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Some Properties of Rings of the Type f(x),g(x)-clean

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ABSTRACT

In [1], the authors introduced a property of rings called the f(x),g(x)-clean property. We investigate some new results about this property. In particular, we prove that the ring of matrices over a ring R is f(x),g(x)-clean if the ring R is f(x),g(x)-clean as well. We demonstrate, among many other things, that a ring's f(x),g(x)-cleanness may always be passed to R[[x]], but not always to its polynomial ring R[x].

MSC..

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1-Introduction:

The definition of clean rings was initially provided in a 1977 publication by the author of [6]. “An element $a \in R$ of a ring R is called clean if $a = v + \eta$, where v is a unit and η is an idempotent, and R is clean if every element of it is clean”. Numerous studies have been conducted on this kind of ring due to its intriguing features see [2], [5], and [7]. Several researchers have provided a broad range of generalizations of these rings; see, for example, [3], [4], and [9]. A novel class of rings known as g(x)-clean was described by the writers of [4]. “Let R be a ring and let $g(x) \in C(R)[x]$ be a polynomial with coefficients in the center C(R) of R. An element $a \in R$ is said to be g(x)-clean if $a = v + \eta$, where v is a unit and $g(\eta) = 0$, and when every element of R is g(x)-clean, we declare R as a g(x)-clean ring”. Rings that are g(x)-clean rings and rings that possesses the clean property are closely related, as demonstrated by the authors of

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[4]. The authors of [1] have recently investigated a property known as f(x),g(x)-clean. Let f(x),g(x)∈C(R)[x] be two polynomials. If $a = v + \eta + \vartheta$, where v is a unit and f(η)=g(ϑ)=0, then the element $a \in R$ is considered to be f(x),g(x)-clean. We proclaim R to be an f(x),g(x)-clean ring when every element in R is f(x),g(x)-clean.

The purpose of this study is to explore some more characteristics and extensions of rings that are f(x),g(x)-clean. Section 1 covers these additional features, whereas Section 2 deals with extensions of f(x),g(x)-clean rings. Throughout this work, R is a ring that has an identity element 1. The following definition is relevant in Proposition 1.4.

Definition [8]: An element $a \in R$ is said to be 3-good if $a = v_1 + v_2 + v_3$ where the v_i's are unit elements, and the ring R is said to be 3-good if every element of R is 3-good.

2- New Properties of f(x),g(x)-clean Rings

First, let's look at the following lemma, which will help us build additional f(x),g(x)-clean rings. We'll also utilize it in the proof of Theorem 1.5.

Lemma 1.1: Let $f(x), g(x) \in C(R)[x]$. Then R is an f(x),g(x)-clean ring if R is f(x)-clean and g(x) has at worst one root in R.

Proof: Based on the presumption, there exists $\vartheta \in R$ where $g(\vartheta) = 0$. For an arbitrary $a \in R$, there is the following decomposition $a - \vartheta = v + \eta$, such that v is a unit while $f(\eta) = 0$ because R is f(x)-clean. Now, we have $a = v + \eta + \vartheta$ so the element a is f(x),g(x)-clean. Because the element a was arbitrary, each member of R is f(x),g(x)-clean, the proof is concluded.

Example 1.2: As mentioned above, we now use the previous lemma to construct more examples of rings that are f(x),g(x)-clean. Let $R = \left\{\frac{a}{b} \in \mathbb{Q}: \gcd(b,7) = 1\right\}$ and let $\mathbb{Z}_3$ be the group with three elements. The author of [9] showed in Theorem 3.1 that $R\mathbb{Z}_3$ is $(x^4 - x)$-clean and because this ring contains roots of the polynomial $(x^2 - x)$ (e.g. the elements 0 and 1), it follows from Lemma 1.1 that $R\mathbb{Z}_3$ is also $(x^2 - x), (x^4 - x)$-clean. If we set $a = b = c = d = 1$ and $m = 1, n = 2$ in Theorem 1.3 below, we conclude that $R\mathbb{Z}_3$ is also $(x^2 + x), (x^4 + x)$-clean.

Theorem 1.3: A ring R is $ax^{2m} - bx, cx^{2n} - dx$-clean iff R is $ax^{2m} + bx, cx^{2n} + dx$-clean, where $m, n \in \mathbb{N}$ and $a, b, c, d \in C(R)$.

Proof: Assume that the ring R is $ax^{2m} - bx, cx^{2n} - dx$-clean, and let $r \in R$. It follows that $-r = v + \eta + \vartheta$, where v is a unit and $a\eta^{2m} - b\eta = 0 = c\vartheta^{2n} - d\vartheta$, so $r = -v - \eta - \vartheta$, where now $-v$ is still a unit while $-\eta$ and $-\vartheta$ are roots of $a(-x)^{2m} - b(-x) = ax^{2m} + bx$ and $c(-x)^{2n} - d(-x) = cx^{2n} + dx$, respectively. Therefore R is $ax^{2m} + bx, cx^{2n} + dx$-clean. The other direction follows exactly from the same reasoning used above.

Proposition 1.4: Consider a ring R and let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{i=0}^{n} b_i x^i \in C(R)[x]$ such that a_0 and b_0 are unit elements. R is 3-good if it is f(x),g(x)-clean.

Proof: By assumption, we can write an arbitrary $r \in R$ as $r = v + \eta + \vartheta$, where v is a unit and η, ϑ are roots for the polynomials f(x) and g(x), respectively. Now, $f(\eta) = 0$ implies that $\eta(a_1 + \cdots + a_m\eta^{m-1}) = (a_1 + \cdots + a_m\eta^{m-1})\eta = -a_0$, because the a_i's are all central. Since $-a_0$ is a unit then so is η. The same argument shows that ϑ is also a unit so the element r is 3-good. The element r was arbitrary so the whole ring is 3-good.

The next theorem exhibits an interesting relationship between rings that has the clean property and certain rings that possesses the f(x),g(x)-clean property.

Theorem 1.5: Let 2 is a nilpotent element in a commutative ring R, and let a,b ∈ R be arbitrary elements, the claims that follow considered equivalent:

1. R possesses the clean property.
2. R possesses the $(x^2\text{-}x),(x^2\text{-}a^2)$-clean property.

3. R possesses the $(x^2\text{-}x),(x^2+b^2)$-clean property.

Proof: (1⟺ 2) If R possesses the clean property, then this means that R is (x^2-x)-clean (an idempotent η is an element such that $\eta^2=\eta$) and because the element a is a root of (x^2-a^2) we conclude from Lemma 1.1 that R possesses the $(x^2\text{-}x),(x^2\text{-}a^2)$-clean property. Conversely, assume that R is $(x^2\text{-}x),(x^2\text{-}a^2)$-clean and pick an arbitrary $r\in R$. Then by assumption we are allowed to write $r-a=v+\eta+\vartheta$, where v is a unit, $\eta^2=\eta$ and $\vartheta^2=a^2$ so $r=v+(\vartheta+a)+\eta$. Note that

$$(\vartheta+a)^2=\vartheta^2+2\vartheta a+a^2=a^2+2\vartheta a+a^2=2(a^2+\vartheta a),$$

which tells us that the element $\vartheta+a$ is a nilpotent because 2 is, therefore the element $v+(\vartheta+a)$ is a unit because it is a unit and nilpotent combination in a ring that is commutative so the element r is clean since η is an idempotent which proves the cleanness of R.

(1⟺ 3) The forward arrow is a repetition of the forward arrow of the equivalence 1⟺ 2. Conversely, assume that R is $(x^2\text{-}x),(x^2+b^2)$-clean ring and let $r\in R$. Now we can decompose $r-b$ as $r-b=v+\eta+\vartheta$ where v is a unit while $\eta^2=\eta$ and $\vartheta^2=-b^2$ therefore $r=v+(\vartheta+b)+\eta$. We have

$$(\vartheta+b)^2=\vartheta^2+2\vartheta b+b^2=-b^2+2\vartheta b+b^2=2\vartheta b,$$

which shows that the element $\vartheta+b$ is a nilpotent because 2 is a nilpotent so the element $v+(\vartheta+b)$ is a unit because it is a unit and nilpotent combination in a commutative ring. As η is idempotent, the element r is clean which indeed proves the cleanness of R and concludes the proof.

The previous type of reasoning can be used to show the impossibility of constructing certain rings with the f(x),g(x)-clean property as demonstrated in the proposition below.

Proposition 1.6: Let R be a ring that is nonzero, commutative and contains 2 as a nilpotent, then R can't be $(x^2+1),(x^2\text{-}1)$-clean.

Proof: On the contrary, suppose that R is $(x^2+1),(x^2\text{-}1)$-clean. We can write $2=v+\eta+\vartheta$, where v is a unit, $\eta^2=1$ and $\vartheta^2=-1$. Now we get $0=v+(\eta-1)+(\vartheta-1)$ with the following observation

$$(\eta-1)^2=\eta^2+2\eta+1=1+2\eta+1=2(\eta+1).$$

As was argued in the previous theorem, the element $\eta-1$ is a nilpotent. Similarly, the element $\vartheta+1$ is also nilpotent from which we deduce that the element $v+(\eta-1)+(\vartheta-1)$ is a unit because it is a sum of a unit and two nilpotents. Set this unit to be $u=v+(\eta-1)+(\vartheta-1)$ so $u=0$ hence $1=0$ a contradiction with R being a nonzero ring.

To create an f(x),g(x)-clean ring from two rings-one of which is an f(x)-clean ring and the other has the g(x)-clean property-one can utilize the direct product of two rings. Given two rings, R and S, and two polynomial $f(x)=\sum_{i=0}^{m}a_ix^i\in C(R)[x]$ and $g(x)=\sum_{i=0}^{n}b_ix^i\in C(S)[x]$, we may embed f(x) and g(x) in $C(R\times S)$ as follows: $f(x)=\sum_{i=0}^{m}(a_i,0)x^i$ and $g(x)=\sum_{i=0}^{n}(0,b_i)x^i$, respectively.

Proposition 1.7: Let R and S be two rings and f(x) and g(x) be two polynomials as discussed above. If R possesses the f(x)-clean property and S possess the g(x)-clean property, then their product $R\times S$ is a ring that is f(x),g(x)-clean.

Proof: Let $(r,s)\in R\times S$. Because R possesses the f(x)-clean property and S possess the g(x)-clean property, we may write $(r,s)=(v+\eta,u+\vartheta)$, where v is an invertible element of R, η is a zero of the polynomial f(x), u is an invertible element of S and ϑ is a zero of g(x). Therefore, we have $(r,s)=(v+\eta,u+\vartheta)=(v,u)+(\eta,0)+(0,\vartheta)$, where (u,v) is an invertible element of $R\times S$ with multiplicative inverse (v^{-1},u^{-1}), while $(\eta,0)$ and $(0,\vartheta)$ are zeros of the polynomials f(x) and g(x), in that order (recall the embedding we mentioned earlier). We concluded that (r,s) is f(x),g(x)-clean so $R\times S$ is f(x),g(x)-clean ring.

3- Extensions of f(x),g(x)-clean Rings

The current part of the paper discuses several extensions of f(x),g(x)-clean rings. Let R and S be two rings and let $\phi: R \longrightarrow S$ be a ring homomorphism. We can build a new ring homomorphism $\overline{\phi}: C(R)[x] \longrightarrow C(S)[x]$ defined by $\overline{\phi}(a_0 + a_1 x + \cdots + a_m x^m) = \phi(a_0) + \phi(a_1)x + \cdots + \phi(a_m)x^m$.

Proposition 2.1: Consider two rings R and S with a surjective ring homomorphism, $\phi: R \longrightarrow S$. Let $f(x)=a_0+a_1x+\ldots+a_mx^m, g(x)=b_0+b_1x+\ldots+b_nx^n \in C(R)[x]$. If R is f(x),g(x)-clean, then S is $f_\phi(x), g_\phi(x)$-clean where $f_\phi(x) = \overline{\phi}(f(x))$ and $g_\phi(x) = \overline{\phi}(g(x))$.

Proof: Let $s \in S$ be an arbitrary element, then $\exists r \in R$ with $\phi(r) = s$. Because R is f(x),g(x)-clean, there is a decomposition $r = v + \eta + \vartheta$ with v an invertible element and $f(\eta) = g(\vartheta) = 0$. Now, $s = \phi(r) = \phi(v + \eta + \vartheta) = \phi(v) + \phi(\eta) + \phi(\vartheta)$ where $\phi(v)$ is a unit of S because it is an image of a unit under ϕ and $\phi(\eta)$ is a root of $f_\phi(x)$ because

$$f_\phi\big(\phi(\eta)\big) = \phi(a_0) + \phi(a_1)\phi(\eta) + \cdots + \phi(a_m)(\phi(\eta))^m$$

$$= \phi(a_0 + a_1\eta + \cdots + a_m\eta^m)$$

$$= \phi(0) = 0,$$

because $f(\eta) = 0$. Similarly, $\phi(\vartheta)$ is a zero for $g_\phi(x)$ so the element s is $f_\phi(x), g_\phi(x)$-clean so the whole ring is $f_\phi(x), g_\phi(x)$-clean.

As an immediate result, we obtain the following statements.

Corollary 2.2: Consider 2 polynomials f(x) and g(x) with integers coefficients (in the subring generated by 1) and consider a family of rings $(R_\lambda)_{\lambda \in \Lambda}$. The ring $\prod_{\lambda \in \Lambda} R_\lambda$ is f(x),g(x)-clean iff each R_λ is f(x),g(x)-clean.

Corollary 2.3: Consider a ring R with an ideal, $J \subseteq R$ and let f(x),g(x) be 2 polynomials in C(R)[x]. The quotient ring R/J possesses the $f_\pi(x), g_\pi(x)$-clean property if R possesses the f(x),g(x)-clean property, where $\pi: R \longrightarrow R/J$ is the homomorphism defined by $\pi(a) = a + J$.

A partial converse of Corollary 2.3 can be constructed. Recall that if $\eta + J$ is a zero for a polynomial $f_\pi(x)$ in C(R/J)[x], then we say that $\eta + J$ lifts module J if there exist $\eta' \in R$ that is a zero for f(x) and with $\eta + J = \eta' + J$ i.e. $\eta - \eta' \in J$.

Theorem 2.4: Consider a ring R, an ideal J of R contained in the Jacobson radical, and let f(x) and g(x) be in C(R)[x]. If R/J possesses the $f_\pi(x), g_\pi(x)$-clean property where all roots (zeros) of the polynomials f(x) and g(x) lifts module J, it must be that R possesses the f(x),g(x)-clean property.

Proof: Chose some arbitrary $a \in R$, then from the assumption $a + J = (v + J) + (\eta + J) + (\vartheta + J)$, where $v + J$ is an invertible element of R/J, while $\eta + J$ and $\vartheta + J$ are roots of $f_\pi(x)$ and $g_\pi(x)$, respectively. Since roots lift module J, there exist roots η' and ϑ' in R for f(x) and g(x), respectively where $\eta + J = \eta' + J$ and $\vartheta + J = \vartheta' + J$. Now, $a + J = (v + J) + (\eta' + J) + (\vartheta' + J)$ so $(a + J) - (\eta' + J) - (\vartheta' + J) = v + J$, and as $v + J$ is a unit, then also $(a - \eta' - \vartheta') + J$ is a unit of R/J. Because the ideal J lies in the Jacobson radical, we deduce that the element $a - \eta' - \vartheta'$ is a unit of R itself so we can write $a = (a - \eta' - \vartheta') + \eta' + \vartheta'$, where the element in the brackets is a unit of R, and the second and third are roots for the polynomials f(x) and g(x), in that order. This proves the f(x),g(x)-clean property and ends the proof.

Next, we prove the f(x),g(x)-clean property of a matrix ring if its ring of entries is of the same type. Recall that the map $a \longrightarrow aI_n$ equips the n by n matrix ring with a structure of an algebra over C(R), where a is a central element of R and I_n is the identity matrix of size n by n.

Theorem 2.5: Consider a ring R with $f(x), g(x) \in C(R)[x]$. Whenever R possesses the f(x),g(x)-clean property, the ring $M_{n\times n}(R)$ possesses the f(x),g(x)-clean property for every positive integer n.

Proof: The proof of this theorem is achieved by mathematical induction as follows. If $n = 1$, the theorem is satisfied because $M_{1\times 1}(R) = R$. Assume the theorem is satisfied for $n - 1 > 1$. Let $K \in M_{n\times n}(R)$, we can put K in the form $K = \begin{bmatrix} K' & A \\ B & k \end{bmatrix}$, where $K' \in M_{n-1\times n-1}(R)$, A is an $n - 1$ column, B is an $n - 1$ row and $k \in R$. Because $M_{n-1\times n-1}(R)$ possesses the f(x),g(x)-clean property, there is an expression $K' = V + H + \Theta$, where V is a unit (invertible matrix) of $M_{n-1\times n-1}(R)$, H is a zero for f(x) in $M_{n-1\times n-1}(R)$ and Θ is a zero for g(x) in $M_{n-1\times n-1}(R)$. Note also that $k - BV^{-1}A \in R$ so by the f(x),g(x)-cleanness of R we can write $k - BV^{-1}A = v + \eta + \vartheta$, where v is an invertible element and $\eta, \vartheta \in R$ such that $f(\eta) = g(\vartheta) = 0$. Putting everything in the described formula for K we get

$$K = \begin{bmatrix} V & A \\ B & v + BV^{-1}A \end{bmatrix} + \begin{bmatrix} H & 0 \\ 0 & \eta \end{bmatrix} + \begin{bmatrix} \Theta & 0 \\ 0 & \vartheta \end{bmatrix}.$$

Note that the second and third matrices are roots of f(x) and g(x), respectively so we only need to show that the first matrix is invertible. One can easily see that

$$\begin{bmatrix} V & A \\ B & v + BV^{-1}A \end{bmatrix} = \begin{bmatrix} I_{n-1} & 0 \\ -BV^{-1} & 1 \end{bmatrix}^{-1} \begin{bmatrix} V & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} I_{n-1} & -V^{-1}A \\ 0 & 1 \end{bmatrix}^{-1},$$

a product of invertible matrices so the first matrix in the decomposition of K is invertible proving the f(x),g(x)-cleanness of K and hence of $M_{n\times n}(R)$. The result is satisfied for all $n \geq 1$.

Example 2.6: By direct computation, one can easily show that the ring $\mathbb{Z}_5$ is $(x^2-3x+2),(x^2-4x+3)$-clean so it follows from Theorem 2.5, the ring of matrices $M_{n\times n}(\mathbb{Z}_5)$ is $(x^2-3x+2),(x^2-4x+3)$-clean for all $n \geq 1$.

We can consider another ring of matrices namely the trivial extension. Consider an R-module N where the ring R is commutative. We construct a ring called the trivial extension of R by N written as R(N) which is the collection

$$R(N) = \left\{ \begin{bmatrix} a & n \\ 0 & a \end{bmatrix} : a \in R \text{ and } n \in N \right\},$$

with the typical and intuitive addition and matrix multiplication. We associate with such ring a homomorphism $\Phi: R[x] \longrightarrow R(N)[x]$ that identify polynomials in the ring R[x] with their images in the ring R(N)[x] and which is defined as

$$\Phi(a_0 + a_1 x + \cdots + a_m x^m) = \begin{bmatrix} a_0 & 0 \\ 0 & a_0 \end{bmatrix} + \begin{bmatrix} a_1 & 0 \\ 0 & a_1 \end{bmatrix} x + \cdots + \begin{bmatrix} a_m & 0 \\ 0 & a_m \end{bmatrix} x^m.$$

Proposition 2.7: Consider a ring R which is commutative with some $f(x),g(x) \in C(R)[x]$. R possesses the f(x),g(x)-clean property iff R(N) possesses the same property.

Proof: Assume that R(N) is f(x),g(x)-clean. The set $\left\{ J = \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix} \text{ with } n \in N \right\}$ is evidently an ideal in the ring R(N) hence $R(N)/J \cong R$ possesses the f(x),g(x)-clean property by Corollary 2.3. Conversely, pick some $\begin{bmatrix} a & n \\ 0 & a \end{bmatrix} \in R(N)$. By the f(x),g(x)-cleanness of R, there is a decomposition $a = v + \eta + \vartheta$, where v is a unit of R, and η,ϑ are roots of f(x) and g(x), respectively. So we end up with the following decomposition

$$\begin{bmatrix} a & n \\ 0 & a \end{bmatrix} = \begin{bmatrix} v & n \\ 0 & v \end{bmatrix} + \begin{bmatrix} \eta & 0 \\ 0 & \eta \end{bmatrix} + \begin{bmatrix} \vartheta & 0 \\ 0 & \vartheta \end{bmatrix},$$

where the second and third matrices are the zeros for f(x) and g(x), at that order while the first matrix is obviously a unit. The proof is achieved.

Lastly, we will show that the ring of power series R[[x]] over a given ring R possesses the f(x),g(x)-clean property whenever R possesses this same property. On the other hand, we can find polynomials f(x) and g(x) for which we get R is f(x),g(x)-clean but the ring of polynomials over the ring R is not.

Proposition 2.8: Consider a ring R and let f(x),g(x)∈C(R)[x]. R possesses the f(x),g(x)-clean property iff R[[x]] possesses this same property.

Proof: Assume that R is f(x),g(x)-clean, and pick some $a_0 + a_1x + a_2x^2 + \cdots \in R[[x]]$. We have $a_0 = v + \eta + \vartheta$, where v is a unit and $f(\eta) = g(\vartheta) = 0$. Therefore, $a_0 + a_1x + a_2x^2 + \cdots = (v + a_1x + a_2x^2 + \cdots) + \eta + \vartheta$. It is known that power series with unit constant terms are themselves units, so $(v + a_1x + a_2x^2 + \cdots)$ is a unit, proving the f(x),g(x)-cleanness of R. Conversely, when R[[x]] is f(x),g(x)-clean, then the map $a_0 + a_1x + a_2x^2 + \cdots \longrightarrow a_0$ is a surjective ring morphism onto R so R possesses the f(x),g(x)-clean property by Proposition 2.1.

Proposition 2.9: Consider a ring R that is commutative and $(x^m\text{-}x),(x^n\text{-}x)$-clean, for a pair $m, n \in \mathbb{N}$, then the ring of polynomials R[x] over R is not an $(x^m\text{-}x),(x^n\text{-}x)$-clean ring.

Proof: With the aim of obtaining a contradiction, say R[x] possesses the $(x^m\text{-}x),(x^n\text{-}x)$-clean property. We can write x as $x = v + \eta + \vartheta$, where v is a unit and $f(\eta) = g(\vartheta) = 0$. As in Example 3.2 of [9] the elements η and ϑ are in R, so $v = a + x$, where $a = -\eta - \vartheta$ is a scalar. Since v is an invertible element and $v = a + x$, we conclude that the element $a + x$ is a invertible, but this is a contradiction since units in R[x] must have nilpotent coefficients for all terms except the constant term (Note that the coefficient of x is 1 which is not a nilpotent).

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