Journal of Al-Qadisiyah for Computer Science and Mathematics VOL.16(1) 2024, pp Math. 29–33

JQCM
Journal Of Al-Qadisiyah For Computer Science And Mathematics
University Of AL-Qadisiyah

Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

Triple-Zero On Weakly Pseudo Primary 2-Absorbing Sub-Module

Omar Hisham Taha [a] , Marrwa Abdullah Salih[b]

[a] *Department of Mathematics, College of Computer Science and Mathematics, Univ. of Tikrit, Iraq .Email: omar.h.tahamm2314@st.tu.edu.iq*

[b] *Department of Mathematics, College of Education for Pure Sciences, Univ. of Tikrit, Iraq .Email: marrwaabdallah1984@tu.edu.iq*

A R T I C L E I N F O

Article history:
Received: 17 /1/2024
Rrevised form: 4 /3/2024
Accepted : 12 /3/2024
Available online: dd /mm/year

Keywords:

triple-zero
free triple-zero
Multiplication Modules.
weakly pseudo-primary-2-absorbing sub-module.

Each keyword to start on a new line

A B S T R A C T

In this paper, we aim to show many results from the weakly pseudo-primary-2-absorbing sub-module, including triple-zero of the weakly pseudo-primary-2-absorbing sub-module. Furthermore, weakly pseudo-primary-2-absorbing triple-zero have been characterized in some module types, including content, finitely generated, and multiplication modules.

MSC..

https://doi.org/10.29304/jqcsm.2024.16.11446

1. Introduction

Unless otherwise noted, all rings in this paper are commutative with identity, and all modules are unitary left modules. Suppose that R be a ring and $\mathbb{D}$ be an R-module. During the last 13 years, the notion of 2-absorbing sub-modules and weakly 2-absorbing sub-modules has been previously investigated by Darani and Soheilinia [1], and its generalizations to 2-absorbing ideal, where 2-absorbing ideal was first introduced in 2007 by Badawi A. [2] . A proper sub-module A of an R–module D is called 2-absorbing (weakly 2-absorbing) if whenever $r_1r_2d \in A$ $(0 \neq r_1r_2d \in A)$ for some $r_1, r_2 \in R$, $d \in D$, then $r_1d \in A$ or $r_2d \in$ A or $r_1r_2 \in [A:_R D]$. Following that, Mostafanasab, Yetkin, Tekir and Daran [3] introduced the concept of primary- 2-absorbing sub-module and weakly primary- 2-absorbing sub-module as generalizations of primary sub-module and weakly primary sub-module, respectively, a proper sub-module A of an R–module D is called a primary-2-absorbing, if $r_1r_2d \in A$, for $r_1,r_2 \in$R, $d \in D$, implies that either $r_1d \in rad_D(A)$ or $r_2d \in rad_D(A)$ or $r_1r_2 \in [A + Soc(D):_R D]$. The concept pseudo-primary-2-absorbing was introduced recently in [4]. A sub-module $A \notin \mathbb{D}$ of an R–module $\mathbb{D}$, is called a pseudo-primary-2-absorbing sub-module $\mathbb{D}$, if $r_1r_2d \in A$, for $r_1,r_2\in$R, $d \in \mathbb{D}$, implies that either $r_1d \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ or $r_2d \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ or $r_1r_2 \in [A + Soc(\mathbb{D}):_R \mathbb{D}]$. The concept of weakly pseudo-primary-2-absorbing sub-module was introduced recently in [5]. A sub-module $A \notin \mathbb{D}$

∗Corresponding author : Omar Hisham Taha

Email addresses: omar.h.tahamm2314@st.tu.edu.iq

Communicated by ‘sub etitor’

of an R–module $\mathbb{D}$, is called a weakly pseudo-primary-2-absorbing sub-module $\mathbb{D}$, if $0 \neq r_1 r_2 d \in A$, for $r_1, r_2 \in R$, $d \in \mathbb{D}$, implies that either $r_1 d \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ or $r_2 d \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ or $r_1 r_2 \in [A + Soc(\mathbb{D}):_R \mathbb{D}]$. An ideal Q of a ring R is said to be a weakly pseudo-primary-2-absorbing ideal of R if Q is a weakly pseudo-2-absorbing R–submodule an R–module R. Where the residual of A be $\mathbb{D}$ and denoted by $[A:_R \mathbb{D}] = \{r \in R: rD \subseteq A\}$ [6]. And the radical of A is the intersection of all prime sub-modules of an R-module $\mathbb{D}$ containing A, denoted by $rad_{\mathbb{D}}(A)$ [6]. In particular the ideal $[0:_R \mathbb{D}] = \{r \in R: rD = (0)\}$ is called annihilator of $\mathbb{D}$ and is denoted by $Ann_R(\mathbb{D})$ [7]. An R-module $\mathbb{D}$ is multiplication if every sub-module A of $\mathbb{D}$ is from $A = QD$ for some ideal Q of R. Equivalent to $A = [A:_R \mathbb{D}]\mathbb{D}$ [8]. Recall an R-module $\mathbb{D}$ is faithful if $Ann_R(\mathbb{D}) = (0)$ [7].

In section two, we introduce the concept of the treble zero on a weakly pseudo-primary-2-absorbing sub-module. Also, we show If A be a weakly pseudo-primary-2-absorbing sub-module of an R-module $\mathbb{D}$ with (r_1, r_2, d), is weakly pseudo-primary-2-absorbing triple-zero for some $r_1 r_2 \in R$, $d \in \mathbb{D}$. Then, the following hold

1. $r_1 r_2 A = (0)$
2. $[A:_R \mathbb{D}] r_1 d = [A:_R \mathbb{D}] r_2 d = (0)$
3. $r_1 [A:_R \mathbb{D}] d = r_2 [A:_R \mathbb{D}] d = (0)$
4. $[A:_R \mathbb{D}] r_1 A = [A:_R \mathbb{D}] r_2 A = r_1 [A:_R \mathbb{D}] A = r_2 [A:_R \mathbb{D}] A = (0)$
5. $[A:_R \mathbb{D}][A:_R \mathbb{D}] A = [A:_R \mathbb{D}]^2 A = (0)$, In particular $[A:_R \mathbb{D}]^3 \subseteq Ann_R(\mathbb{D})$

See Theorem 2.3, and steady the concept in some types of modules.

2. Treble zero on weakly pseudo-primary-2-absorbing

Definition 2.1. Let A be a weakly pseudo-primary-2-absorbing sub-module of an R- module $\mathbb{D}$. And $r_1, r_2 \in R$, $d \in \mathbb{D}$ we say (r_1, r_2, d) is weakly pseudo-primary-2-absorbing triple-zero of A if $r_1 r_2 d = 0$, $r_1 d \notin rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $r_2 d \notin rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $r_1 r_2 \notin [A + Soc(\mathbb{D}):_R \mathbb{D}]$.

Example 2.2.

1. Suppose that $R = Z_3$, $D = Z_{81}$, and the sub-module $A = (0)$, (9,9,1) is weakly pseudo-primary-2-absorbing triple-zero of A since A is weakly pseudo-primary-2-absorbing by definition with $9.9.1 = 0 \in A$, $9.1 = 9 \notin rad_{\mathbb{D}}(A) + Soc(\mathbb{D}) = 0 + (27) = (27)$ and $9.9 = 81 \notin [A + Soc(\mathbb{D}):_R \mathbb{D}] = 27Z_3$.
2. It's clear that $12\mathbb{Z}$ is a proper sub-module of $\mathbb{Z}$ -module $\mathbb{Z}$, $12\,\mathbb{Z}$ is weakly pseudo-primary-2-absorbing sub-module of $\mathbb{Z}$ -module $\mathbb{Z}$, but has not weakly pseudo-primary-2-absorbing triple-zero

Remark 2.3. Let A is a weakly pseudo-primary-2-absorbing sub-module of an R- module $\mathbb{D}$. Then, the following holds.

1. If A is not a pseudo-primary-2-absorbing sub-module of an R- module $\mathbb{D}$. Then A has triple-zero.
2. If A has not triple-zero. Then A pseudo-primary-2-absorbing sub-module of an R- module $\mathbb{D}$.

Proof. It is clear by Definition 2.1 □

Example 2.4

3. By Example 2.2. (1) A is not pseudo-primary-2-absorbing sub-module then A has triple-zero.
4. By Example 2.2. (2) $12\mathbb{Z}$ has not triple-zero then $12\mathbb{Z}$ is a pseudo-primary-2-absorbing sub-module .

Theorem 2.5. Let A be a weakly pseudo-primary-2-absorbing sub-module of an R-module $\mathbb{D}$ with (r_1, r_2, d), is weakly pseudo-primary-2-absorbing triple-zero for some $r_1 r_2 \in R$, $d \in \mathbb{D}$. Then, the following holds.

1. $r_1 r_2 A = (0)$.

2. $[A:_R \mathbb{D}]r_1 d = [A:_R \mathbb{D}]r_2 d = (0)$.
3. $r_1[A:_R \mathbb{D}]d = r_2[A:_R \mathbb{D}]d = (0)$.
4. $r_1[A:_R \mathbb{D}]d = r_2[A:_R \mathbb{D}]d = (0)$.
5. $[A:_R \mathbb{D}]r_1 A = [A:_R \mathbb{D}]r_2 A = r_1[A:_R \mathbb{D}]A = r_2[A:_R \mathbb{D}]A = (0)$.
6. $[A:_R \mathbb{D}][A:_R \mathbb{D}]A = [A:_R \mathbb{D}]^2 A = (0)$, *In particular* $[A:_R \mathbb{D}]^3 \subseteq Ann_R(\mathbb{D})$.

Proof.

(1) Assume that $r_1 r_2 A \neq (0)$, then $r_1 r_2 a \neq 0$ for some $a \in A$ since (r_1, r_2, d). is weakly pseudo-primary-2-absorbing triple-zero of A then $r_1 r_2 d = o$, $r_1 d \notin rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $r_2 d \notin rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $r_1 r_2 \notin [A + Soc(\mathbb{D}):_R \mathbb{D}]$. Since $0 \neq r_1 r_2 a \in A$ and A be a weakly pseudo-primary-2-absorbing sub-module, and $r_1 r_2 \notin [A + Soc(\mathbb{D}):_R \mathbb{D}]$, then either $r_1 a \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$, or $r_2 a \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ Now, $0 \neq r_1 r_2 (d + a) = r_1 r_2 d + r_1 r_2 a = r_1 r_2 a \in A$ and $r_1 r_2 \notin [A + Soc(\mathbb{D}):_R \mathbb{D}]$, since A be a weakly pseudo-primary-2-absorbing, then either $r_1(d + a) = r_1 d + r_1 a \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ or $r_2(d + a) = r_2 d + r_1 a \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$. If $r_1(d + a) = r_1 d + r_1 a \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $r_1 a \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$. We get $r_1 d \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$, its contradiction. If $r_2(d + a) = r_2 d + r_1 a \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $r_2 a \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$. We get $r_2 d \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$, its contradiction. Hence $r_1 r_2 A = (0)$.

(2) Assume that $[A:_R \mathbb{D}]r_1 d \neq (0)$, then $x r_1 d \neq 0$ for some $x \in [A:_R \mathbb{D}]$ since (r_1, r_2, d) is weakly pseudo-primary-2-absorbing triple-zero of A then $r_1 r_2 d = o$, $r_1 d \notin rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $r_2 d \notin rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $r_1 r_2 \notin [A + Soc(\mathbb{D}):_R \mathbb{D}]$. Since $0 \neq x r_1 d \in A$ and A be a weakly pseudo-primary-2-absorbing sub-module, then either $xd \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$, or $r_1 d \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ or $x r_1 \in [A + Soc(\mathbb{D}):_R \mathbb{D}]$ Now, $0 \neq (r_2 + x) r_1 d = r_1 r_2 d + x r_1 d = x r_1 d \in A$, since A be a weakly pseudo-primary-2-absorbing then either $(r_2 + x)d = r_2 d + xd \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ or $r_1 d \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ (its contradiction) or $(r_2 + x) r_1 \in [A + Soc(\mathbb{D}):_R \mathbb{D}]$. If $(r_2 + x)d = r_2 d + xd \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $xd \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$, we get $r_2 d \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$,its contradiction. If $(r_2 + x) r_1 \in [A + Soc(\mathbb{D}):_R \mathbb{D}]$ and $x r_1 \in [A + Soc(\mathbb{D}):_R \mathbb{D}]$, we get $r_1 r_2 \in [A + Soc(\mathbb{D}):_R \mathbb{D}]$,its contradiction. Hence $[A:_R \mathbb{D}]r_1 d = (0)$.

Similarly, we can prove $[A:_R \mathbb{D}]r_2 d = (0)$.

(3) Assume that $r_1[A:_R \mathbb{D}]d \neq (0)$, then $r_1 x d \neq 0$ for some $x \in [A:_R \mathbb{D}]$ since (r_1, r_2, d) is weakly pseudo-primary-2-absorbing triple-zero of A then $r_1 r_2 d = o$, $r_1 d \notin rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $r_2 d \notin rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $r_1 r_2 \notin [A + Soc(\mathbb{D}):_R \mathbb{D}]$. Since $0 \neq r_1 x d \in A$ and A be a weakly pseudo-primary-2-absorbing sub-module, then either $r_1 d \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$, or $xd \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ or $r_1 x \in [A + Soc(\mathbb{D}):_R \mathbb{D}]$ Now, $0 \neq r_1(x + r_2)d = r_1 x d + r_1 r_2 d = r_1 x d \in A$, since A be a weakly pseudo-primary-2-absorbing then either $r_1 d \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ (its contradiction) or $(x + r_2)d = xd + r_2 d \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ or $r_1(x + r_2)d = r_1 x + r_1 r_2 \in [A + Soc(\mathbb{D}):_R \mathbb{D}]$. If $(x + r_2)d = xd + r_2 d \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $xd \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$, we get $r_2 d \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$,its contradiction. If $r_1(x + r_2)d = r_1 x + r_1 r_2 \in [A + Soc(\mathbb{D}):_R \mathbb{D}]$. and $r_1 x \in [A + Soc(\mathbb{D}):_R \mathbb{D}]$, we get $r_1 r_2 \in [A + Soc(\mathbb{D}):_R \mathbb{D}]$,its contradiction. Hence $r_1[A:_R \mathbb{D}]d = (0)$.

Similarly, we can prove $r_2[A:_R \mathbb{D}]d = (0)$.

(4) Assume that $[A:_R \mathbb{D}]r_1 A \neq (0)$, then $r_1 x a \neq 0$ for some $x \in [A:_R \mathbb{D}]$ and $a \in a$. Since (r_1, r_2, d) is weakly pseudo-primary-2-absorbing triple-zero of A then $r_1 r_2 d = o$, $r_1 d \notin rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $r_2 d \notin rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $r_1 r_2 \notin [A + Soc(\mathbb{D}):_R \mathbb{D}]$. Since $0 \neq r_1 x a \in A$ and A be a weakly pseudo-primary-2-absorbing sub-module, then either $r_1 a \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$, or $xa \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ or $r_1 x \in [A + Soc(\mathbb{D}):_R \mathbb{D}]$ Now, $0 \neq (r_2 + x) r_1 (a + d) = r_1 r_2 a + r_1 r_2 d + x r_1 a + x r_1 d = x r_1 a \in A$, (because $r_1 r_2 d = 0$ and $r_1 r_2 a = 0$,$x r_1 a = 0$ by (1). and $x r_1 d = 0$ by (2)). since A be a weakly pseudo-primary-2-absorbing then either $(r_2 + x)(a + d) = r_2 a + r_2 d + xa + xd \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ (its contradiction) or $r_1(a + d) = r_1 a + r_1 d \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ (its contradiction) or $(r_2 + x) r_1 = r_1 r_2 + x r_1 \in [A + Soc(\mathbb{D}):_R \mathbb{D}]$.(its contradiction) Hence $[A:_R \mathbb{D}]r_1 A = (0)$. Similarly, we can prove $[A:_R \mathbb{D}]r_2 A = r_1[A:_R \mathbb{D}]A = r_2[A:_R \mathbb{D}]A = (0)$.

(5) Assume that $[A:_R \mathbb{D}][A:_R \mathbb{D}]A \neq (0)$, then $x_1 x_2 a \neq 0$ for some $x_1, x_2 \in [A:_R \mathbb{D}]$ and $a \in A$. Since (r_1, r_2, d) is weakly

pseudo-primary-2-absorbing triple-zero of A then $r_1r_2d = o$, $r_1d \notin rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $r_2d \notin rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $r_1r_2 \notin [A + Soc(\mathbb{D}):_R \mathbb{D}]$. Since $0 \neq x_1x_2a \in A$ and A be a weakly pseudo-primary-2-absorbing sub-module, then either $x_1a \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$, or $x_2a \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ or $x_1x_2 \in [A + Soc(\mathbb{D}):_R \mathbb{D}]$ Now, $0 \neq (x_1 + r_2)(x_2 + r_1)(a + d) = x_1x_2a + x_1x_2d + x_1r_1a + x_1r_1d + r_2x_2a + r_2x_2d + r_2r_1a + r_2r_1d = x_1x_2a \in A$, (because $r_1r_2d = 0$ and $r_1r_2a = 0$ by (1), $x_1r_1a = 0$, r_2x_2a by (4), and $x_1r_1d = 0$ (2)). since A be a weakly pseudo-primary-2-absorbing then either $((x_1 + r_2)(a + d) = x_1a + x_1d + r_2a + r_2d \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ (its contradiction) or $(x_2 + r_1)(a + d) = x_2a + x_2d + r_1a + r_1d \in rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ (its contradiction) or $(x_1 + r_2)(x_2 + r_1) = x_1x_2 + x_1r_1 + r_2x_2 + r_2r_1 \in [A + Soc(\mathbb{D}):_R \mathbb{D}]$. (its contradiction).

Then we get $[A:_R \mathbb{D}]\,[A:_R \mathbb{D}]A = [A:_R \mathbb{D}]^2A = (0)$. We get $[A:_R \mathbb{D}]^3 \subseteq [[A:_R \mathbb{D}]^2A:_R \mathbb{D}] = [(0):_R \mathbb{D}] = Ann(\mathbb{D})$. □

Proposition 2.6. Let $\mathbb{D}$ be a multiplication R-module and A is a weakly pseudo-primary-2-absorbing sub-module of an R-module $\mathbb{D}$ that is not a pseudo-primary-2-absorbing sub-module. Then $A^3 = (0)$.

Proof. We have that $[A:_R \mathbb{D}]\mathbb{D} = A$ since $\mathbb{D}$ be a multiplication R-module. Then $A^3 = [A:_R \mathbb{D}]^3D = [A:_R \mathbb{D}]^2A = (0)$. (by Theorem 3(5)). Consequently $A^3 = (0)$. □

Proposition 2.7. Let $\mathbb{D}$ be a faithful multiplication R-module, and $A \neq \mathbb{D}$ be a sub-module of $\mathbb{D}$. If A be a weakly pseudo-primary-2-absorbing sub-module of an R-module $\mathbb{D}$, but it's not a pseudo-primary-2-absorbing sub-module. Then $A \subseteq rad_{\mathbb{D}}(0)$. In addition $rad_{\mathbb{D}}(N) = rad_{\mathbb{D}}(0)$.

Proof. by Theorem 2.5 (5) $[A:_R \mathbb{D}]^3 \subseteq Ann(\mathbb{D})$, since $\mathbb{D}$ is faithful, then $[A:_R \mathbb{D}]^3 = 0$. Assume that $x \in [A:_R \mathbb{D}]$, then $x^3 = 0$, so $a \in \sqrt{0}$. Hence $[A:_R \mathbb{D}] \subseteq \sqrt{0}$ and thus $A = [A:_R \mathbb{D}]\mathbb{D} \subseteq rad_{\mathbb{D}}(0)$ (since $\mathbb{D}$ is multiplication) In addition, by Theorem 2.5 (5), $[A:_R \mathbb{D}]^2A = (0)$. Then $[A:_R \mathbb{D}]^3 = [A:_R \mathbb{D}]^2[A:_R \mathbb{D}] \subseteq [[A:_R \mathbb{D}]^2D:_R \mathbb{D}] = Ann(\mathbb{D})$ and so $[A:_R \mathbb{D}] \subseteq \sqrt{Ann(\mathbb{D})}$. Thus $\sqrt{[A:_R \mathbb{D}]} = \sqrt{Ann(\mathbb{D})}$. Hence $rad_{\mathbb{D}}(N) = \sqrt{[A:_R \mathbb{D}]}\mathbb{D} = \sqrt{Ann(\mathbb{D})}\mathbb{D} = rad_{\mathbb{D}}(0)$ □

Lemma 2.8. Let A is a weakly pseudo-primary-2-absorbing sub-module of an R-module $\mathbb{D}$ and let $r_1r_2T \subseteq A$ for some $r_1, r_2 \in R$ and some sub-module T of $\mathbb{D}$ such that (r_1, r_2, t) is not weakly pseudo-primary-2-absorbing triple-zero for every $t \in T$. If $r_1r_2 \notin [A + Soc(\mathbb{D}):_R \mathbb{D}]$, then either $r_1T \subseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ or $r_2T \subseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$.

Proof. Assume that (r_1, r_2, t) is not weakly pseudo-primary-2-absorbing triple-zero for every $t \in T$. Let $r_1T \nsubseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$, and $r_2T \nsubseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$. Then $r_1t \notin rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $r_2t \notin rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ for some $t \in T$. If $0 \neq r_1r_2t \in A$ with $r_1r_2 \notin [A + Soc(\mathbb{D}):_R \mathbb{D}]$ and $r_1t \notin rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$,and $r_2t \notin rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ it contradiction since A be a weakly pseudo-primary-2-absorbing sub-module. If $r_1r_2t = 0$ with $r_1t \notin rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$,$r_2t \notin rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$, $r_1r_2 \notin [A + Soc(\mathbb{D}):_R \mathbb{D}]$ and (r_1, r_2, t) is not weakly pseudo-primary-2-absorbing triple-zero, that is a contradiction. Hence either $r_1T \subseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ or $r_2T \subseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ □

Definition 2.9. Let A be a weakly pseudo-primary-2-absorbing sub-module of an R-module $\mathbb{D}$ and let $(0) \neq Q_1Q_2T \subseteq A$ for some Q be an ideal of R and some sub-module T of $\mathbb{D}$. A is called weakly pseudo-primary-2-absorbing free triple-zero in regard to Q_1, Q_2, T if (q_1, q_2, t) is not triple-zero for every $q_1 \in Q_1$,$q_2 \in Q_2$ and $t \in T$

Theorem 2.10. Let A is a weakly pseudo-primary-2-absorbing sub-module of an R-module $\mathbb{D}$, and $(0) \neq Q_1Q_2T \in A$, for some ideal Q_1, Q_2 in R, some sub-module T of $\mathbb{D}$, where A is free triple-zero in regard to Q_1, Q_2, T. Then either $Q_1T \subseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ or $Q_2T \subseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ or $Q_1Q_2 \subseteq [A + Soc(\mathbb{D}):_R \mathbb{D}]$.

Proof. Assume that A be a weakly pseudo-primary-2-absorbing sub-module of an R-module $\mathbb{D}$, and $(0) \neq Q_1Q_2T \in A$, for some ideal Q_1, Q_2 in R, some sub-module T of $\mathbb{D}$, where A is free triple-zero in regard to Q_1, Q_2, T. Suppose that $Q_1Q_2 \nsubseteq [A + Soc(\mathbb{D}):_R \mathbb{D}]$. Now, we must show that either $Q_1T \subseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ or $Q_2T \subseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$. Let $Q_1T \nsubseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $Q_2T \nsubseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$. Then there exist $x \in Q_1$ and $y \in Q_2$ such that $xT \nsubseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $yT \nsubseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$. Where $x \in Q_1$ and $y \in Q_2$. Since A be a weakly pseudo-primary-2-absorbing sub-module and by Lemma 2.6, we get $xy \in [A + Soc(\mathbb{D}):_R \mathbb{D}]$ since $xy\mathbb{D} \subseteq A + Soc(\mathbb{D})$, $xT \nsubseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $yT \nsubseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$. By our assumption $(0) \neq Q_1Q_2T \in A$, there exist $a \in Q_1$ and $b \in Q_2$, with $ab \notin [A + Soc(\mathbb{D}):_R \mathbb{D}]$.by Lemma 2.6, we get either $aT \subseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ or $bT \subseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$, as $(0) \neq abT \in A$ and $ab \notin [A + Soc(\mathbb{D}):_R \mathbb{D}]$. We investigate (3) cases.

Case 1: Assume that $aT \subseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $bT \nsubseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$. Since $xbT \subseteq A$ and $xT \nsubseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $bT \nsubseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$. by Lemma 2.8, we get $xb \notin [A + Soc(\mathbb{D}):_R \mathbb{D}]$. We have $(x + a)T \nsubseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ as $(0) \neq (x + a)bT \in A$, $xT \nsubseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$ and $aT \subseteq rad_{\mathbb{D}}(A) + Soc(\mathbb{D})$. Since $bT \nsubseteq rad_{\mathbb{D}}(A) +$

$Soc(\mathbb{D})$ and $(x+a)T \nsubseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$, then we obtain $b(x+a) \in [A+Soc(\mathbb{D}):_R \mathbb{D}]$, by Lemma 2.8. Thus since $b(x+a)=bx+ba \in [A+Soc(\mathbb{D}):_R \mathbb{D}]$ and $xb \in [A+Soc(\mathbb{D}):_R \mathbb{D}]$, then $xb \in [A+Soc(\mathbb{D}):_R \mathbb{D}]$, which is a contradiction .

Case 2: Assume that $aT \nsubseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$ and $bT \subseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$. It is clearly shown similarly to case (1).

Case 3: Assume that $aT \subseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$ and $bT \subseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$. Then $(y+b)T \nsubseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$ as $yT \nsubseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$ and $bT \subseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$.by Lemma 2.8, we get $x(y+b) \in [A+Soc(\mathbb{D}):_R \mathbb{D}]$ since $(0) \neq x(y+b)T \in A, xT \nsubseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$ and $(y+b)T \nsubseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$. Then $xb \in [A+Soc(\mathbb{D}):_R \mathbb{D}]$ since $x(y+b) \in [A+Soc(\mathbb{D}):_R \mathbb{D}]$ and $xy \in [A+Soc(\mathbb{D}):_R \mathbb{D}]$. As $aT \nsubseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$ and $xT \nsubseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$, then $(x+a)T \nsubseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$. As $(0) \neq (x+a)yT \subseteq A$, and $xT \nsubseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$, then $(x+a)T \nsubseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$. Since $(0) \neq (x+a)yT \subseteq A$, $yT \nsubseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$ and $(x+a)T \nsubseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$. by Lemma 2.8, we get $(x+a)y = xy+ay \in [A+Soc(\mathbb{D}):_R \mathbb{D}]$. Since $xy \in [A+Soc(\mathbb{D}):_R \mathbb{D}]$ and $xy+ay \in [A+Soc(\mathbb{D}):_R \mathbb{D}]$ then $ay \in [A+Soc(\mathbb{D}):_R \mathbb{D}]$. by Lemma 2.6, we get $(x+a)(y+b) = xy+xb+ay+ab \in [A+Soc(\mathbb{D}):_R \mathbb{D}]$, since $(0) \neq (x+a)(y+b)T \subseteq A, (x+a)T \nsubseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$ and $(y+b)T \nsubseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$. As $xy, xb, ay, ab \in [A+Soc(\mathbb{D}):_R \mathbb{D}]$, then $xy+xb+ay+ab \in [A+Soc(\mathbb{D}):_R \mathbb{D}]$. Thus $ab \in [A+Soc(\mathbb{D}):_R \mathbb{D}]$, since $xy+xb+ay+ab \in [A+Soc(\mathbb{D}):_R \mathbb{D}]$ and $xy+xb+ay+ab \in [A+Soc(\mathbb{D}):_R \mathbb{D}]$ a contradiction . Hence either $Q_1T \subseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$ or $Q_2T \subseteq rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$ □

Proposition 2.11. [5] Let A is a weakly pseudo-primary-2-absorbing sub-module of an R-module $\mathbb{D}$ with $\mathbb{D}$ is semisimple, and T is sub-module of $\mathbb{D}$ with $T \subseteq A$, then $\frac{A}{T}$ is weakly pseudo-primary-2-absorbing sub-module of an R-module $\frac{\mathbb{D}}{T}$.

Proposition 2.12. Let A is a weakly pseudo-primary-2-absorbing sub-module of an R-module $\mathbb{D}$ with $\mathbb{D}$ is semisimple and T is sub-module of $\mathbb{D}$ with $T \subseteq A$, then for any $r \in R$ and $d \in \mathbb{D}$, (r_1, r_2, d) be a triple-zero of A if and only if $(r_1, r_2, d+T)$ is a triple-zero of $\frac{A}{T}$.

Proof. Assume that (r_1, r_2, d) be a triple-zero of A for some $r \in R$ and $d \in \mathbb{D}$. Then $r_1r_2d = o, r_1d \notin rad_{\mathbb{D}}(A)+(\mathbb{D})$ and $r_2d \notin rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$ and $r_1r_2 \notin [A+Soc(\mathbb{D}):_R \mathbb{D}]$. by Proposition 2.11, we get that $\frac{A}{T}$ is weakly pseudo-primary-2-absorbing sub-module of an R-module $\frac{\mathbb{D}}{T}$. Thus $r_1r_2(d+T) = T$, $r_1d \notin rad_{\mathbb{D}}\left(\frac{A}{T}\right)+Soc\left(\frac{\mathbb{D}}{T}\right)$ and $r_2d \notin rad_{\mathbb{D}}\left(\frac{A}{T}\right)+Soc\left(\frac{\mathbb{D}}{T}\right)$ and $r_1r_2 \notin \left[\frac{A}{T}+Soc\left(\frac{\mathbb{D}}{T}\right):_R \frac{\mathbb{D}}{T}\right]$. Hence $(r_1, r_2, d+T)$ be a triple-zero of $\frac{A}{T}$.

Conversely, Assume that $(r_1, r_2, d+T)$ be a triple-zero of $\frac{A}{T}$. Suppose that $r_1r_2d \neq o$. Then $r_1r_2d \in A$ since $r_1r_2(d+T) = T$. Thus $r_1d \in rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$ or $r_2d \in rad_{\mathbb{D}}(A)+Soc(\mathbb{D})$ or $r_1r_2 \in [A+Soc(\mathbb{D}):_R \mathbb{D}]$ as A be a weakly pseudo-primary-2-absorbing, a contradiction. So it must be $r_1r_2d = 0$. Consequently, (r_1, r_2, d) is a triple-zero of A. □

References

[1] A. Y. a. F. S. Darani, "2-absorbing and weakly 2-absorbing submodules," *Thai J. Math,* vol. 9.3, pp. 577-584, 2011.

[2] A. Badawi, "On 2-absorbing ideals of commutative rings," *Bulletin of the Australian Mathematical,* vol. 75.3, pp. 417-429, 2007.

[3] H. E. Y. Ü. T. a. A. Y. D. Mostafanasab, "On 2-absorbing primary submodules of modules over commutative rings," *Analele științifice ale Universității" Ovidius" Constanța. Seria Matematică,* vol. 1, pp. 335-351, 2016.

[4] O. A. a. M. H. K. Abdulla, "Pseudo Primary-2-Absorbing sub-modules and Some Related Concepts," *Ibn Al-Haitham Journal for Pure and Applied Sciences,* vol. 32, no. 3, 2019.

[5] O. H. T. a. M. A. Salih, "Weakly pseudo primary 2-absorbing sub-modules," *Algebra Universalis (Under preparation).*

[6] J. a. P. F. S. Jenkins, "On the prime radical of a module over a commutative ring," *Communications in Algebra,* no. 20.12, pp. 3593-3602, 1992.

[7] C. U. Jensen, Model theoretic algebra with particular emphasis on fields, rings, modules, Routledge, 2022.

[8] D. D. a. A. T. a. T. U. a. K. S. Anderson, "On S-multiplication modules," *Communications in Algebra,* vol. 48, no. 8, pp. 3398--3407, 2020.