_2^*$ is σ_{2_A}-D-set in A
Then A∩G = $A^* = A_1^* \cup A_2^*$ is $\tau_A\sigma_A$-D-set in A . Since $x \in G$, $x \in A$ thus $x \in A^*$ and y∉G , y∈ A thus y∉A*. we have hence (A , σ_{1_A} , σ_{2_A}) is D_o .

The following example shows that converse of Theorem(3.4) is not true

Example (3.2)
Let Y ={i , g , h , k } , σ_1={ ∅ , X ,{g}} and σ_2 = { ∅ , X , {g , h}} . Let A={i} then σ_{1_A} = σ_{2_A}={ ∅, X} . It is clear that (A , σ_{1_A},σ_{2_A}) is D_o . But i≠ k and ∄ U is $\sigma_1\sigma_2$-D-set and i ∈ U , k∉U hence (Y , σ_1 , σ_2) is not D_o .

Definition (3.4)
Bitopology (Y , σ_1 , σ_2) is said to be D_1 if and only if for each distance points m and n there are H and K $\sigma_1\sigma_2$-D-sets such that $m \in H$, $n \notin H$ and $m \notin$K , $n \in K$.

Theorem (3.5)
If (Y ,σ_1) and (Y , σ_2) are D_1 then (Y , σ_1 , σ_2) is D_1.
Proof :
Suppose i ≠ j in X . Since (Y , σ_1) , (Y , σ_2) are D_1 then there exist U_1 and U_2 are σ_1-D-sets such that ($i \in U_1$, j∉U_1 and $i \notin U_2$, j∈ U_2) and V_1 and V_2 are σ_2-D-sets such that ($i \in V_1$, j∉V_1 and $i \notin V_2$, j∈ V_2) . Let U = $U_1 \cup U_2$ and V= $V_1 \cup V_2$. Hence U and V are $\sigma_1\sigma_2$-D-sets and $i \in$ U , j∉U , i∉V and $j \in V$. Thus (Y , σ_1 , σ_2) is D_1 .
The following example converse of Theorem(3.4) is not true for

Example (3.3)
Suppose Y ={ u , v , w }} , σ_1={ ∅ , Y ,{u , v},{u}} and σ_2 = { ∅ , Y ,{ v, w},{w}} . It is clear that (Y , σ_1 , σ_2) is D_1. But u≠w and ∄ C_1 , C_2 are σ_1-D-sets as u∈ C_1 , v∉C_1 , u∉C_2 , v∈ C_2 .Thus (Y ,σ_1) not D_1. Similarity we have (Y , σ_2) is not D_1 .

Theorem (3.6)
If f:(Y_1 ,σ_1,σ_2)→ (Y_2 , ρ_1, ρ_2) is continuous and one to one and Y is D_1-space then Y_1 D_1 -space .
Proof :
Let Y_2 is D_1 and $x \neq y$ in Y_1 . Then there exist a , $b \in Y_2$ where $a \neq b$ f(x)=u , f(y)=v . Since f is one to one hence $f(x)$ is not equal to (y) . Since Y_2 is D_1 then there exist E and H are $\rho_1\rho_2$-D containing in Y and $u \in$ E , $v \notin$E , $u \notin$H and v ∈H . The inverse image of E and H are $\sigma_1\sigma_2$-D-sets in Y_1 because f is continuous . And the inverse image of E containing u not v .We have (Y_1 ,σ_1,σ_2) is D_1 .

Theorem(3.7)
Every subspace of D_1-bitopological space is D_1-bitopological space
Proof :
Suppose (A , σ_{1_A} ,σ_{2_A}) is subspace in (M , σ_1,σ_2) . Let u and v are two different points in A thus u , $v \in$ M . Since M is D_1 then there exist G and W are $\sigma_1\sigma_2$-D-sets in M and ($u \in$ G , $v \notin$G , $u \notin$ W and $v \in$ W) . Since A∩G and A∩W are $\sigma_{1_A}\sigma_{2_A}$-D-sets in A . Let G* = A∩G and W* = A∩W. Therefore G* containing u not v and W* containing v not u then (A , σ_{1_A} ,σ_{2_A}) is D_1

Theorem (3.8)
Every D_1-bitopological space is D_o .
Proof :
Let $a \neq b$ in X . Since (X ,σ_1,σ_2) is D_1 then then there exist G = { ∈ G , b∉G } and W ={ a∉ W , $b \in$ W } are $\sigma_1\sigma_2$-D .Then (X ,σ_1,σ_2) is D_o .

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4. Pairwise and weak pairwise D_k - bitopological spaces

Definition(4.1)
(N , σ_1,σ_2) is said to be
1- pairwise T_o if and only if ∀ c , d ∈ N and c ≠ d ∃ K_1 is σ_1-open (c ∈ K_1 , d∉K_1) or ∃ K_2 is σ_2-open ($d \in K_2$, c∉K_2) . [4]
2- pairwise D_o if and only if ∀ u ,v ∈ N u ≠ v ∃ G_1 is σ_1-D-set (u ∈ G_1 , d∉G_1) or ∃ G_2 σ_2-D ($v \in G_2$, u∉G_2) .

Theorem (4.1)
Bitopological (X , σ_1,σ_2) pairwise T_o if and only if pairwise D_o
Proof:
Let $x \neq y$ in X and (X , σ_1,σ_2) is pairwise T_o then there exist either A is σ_1-open A containing x not y . Or there is B is σ_2-open set containing x not y thus σ_1-D-set containing x not y . Therefore X is pairwise D_o .
Let $m \neq n$ in X and (X, σ_1,σ_2) is pairwise D_o then either A is σ_1-D-set , $m \in$ A and $n \notin$A . Thus A= $A_1 - A_2$ such that $A_1 \neq$X where A_1, A_2 are σ_1-open hence $m \in A_1$ and $y \notin A_1$. Or B is σ_2-D-set such that $m \notin$B and $n \in$ B thus B= $B_1 - B_2$ such that $B_1 \neq$X and B_1 , B_2 are σ_2-open hence $m \notin B_1$and $n \in B_1$ .Therefore X is pairwise T_o .

Theorem (4.2)
Every D_o is pairwise D_o .
Proof:
Let $x \neq y$ in X and (X , σ_1, σ_2) is D_o thus ∃G is $\sigma_1\sigma_2$-D-set where G = {$m \in$ G and $n \notin$G } .Thus G =A∪B such that A is σ_1-D-set and B is σ_2-D-set thus x containing in A or not containing in B and y not containing in A . Thus there exist A is σ_1-D-set , A={ $x \in$ A and $y \notin$A } or there exist B is σ_2-D-set , B{ $x \notin$B and $y \in$ B } . Hence (X , σ_1,σ_2) is pairwise D_o .

Theorem(4.3)
If (M , σ_1 , σ_2) is pairwise- D_o and A subset of X therefore (A, σ_{1_A} , σ_{2_A}) is pairwise- D_o
proof
Suppose c , d are different points in A thus $x \neq y$ in M . Since M is pairwise- D_o then either G is (σ_1-D-set or is σ_2-D) in M where G ={ c ∈G and d∉G} or G ={ c∉G and d ∈G} . It is clear that G∩A is σ_{1_A}-D-set or is σ_{2_A}-D-set in A , let G* = G∩A .We have that c ∈G* and d∉G* or c∉G* and d∈G* . Therefore (A , σ_{1_A} , σ_{2_A}) is pairwise- D_o .

Definition (4.2)[1]
A function f:(X ,σ_1,σ_2)→ (Y, ρ_1, ρ_2) said to be pairwise continuous if f^{-1}(A) ∈ $\sigma_1 \cup \sigma_2$ for all A ∈ $\rho_1 \cup \rho_2$

Theorem (4.4)
If g:(N ,σ_1,σ_2)→ (H , ρ_1, ρ_2) is pairwise continues and one to one and Y is D_o -space then X D_o -space .
Proof :
Let H pairwise- D_o where $x \neq y$ in X . We have ∃ a, b in Y , $a \neq b$ and g(x)=a , g(y)=b . Since g is one to one hence $g(x) \neq g(y)$. Since N is pairwise-D_o then there exist G is (ρ_1-D-set suchthat $a \in G$ and $b \notin$ G) or (G is ρ_2-D-set suchthat $a \notin$ G and $b \in G$. Hence then there exist S_1 , S_2 are $\rho_1$open or ρ_2-open suchthat S_1 ≠H and G = $S_1 - S_2$. We have S_1 , S_2 ∈ $\rho_1 \cup \rho_2$. Since f is pairwise continuous thus the inverse image of S_1 , S_2 are σ_1-open or σ_2-open sets in N . Therefore the inverse image of S_1 , S_2 containing in $\tau_1 \cup \sigma_1$ suchthat $f^{-1}(G) = f^{-1}(U) - f^{-1}(V)$ thus $f^{-1}(G)$ not equal to N thus $f^{-1}(G)$ is (σ_1-D-set suchthat $f^{-1}(G)$ ={ $x \in f^{-1}(G)$ and $y \notin f^{-1}$(G) } or (G is σ_2-D-set suchthat $x \notin$ G and $y \in G$) . Hence Let X is pairwise- D_o .

Definition(4.3)
A space (K ,σ_1,σ_2) is said to be
i) Weak pairwise T_o if and only if ∀ m and n are distinct points in k ∃ U is σ_1-open set or σ_2-open , m ∈U and n∉U . [4]
ii)Weak pairwise D_o if and only if ∀ m and n are distinct points in K ∃ U is σ_1-D-set or σ_2-D-set m ∈U and n∉U .

Remark(4.1)
Every pairwise D_o is weak pairwise D_o

Theorem(4.5)
A bitopological space (X , σ_1 , σ_2) is weak pairwise T_o if and only if weak pairwise D_o
Proof :
Similarity to Theorem (4.1)

Remark(4.2)
Every D_o is Weak pairwise D_o .

Theorem (4.6)
If (M ,σ_1) or (M, σ_2) D_o then (M , σ_1 , σ_2) is waek pairwise- D_o .
Proof:
Suppose x and y are different points in X . Since (M , σ_1) or (M , σ_2) are D_o thus there exist W is σ_1-D-set or σ_2-D-set such that x ∈W and y⍰W . Therefore (M , σ_1 , σ_2) is weak pairwise- D_o .

But the converse is not true for example

Example (4.1)
Let M ={ c , d , e } such that σ_1 ={ ∅ , M ,{c}} and σ_2= { ∅ , M ,{d}} . It is clear that M is weak pairwise-D_o but (M , σ_1) and (M , σ_2) not D_o

Theorem(4.7)

If (X , σ_1 , σ_2) is waek pairwise- D_o and A subset of X then (A ,σ_{1_A} , σ_{2_A}) is weak pairwise- D_o

Proof : Clear

Remark (2.1)
If σ_1 or σ_2 discrete then (X , σ_1 , σ_2) is weak pairwise- D_o

Definition(4.4)
(M , σ_1 , σ_2) is said to be
i) pairwise D_1-space if and only if ∀ i , j ∈M where i≠j . ∃ σ_1-D-set U and σ_2-D-set V where U ={ i ∈U , j∉U } and V {i∉V , j ∈V }

ii) Weak pairwise D_1-space if and only if ∀ i , j ∈M where i≠j . ∃ σ_1-D-set A and σ_2-D-set B such that either i ∈ A , j∉A and j ∈ B , i∉B or i∈ B , j∉B and i∉A , j∈ A .

Theorem (4.8)

1) Every D_1 is pairwise D_o .
2) Every pairwise D_1 is weak pairwise D_o .
3) Every weak pairwise D_1 is pairwise D_o .

Proof :

1) Let $x \neq y$ in X . And (X, σ_1, σ_2) is D_1 thus there exist A and B are $\sigma_1\sigma_2$-D-sets **,** A containing x not y , B containing y not x, y ∈ B . Then A = S ∪ O and B= G ∪ W such that S and G are σ_1-D-sets , O and W are σ_2-D-sets hence there exits S is σ_1-D-set such that $x \in$ S , $y \notin$S or O is σ_2-D-set such that $x \notin$O , $y \in$O . We have X is pairwise D_o .

2) Suppose $x \neq y$ in W . And (W, σ_1, σ_2) is pairwise D_1 thus there exist A is σ_1-D-set and B is σ_2-D-set such that $x \in$A , $y \notin$A and $x \notin$B , $y \in$B . We have W is pairwise D_o .

3) Let $x \neq y$ in X . And (X , σ_1, σ_2) is weak pairwise D_1 thus there exist K is σ_1-D-set and G is σ_2-D-set such that either K={$x \in$K , $y \notin$K } and G={ $x \notin$G , $y \in$ G } or G={ $x \in$G , $y \notin$G } and K={ $x \notin$K , $y \in$ K }. Hence there exist K is σ_1-D-set such that $x \in$K , $y \notin$K or there exist G is σ_2-D-set such that $x \notin$G , $y \in$ G . We have (X, σ_1, σ_2) is pairwise D_o

Theorem (4.9)

If h:(M,σ_1,σ_2)→ (N, ρ_1, ρ_2) is pairwise continues and one to one and N is D_1 then M is D_1 .

Proof :

Similarity to Theorem (4.4)

Theorem (4.10)

If (X ,σ_1,σ_2)is waek pairwise- D_1 and A subset of X then (A ,τ_A , σ_A) is weak pairwise- D_1

Proof: clear

The following diagram explain the relations of types of D_K for K = 0 , 1, as shown in Fig. 1.

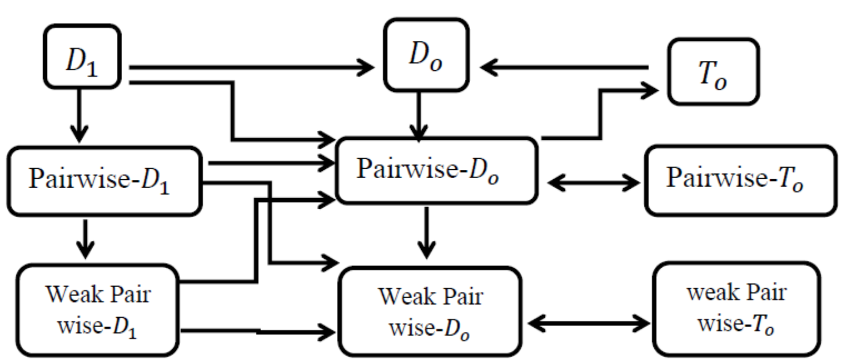

Fig. 1. Relationship of types of D_K

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