Journal of Al-Qadisiyah for Computer Science and Mathematics VOL.16(1) 2024, PP Math. 65–78

University Of AL-Qadisiyah

Available online at www.qu.edu.iq/journalcm

JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS

ISSN:2521-3504(online) ISSN:2074-0204(print)

Fuzzy Anti-normed Spaces

Dunia Ameen Abd alsaheb , Noori Farhan Al Mayahi *

College of Computer Science and Information Technology . Al-Qadisah University , Diwaniyah-Iraq .Email:dunia.ameen@qu.edu.iq

Department of Mathematics,""College of Science""University of Al-Qadisiyah,Diwaniyah -Iraq.Email: nfam60@yahoo.com

ARTICLE INFO

Article history:
Received: 11 /1/2024
Revised form: 3 /3/year
Accepted: 13 /3/2024
Available online: 30 /3/2024

Keywords: fuzzy anti-normed space, fuzzy anti-convergence, fuzzy anti-continuity.

ABSTRACT

In this paper the definition of fuzzy anti-norm is modified the authors have studied some properties of convergence and continuity in fuzzy anti-normed spaces. Firstly, the authors have given some definitions and theorems, such as fuzzy anti-normed, fuzzy anti–continuous, and fuzzy anti-convergent sequence and Cauchy sequence on fuzzy anti-normed space and definitions of open ball and closed ball. The authors have presented some examples by using these definitions. Additionally, the authors have given a definition, sequentially fuzzy anti-continuous. and some of their properties are studied.

MSC..

https://doi.org/10.29304/jqcsm.2024.16.11449

1. Introduction

When the data are imprecise or ambiguous, fuzzy set theory can be a helpful tool to describe the situation. To handle such situations, fuzzy sets assign an object a degree of belonging to a set. The concept of the fuzzy norm was first proposed by Katsaras [4]. A fuzzy norm on a linear space was defined by Felbin [6], and its corresponding fuzzy, The type of metric is Kaleva-Seikkala [8]. A fuzzy norm on a linear space with a Kramosil and Michalek-type associated metric was first proposed by Cheng and Mordeson [5], [7].

A fuzzy norm was defined by Bag and Samanta in [1] in a way that makes the corresponding fuzzy metric of the Kramosil and Michalek type [7]. In [2] and [3], they also looked at a few fuzzy norm properties. Bag and Samanta talked about the concepts of Cauchy sequence and convergent sequence in fuzzy normed in [1]'s linear space. Additionally, they conducted a comparison of the fuzzy norms defined by Bag and Samanta [1], Felbin [6], and Katsaras [4] in [3].

Following an introduction to fuzzy norms, we present a fuzzy anti-norm space in this work, drawing on the concept of fuzzy anti-norm as first presented by Bag and Samanta [3] and examining some of its significant characteristics. Next, we will discuss the concepts of the Cauchy sequence in fuzzy anti-normed systems and convergent sequences.

∗Corresponding author: Dunia Ameen Abd alsaheb

Email addresses: dunia.ameen@qu.edu.iq

Communicated by 'sub etitor'

We now outline the work in the article as follows. Section 1, is the introduction part containing a brief history and some concepts, Section 2, contains the basic definitions and results that are useful in the work, Section 3, contains some new definitions with examples, and Section 4, is devoted to the main theorems and results that established the relationships between studied notions in Section.

2. Fuzzy Anti-normed Spaces

This section defines fuzzy anti-normed space and looks into some of its key characteristics.

2.1. Definition[4].

Allow X to anon-empty set, where $F(X)$ is the collection among every fuzzy set in X. if $f \in F(X)$ then f={(x,α) :x∈X and α∈(0,1]}.It appears that f is a limited function for$|f(x)| \leq 1$. allow F to be the real number's space, then $F(X)$ is a vector space above the field F the addition and scalar multiplication definitions are

f+g={(x,α)+(y,β)}={(x+y,αΛβ):(x,α)∈f ,(y,β)∈g}.

And

kf={k(x,α):(x,α)∈f .where k∈F}.

It is said that The normed space is the linear space F(X). if for every $f \in F(X)$,

A function$\|.\| : F(X) \to R$ is referred to as norm on $F(X)$ if the following is met :

- $\|f\| \geq 0$ for all $f \in F(X)$.
- $\|f\| = 0$ if and only if $f = 0$.
- $\|kf\| = |k|\|f\|$, $k \in F$.
- $\|f + g\| \leq \|f\| + \|g\|$ for every $f, g \in F(X)$.

$(F(X), \|.\|)$ is normed linear space.

2.2. Definition[4]

Over field F, let F(X) be a vector space. a function$\mathcal{N}$:F(X)×R→[0,1], (R, the real number set) is known as the fuzzy norm function on $F(X)$ if satisfies the following:

- $\forall\ t \in R$ with t≤0 ,N(f,t)=0
- $\forall\ t \in R$ with t>0 ,N(f,t)=1 if and only if $f = 0$
- $\forall\ t \in R$ with$t \geq 0$,N(cf,t)=N$\left(f, \frac{t}{|c|}\right)$,c≠0 ,c∈F
- ∀ r ,q∈R,N(f+g,r+q)≥min {N(f,r),N(g,q)}
- N(f, .):(0,∞)⟶[0,1] is continuous
- $\lim_{t \to \infty} N(f,t)=1$

the pair (F(X),N) denoted a fuzzy normed space on $F(X)$.

2.3. Definition

let $F(X)$ be vector space over a real field $\boldsymbol{F}$. a function

N^* :$F(X) \times R \longrightarrow [0,1]$ is known as a fuzzy anti-norm function on $F(X)$ if the following is met :

- for all f,g∈F(X)
- for all $r \in R$, $t \leq 0$, $N^*(f,r)=1$
- for all $r \in R$with $t > 0$, $N^*(f,r)=0$ if and only if $f = 0$
- for all $r \in R$with $t \geq 0$, $N^*(cf,r)=N^*\left(f,\frac{r}{|c|}\right)$, c∈F/0
- for all r,q∈R , $N^*(f+g,r+q)\leq Max\{N^*(f,r), N^*(g,q)\}$
- $\lim_{t\to\infty} N^*(f,t)=0$.

The pair $(F(X),N^*)$ is called fuzzy anti-normed space.

2.4. Lemma

Consider $(F(X),N^*)$ fuzzy anti-normed space. then $N^*(f,.)$is not increasing to t for each $f \in F(X)$.

$$N^*(f\text{-}g, t)=N^*(g\text{-}f,t) .$$

Proof :

let $t < s$. Then $k = s - t > 0$ we have $N^*(f,t)=Max\{N^*(f,t),0\}= Max\{N^*(f,t),N^*(0,k)\}\geq N^*(f,s)$.

for f,g∈F(X) and t∈(0,∞),

$$N^*(f\text{-}g, t)=N^*(-(g\text{-}f), t) =N^*\left(g\text{-}f, \frac{t}{|-1|}\right)=N^*(g\text{-}f, t).$$

2.5. Definition

Let be fuzzy anti-normed space$(F(X), N^*)$. We clarify what the open ball is B(f,r,t)with center $f \in F(X)$ with radius r, r∈(0,1), t>0, as $B(f,r,t)=\{g\in F(X): N^*(f\text{-}g,t)<r\}$.

2.6. Theorem

let $B(f,r_1,t)$ and $B(f,r_2,t)$ have the same center and be open balls.$f \in F(X)$, $t > 0$. and radius $r_1,r_2\in(0,1)$ then we either have

$B(f,r_1,t)\subseteq B(f,r_2,t)$ *or* $B(f,r_2,t) \subseteq B(f,r_1,t)$.

Proof: let $f \in F(X)$ and $t > 0$. consider the open balls

$B(f,r_1,t)$ and $B(f,r_2,t)$ *, with* $0 < r_1 < 1$ *and* $0<r_2<1$,

if $r_1 = r_2$ so that the theorem holds . next, we believe $r_1 \neq r_2$ We can presume that,

without sacrificing generated. $0 < r_1 < r_2 < 1$, *then now let* $g\in B(f,r_1,t) \Rightarrow N^*(f\text{-}g,t)<r_1<r_2$.

$$\Rightarrow g\in B(f,r_2,t) \Rightarrow B(f,r_1,t)\subseteq B(f,r_2,t)$$

By assume $0<r_2<r_1<1$, we can similarly show

$$B(f,r_2,t) \subseteq B(f,r_1,t) .$$

3. Fuzzy Anti -Convergence in Fuzzy Anti-normed Space

3.1.Definition

allow $(F(X),N^*)$ to be fuzzy anti-normed space we can say:

- The sequence $\{f_n\}$ in $F(X)$ is fuzzy anti-converges to f in $F(X)$ if given $t>0, r\in(0,1)$, there exists an integer $n_0\in Z^+$, such that $N^*(f_n-f,t)<r$, for all $n \geq n_0$.
- The sequence $\{f_n\}$ in $F(X)$is fuzzy anti-Caushy if given$t > 0, r \in (0,1)$, there exists an integer $n_0\in Z^+$, such that $N^*(f_n-f_m,t)<r$, for all n, $m\geq n_0$.
- a fuzzy anti-normed space is said to complete if every fuzzy anti-Cuchy sequence is fuzzy anti-converge.

3.2.Theorem

In the fuzzy anti-normed space $(F(X),N^*)$ the sequence $\{f_n\}$ is fuzzy anti-converge to $f \in F(X)$ if and only if $\lim_{n\to\infty}(f_n-f,t)=0$, for all $t > 0$.

Proof : let t>0, suppose $\{f_n\}$ converges to $f \in F(X)$ then for a given $r \in (0,1)$, there exist $n_0 \in Z^+$ such that:

$N^*(f_n-f,t)<r$, forall $n\geq n_0$. and hence

$$N^*(f_n-f,t)\to 0, \text{ as } n\to\infty .$$

Conversely , if for each $t>0, N^*(f_n-f,t)\to 0$, as $n\to\infty$, then for every $s\in(0,1)$, there exist $n_0 \in Z^+$ such that :

$N^*(f_n-f,t)<s$, forall $n\geq n_0$. hence $\{f_n\}$ converges to f *in* $F(X)$.

3.3.Theorem

In the fuzzy anti-normed space $(F(X), N^*)$ the sequence $\{f_n\}$ is afuzzy anti-Cauchy sequence in F(X) if and only if $\lim_{n,m\to\infty} N^*(f_n-f_m,t)=0$,for all $t > 0$.

Proof : let t>0, suppose $\{f_n\}$ fuzzy anti -Cauchy sequence in $F(X)$ then

For given $r\in(0,1)$ and there exist $n_0 \in Z^+$ such that:

$N^*(f_n-f_m,t)<r$, for all $n,m\geq n_0$. and hence $N^*(f_n-f_m,t)\to 0$, As $n \to \infty$.

Conversely, if for each $t > 0, N^*(f_n-f_m,t)\to 0$, as $n \to \infty$. then for every $r\in(0,1)$, there exist $n_0 \in Z^+$ such that :

$N^*(f_n-f_m,t)<r$,for all $n,m\geq n_0$.

Hence $\{f_n\}$ is a fuzzy anti-Cauchy sequence in $F(X)$.

3.4.Theorem

let $(F(X),N^*)$ fuzzy anti-normed space and f_n ,g_n be two sequences in $F(X)$

Each fuzzy anti-converge sequence is a fuzzy anti-Cauchy sequence.

In F(X), every sequence has a unique fuzzy limit.

In case $f_n \longrightarrow f$ then $rf_n\longrightarrow rf$, $r\in F-\{0\}$.

In case $f_n\longrightarrow f$, $g_n\longrightarrow g$, then $f_n + g_n \longrightarrow f + g$.

Proof :

let $\{f_n\}$ be a sequence in $F(X)$ in which $f_n \longrightarrow f$ then

$\lim_{n\to\infty} N^*\left(f_n-f,\frac{t}{2}\right)=0$, *for all* $t>0$.*let* p=1,2,3,.....

$N^*(f_n-f_{n+p},t)=N^*((f_n-f)-(f_{n+p}-f),t)\leq Max\left\{N^*\left(f_n-f,\frac{t}{2}\right),N^*\left(f_{n+p}-f,\frac{t}{2}\right)\right\}$ *by taking limit :*

$$\lim_{n\to\infty} N^*(f_n-f_{n+p},t) \leq Max\left\{\lim_{n\to\infty} N^*\left(f_n-f,\frac{t}{2}\right),\lim_{n\to\infty} N^*\left(f_{n+p}-f,\frac{t}{2}\right)\right\}=0$$

$\Longrightarrow \lim_{n\to\infty} N^*(f_n-f_{n+p},t) \leq 0$,*but* $\lim_{n\to\infty} N^*(f_n-f_{n+p},t)\geq 0$,

$\Longrightarrow \lim_{n\to\infty} N^*(f_n-f_{n+p},t)=0$.

Therefore $\{f_n\}$ is the fuzzy anti-Cauchy sequence in $F(X)$.

allow $\{f_n\}$ to be the sequence in $F(X)$ so that $f_n \longrightarrow f$, $f_n \longrightarrow p$ as $n \longrightarrow \infty$ and $f \neq g$, Also let t,s>0,

Now $\lim_{n\to\infty} N^*(f_n-f,s)=0$, $\lim_{n\to\infty} N^*(f_n-p,t-s)=0$,

$N^*(f-p,t)= N^*(f-f_n+f_n-p,t-s+s) \leq Max\{N^*(f_n-f,s),N^*(f_n-p,t-s)\}$

Taking limit, we have

$$N^*(f-p,t)\leq Max\left\{\lim_{n\to\infty} N^*(f_n-f,s),\lim_{n\to\infty} N^*(f_n-p,t-s)\right\}=0,$$

$\Longrightarrow N^*(f-p,t)\leq 0$, *but* $N^*(f-p,t)\geq 0$

$\Longrightarrow N^*(f-p,t)=0$ then by condition$(N_2{}^*)$

$$f-p=0 \Longrightarrow f=p .$$

since $f_n \longrightarrow f$ then, if it was believed t>0,r∈(0,1), there exist

$n_0 \in Z^+$ *such that* $N^*(f_n-f,t)<r$, for all $n\geq n_0$

Put $t=\frac{t_1}{|c|}$ *such that* $t_1>0$,

$N^*(cf_n-cf,t_1)=N^*\left(f_n-f,\frac{t_1}{|c|}\right)=N^*(f_n-f,t)<r$,

Then $cf_n \longrightarrow cf$.

since $f_n \longrightarrow f$ then given t>0,r∈(0,1), there exist

an integer $n_1 \in Z^+$ such that $N^*\left(f_n-f,\frac{t}{2}\right)<r$, for all $n\geq n_1$,

since $g_n \longrightarrow g$ then for given t>0,r∈(0,1), there exist an integer

$n_2 \in Z^+$ *such that* $N^*\left(g_n-g,\frac{t}{2}\right)<r$, for all $n\geq n_2$,

take $n_0=\min\{n_1,n_2\}$, *such that :*

$$N^*\left((f_n+g_n)-(f+g),t\right)=N^*\left((f_n-f)+(g_n-g),t\right)\leq \mathrm{Max}\left\{N^*\left(f_n-f,\frac{t}{2}\right),N^*\left(g_n-g,\frac{t}{2}\right)\right\}<\mathrm{Max}\{r,r\}=r$$

for all $n \geq n_0$. *then* $f_n + g_n \longrightarrow f + g$.

3.5.Theorem

Two fuzzy anti-normed spaces are $(F(X), N^*)$, and $(F(Y), N^*)$ and $f_n \longrightarrow f$, $g_n \longrightarrow g$ such that $\{f_n\}$ and $\{g_n\}$ $\alpha, \beta \in F - \{0\}$, two sequences in $F(X)$. then

$$\alpha T(f_n) + \beta\psi(g_n) \longrightarrow \alpha T(f) + \beta\psi(g)$$

whenever T ,ψ are two identity fuzzy functions.

proof: since $f_n \longrightarrow f$ then for a given $t>0, r_1 \in (0,1)$, there exist

an integer $n_1 \in Z^+$ such that $N^*\left(f_n-f,\frac{t}{2|\alpha|}\right)<r_1$, for all $n\geq n_1$,

since $g_n \longrightarrow g$ then for a given $t>0, r_2 \in (0,1)$, there exist an integer $n_2 \in Z^+$ such that $N^*\left(g_n-g,\frac{t}{2|\beta|}\right)<r_2$, for all $n\geq n_2$,

take $n_0=\min\{n_1,n_2\}$,$n\geq n_0$ such that :

$$N^*\left((\alpha T(f_n)+\beta\psi(g_n))-(T(f)+\psi(g)),t\right)$$

$$=N^*\left(\alpha(T(f_n)-T(f))+\beta(\psi(g_n)-\psi(g)),t\right)$$

$$\leq \mathrm{Max}\left\{N^*\left(T(f_n)-T(f),\frac{t}{2|\alpha|}\right),N^*\left(\psi(g_n)-\psi(g),\frac{t}{2|\beta|}\right)\right\}$$

$$= \mathrm{Max}\left\{N^*\left(f_n-f,\frac{t}{2|\alpha|}\right),N^*\left(g_n-g,\frac{t}{2|\beta|}\right)\right\}< \mathrm{Max}\{r_1,r_2\}=r$$

$$\therefore \ \alpha T(f_n) + \beta\psi(g_n) \longrightarrow \alpha T(f) + \beta\psi(g) .$$

3.6.Theorem

$(F(X), N^*)$ is a fuzzy anti-normed space that is complete. if each fuzzy anti-Cauchy sequence $\{f_n\}$ in $F(X)$ has fuzzy anti-convergent sub-sequence.

proof: in $F(X)$ let $\{f_n\}$ be a fuzzy anti-Cauchy sequence and $\{f_{nm}\}$ be the subsequence that of $\{f_n\}$

so that $f_{nm} \longrightarrow f$, $f \in F(X)$

Right now we need to prove $f_n \longrightarrow f$ for $t > 0$

Since f_n is a fuzzy anti-Cauchy sequence then for the given

$t>0$,$r_1 \in (0,1)$ there exist $n_0 \in Z^+$ such that

$$N^*\left(f_n-f_m,\frac{t}{2}\right)<r_1 \quad \text{for all } n,m\geq n_0$$

Since $\{f_{nm}\}$ is fuzzy anti-convergent to f, there exists $im \geq n_0$ such that

$$N^*\left(f_{im}-f,\frac{t}{2}\right)<r_2 \qquad , r_2\in(0,1)$$

$$N^*(f_n-f,t)= N^*\left((f_n-f_{im})+(f_{im}-f),\frac{t}{2}+\frac{t}{2}\right)$$

$$\leq Max\left\{ N^*(f_n-f_{im},\tfrac{t}{2}), N^*\left(f_{im}-f,\tfrac{t}{2}\right)\right\}<Max\{r_1 ,r_2\}=r.$$

4. Fuzzy Anti-Continuous in Fuzzy Anti-Normed Spaces

4.1.Definition.

Let's say there are two fuzzy anti-normed spaces $(F(X), N^*)$ and $(F(Y),N^*)$. a purpose $T: F(X) \longrightarrow F(Y)$ is define fuzzy anti-continuous at $f_0 \in F(X)$ if for any given t>0 ,ε∈(0,1),there exist $s > 0$and δ∈(0,1) such that for all $f \in F(X)$

$$N^*(f-f_0 ,s)<\delta \quad \textit{implies} \ N^*(T(f)-T(f_0),t)<\varepsilon$$

If f is fuzzy anti-continuous at every point in $F(X)$, then it is defined as a fuzzy anti-continuous function.

4.2.Theorem.

In fuzzy anti-normed space, every identity fuzzy function is fuzzy anti-continuous.

Proof : $\forall$ ε∈(0,1) and $r > 0$ there are $s = r$ and $\delta < \varepsilon$

such that δ∈(0,1) and $N^*(f_n-f,s)<\delta$

$N^*(T(f_n)-T(f),r)=N^*(f_n-f,s)<\delta<\varepsilon$ there fore f is a fuzzy anti-continuous at f , It follows that T is a fuzzy anti-continuous function since f is an arbitrary point.

4.3.Theorem.

let $(F(X),N^*)$ be fuzzy anti-normed space over F. Then The functions

$$T:G(X)\times G(X)\longrightarrow G(X) \ , \ T(s,h)=s+h \ \text{ and } \ \psi:F\times G(X)\longrightarrow G(X), \quad \psi(\lambda,g)=\lambda g$$

are fuzzy anti-continuous functions.

Proof : let s ,h∈G(X) and $\{s_n\},\{h_n\}$ in G(X) so that $s_n \longrightarrow s$ and

$h_n \longrightarrow h$ as n⟶∞ , then for given ε∈(0,1) and $t > 0$ there exists $n_1 \in Z^+$

such that $N^*\left(s_n-s,\frac{t}{2}\right)<\varepsilon$ for all $n \geq n_1$,

and for given ε∈(0,1) and t>0 there exists $n_2 \in Z^+$

such that $N^*\left(h_n-h,\frac{t}{2}\right)<\varepsilon$ for all $n\geq n_2$,

put $n_0= \min\{ n_1,n_2\}$

$$N^*((s_n+h_n)-(s+h),t)= N^*\left((s_n-s)+(h_n-h),\frac{t}{2}+\frac{t}{2}\right)$$

$$\leq Max\left\{ N^*\left(s_n-s,\frac{t}{2}\right), N^*\left(h_n-h,\frac{t}{2}\right)\right\}<Max\{\varepsilon ,\varepsilon\}=\varepsilon$$

$for\ all\ n \geq n_0$, therefore

$T(s_n , h_n) \longrightarrow T(s,h)$ as $n \longrightarrow \infty$,T is a fuzzy anti-continuous function

at (s,h) since (s,h) is any point in $F(X) \times F(X)$.

let $g \in G(X), \lambda \in F$ and $\{g_n\}$ in $G(X)$ and $\{\lambda_n\}$ in $F/\{0\}$ such that

$g_n \longrightarrow g$ and $\lambda_n \longrightarrow \lambda$ as $n \longrightarrow \infty$, for each $\frac{t}{2|\lambda_n|} > 0$,

$\lim_{n\to\infty} N^*\left(g_n - g, \frac{t}{2|\lambda_n|}\right) = 0$,$|\lambda_n - \lambda| \longrightarrow 0$ as $n \longrightarrow \infty$,

$$N^*(T(\lambda_n,g_n)-T(\lambda,g),t) = N^*(\lambda_n g_n - \lambda g,t)$$

$$= N^*\big((\lambda_n g_n - \lambda_n g)+(\lambda_n g_n - \lambda g),t\big)$$

$$\leq Max\left\{ N^*(\lambda_n (g_n - g), \frac{t}{2}) , N^*(g(\lambda_n - \lambda), \frac{t}{2})\right\}$$

$$= Max\left\{ N^*(g_n - g, \frac{t}{2|\lambda_n|}), N^*(g, \frac{t}{2|\lambda_n - \lambda|})\right\}$$

By taking limit :

$\lim_{n\to\infty} N^*(T(\lambda_n,g_n)-T(\lambda,g),t) \leq$

$$Max\left\{\lim_{n\to\infty} N^*\left(g_n - g, \frac{t}{2|\lambda_n|}\right), \lim_{n\to\infty} N^*\left(g, \frac{t}{2|\lambda_n - \lambda|}\right)\right\} =$$

$Max\{0,0\} = 0$, *but* $\lim_{n\to\infty} N^*(T(\lambda_n,g_n)-T(\lambda,g),t) \geq 0$,

Then $\lim_{n\to\infty} N^*(T(\lambda_n,g_n)-T(\lambda,g),t) = 0$ *then*

$T(\lambda_n,g_n) \longrightarrow T(\lambda,g)$ as $n \longrightarrow \infty$,T*is fuzzy anti-continuous at* (λ,g)

since (λ, g) *is any point in* $F \times G(X)$.

4.4.Theorem

let $(F(X),N^*)$ be fuzzy anti-normed space and let T be a linear function that is either fuzzy anti-continuous at all points in $F(X)$ or not at all, then T is fuzzy anti-continuous. at every point or no point of F(X).then T is fuzzy anti-continuous

Proof: any two point f_1 and f_2 of $F(X)$ and let T fuzzy anti-continuous at f_1. Then given t>0 ,$\varepsilon \in (0,1)$ there are

$\delta \in (0,1)$,s>0 , such that $f \in F(X)$

$N^*(f-f_1 ,s) < \delta$ *implies* $N^*(T(f)-T(f_1) ,t) < \varepsilon$

Now : $N^*(f-f_2 ,s) < \delta \Rightarrow N^*\big((f+f_1-f_2)-f_1 ,s\big) < \delta$

$$\Rightarrow N^*(T(f+f_1-f_2)-T(f_1) ,t) < \varepsilon$$

$$\Rightarrow N^*(T(f)+T(f_1)-T(f_2)-T(f_1) ,t) < \varepsilon$$

$$\Rightarrow N^*(T(f)-T(f_2),t) < \varepsilon$$

T is fuzzy anti-continuous at, since f_2 arbitrary point. then T is

a fuzzy anti-continuous.

4.5.Corollary

Let $(F(X), N^*)$, $(F(Y), \text{and } N^*)$ be fuzzy anti-normed spaces. and T be a linear function, T is fuzzy anti-continuous at all points if it is fuzzy anti-continuous at 0.

Proof: allow $\{f_n\}$ to be a sequence in $F(X)$ so that there exists f_0 , and

$f_n \longrightarrow f_0$, since f_n a fuzzy anti-continuous at 0 then :

To any given $t>0, \varepsilon \in (0,1)$ there exist $\delta \in (0,1), s>0$

$$(f_n\text{-}f_0) \in F(X),\ N^*((f_n\text{-}f_0)\text{-}0,s)<\delta \Rightarrow N^*(T(f_n\text{-}f_0)\text{-}T(0),t)<\varepsilon$$

$$\Rightarrow N^*(T(f_n)\text{-}T(f_0),t)<\varepsilon,\ N^*(f_n\text{-}f_0,s)<\delta \Rightarrow N^*(T(f_n)\text{-}T(f_0),t)<\varepsilon$$

$f_n \longrightarrow f_0 \Rightarrow T(f_n) \longrightarrow T(f_0)$ therefore T is fuzzy anti continuous at f_0

T is a fuzzy anti-continuous function Since f_0 arbitrary point

4.6.Theorem

Consider that $(F(X), N^*)$, $(F(Y), \text{and } N^*)$ are fuzzy anti-normed spaces. the function $T: F(X) \longrightarrow F(Y)$ is fuzzy anti-continuous at $f_0 \in F(X)$ Only if every sequence $\{f_n\}$ fuzzy anti-converge to f_0 in $F(X)$ Then the sequence $\{T(f_n)\}$ is fuzzy anti-Convergent to $T(f_0)$ in $F(Y)$.

Proof. Assume that T fuzzy anti-continuous function in f_0 and let $\{f_n\}$

a sequence in $F(X)$ so that $f_n \longrightarrow f_0$, let $\varepsilon \in (0,1)$,$t>0$,

since T fuzzy anti-continuous in f_0 then There are $\delta \in (0,1)$,$s>0$, so that for all $f \in F(X)$:

$N^*(f\text{-}f_0,s)<\delta \Rightarrow N^*(T(f)\text{-}T(f_0),t)<\varepsilon$

Since $f_n \longrightarrow f_0$,$\delta \in (0,1), s>0$ then There are $k \in Z^+$ so that

$N^*((f_n\text{-}f_0),s)<\delta$ for all $n \geq k$ hence $N^*(T(f_n)\text{-}T(f_0),t)<\varepsilon$

for all $n \geq k$ therefore $T(f_n) \longrightarrow T(f_0)$.

On the other hand. Assume the theorem's condition is true, and T is not fuzzy anti-continuous at f_0 Then there exist $\varepsilon \in (0,1), t>0$ such that for all $\delta \in (0,1), s>0$ there exist $f \in F(X)$ and $N^*((f_n\text{-}f_0),s)<\delta$

$$\Rightarrow N^*(T(f_n)\text{-}T(f_0),t) \geq \varepsilon$$

$\Rightarrow$for all $n \in Z^+$ there exist $f_n \in F(X)$ such that

$N^*(f_n\text{-}f_0,s)<\frac{1}{n} \Rightarrow N^*(T(f_n)\text{-}T(f_0),t) \geq \varepsilon$ that is mean

$f_n \longrightarrow f_0$ in $F(X)$ but $T(f_n) \nrightarrow T(f_0)$ in $F(Y)$. that is a contradiction,

$\therefore T$ is fuzzy anti-continuous.

4.7. Theorem

let $(F(X), N_1^*)$,$(F(Y),N_2^*)$ *be fuzzy anti-normed spaces , if the function* $T : F(X) \longrightarrow F(Y)$ *and* $\psi : F(X) \longrightarrow F(Y)$ *are two fuzzy anti-continuous functions then :*

$$T + \psi \qquad kT \quad \text{where } k \in F/\{0\}$$

over the same field F, are also fuzzy anti-continuous functions.

Proof :

let $\{f_n\}$ be a sequence in $F(X)$ such that $f_n \longrightarrow f$. Since T ,ψ are two

fuzzy anti continuous at f thus for any given t>0,ε∈(0,1) , there exist

δ∈(0,1) and s>0 ,such that for all $f \in F(X)$:

$$N_1^*(f_n\text{-}f,s)<\delta \quad \text{implies } N_2^*\left(T(f_n)\text{-}T(f),\frac{t}{2}\right)<\varepsilon \quad \text{and}$$
$$N_1^*(f_n\text{-}f,s)<\delta \quad \text{implies } N_2^*\left(\psi(f_n)\text{-}\psi(f),\frac{t}{2}\right)<\varepsilon$$

Now $N_2^*((T+\psi)(f_n)\text{-}(T+\psi)(f),t)=$

$N_2^*(T(f_n)\text{-}T(f)+\psi(f_n)\text{-}\psi(f),t)\leq$

$$\text{Max}\left\{N_2^*\left(T(f_n)\text{-}T(f),\frac{t}{2}\right),N_2^*\left(\psi(f_n)\text{-}\psi(f),\frac{t}{2}\right)\right\}< \text{Max}\{\varepsilon,\varepsilon\}=\varepsilon$$

Then $T + \psi$ is fuzzy anti-continuous.

let $\{f_n\}$ is a sequence in $F(X)$ such that for any given t>0 ,

ε∈(0,1), there exist δ∈(0,1),s>0 such that :

$$N_1^*(f_n\text{-}f,s)<\delta \quad \text{implies } N_2^*(T(f_n)\text{-}T(f),t)<\varepsilon$$

Take $t_1 = t$. then for all $\varepsilon \in (0,1)$ and $t_1 > 0$, there exist $\delta \in (0,1)$

and $s > 0$ such that :

$N_1^*(f_n\text{-}f,s)<\delta$ implies $N_2^*((kT)(f_n)\text{-}(kT)(f),t_1)=$

$N_2^*(k(T(f_n)\text{-}T(f)),t_1)= N_2^*\left(T(f_n)\text{-}T(f),\frac{t_1}{|k|}\right)=$

$$N_2^*(T(f_n)\text{-}T(f),t)<\varepsilon$$

Then kT is a fuzzy anti-continuous function.

4.8. Definition

a function T :$(F(X), N_1^*)\longrightarrow(F(Y),N_2^*)$ is define Sequentially fuzzy anti-continuous at f_0∈F(X) if for any$\{f_n\}_n$, $f_n \in F(X)$for all n, with $f_n \longrightarrow f_0$ implies $T(f_n) \longrightarrow T(f_0)$ in $F(Y)$ that is for all $t > 0$,i.e.

$$\lim_{n\to\infty} N_1^*(f_n\text{-}f_0,t)=0, \quad \text{for all } t>0 \quad \Longrightarrow \lim_{n\to\infty} N_2^*(T(f_n)\text{-}T(f_0),t)=0, \text{ for all } t>0$$

4.9. Theorem

let T :$(F(X),N_1^*)\longrightarrow(F(Y),N_2^*)$ be a function where $(F(X),N_1^*)$,$(F(Y),N_2^*)$ are fuzzy anti-normed spaces then ***T*** is fuzzy anti continuous if and only if it is sequentially fuzzy anti continuous.

Proof**:** suppose T is fuzzy anti-continuous at $f_0 \in F(X)$,let $\{f_n\}_n$ be a sequence in $F(X)$ such that $f_n \longrightarrow f_0$. let $t > 0$ be given and choose ε∈(0,1) .since T is fuzzy anti continuous at f_0 then there exist $s > 0$

and δ∈(0,1)such that for all $f \in F(X)$

$$N_1^*(f-f_0,s)<\delta \text{ implies } N_2^*(T(f)-T(f_0),t)<\varepsilon$$

Since $f_n \longrightarrow f_0$ in $F(X)$ then there exists $n_0 \in Z^+$ such that

$$N_1^*(f_n-f_0,t)<\delta \qquad \text{for all } n\geq n_0$$

then $N_2^*(T(f_n)-T(f_0),t)<\varepsilon$ for all $n \geq n_0$

So for any given $t > 0$ and ε∈(0,1) there exist $n_0 \in Z^+$ such that

$N_2^*(T(f_n)-T(f_0),t)<\varepsilon$ for all $n \geq n_0$

This implies $\lim_{n\to\infty} N_2^*(T(f_n)-T(f_0),t)=0$

Since $t > 0$ is arbitry , thus $T(f_n) \longrightarrow T(f_0)$ in $(F(Y),N_2^*)$.

The next step is to assume that T is sequentially fuzzy anti-continuous in f_0 If this cannot be achieved, we should assume that T is not fuzzy anti-continuous atf_0, in which case there exist

$t > 0$ and ε∈(0,1) such that for any $s > 0$ and $\delta \in (0,1)$ there exist

$f \in F(X)$ such that :

$$N_1^*(f_0-f,s)<\delta \text{ implies } N_2^*(T(f)-T(f_0),t)\geq\varepsilon \qquad (1)$$

thus for $s=1-\frac{1}{n+1}, \delta=\frac{1}{n+1}$,n=1,2,3,….

There exist f_n such that

$$N_1^*\left(f_0-f_n,1-\frac{1}{n+1}\right)<\frac{1}{n+1} \text{ implies } N_2^*(T(f_0)-T(f_n),t)\geq\varepsilon$$

Taking $s > 0$,there exist n_0 such that $s > 1-\frac{1}{n+1}$,for all $n \geq n_0$ then

$N_1^*(f_0-f_n,s)< N_1^*\left(f_0-f_n,1-\frac{1}{n+1}\right)<\frac{1}{n+1}$, for all $n \geq n_0$

$\lim_{n\to\infty} N_1^*(f_0-f_n,s)=0 \Longrightarrow f_n \longrightarrow f_0$,

But from (1) $N_2^*(T(f_0)-T(f_n),t) \geq \varepsilon$, so $N_2^*(T(f_0)-T(f_n),t) \nrightarrow 0$

as $n \longrightarrow \infty$. thus $T(f_n)$ does not fuzzy anti-convergent to $T(f_0)$ where

$f_n \longrightarrow f_0$. thus would be in opposition to the premise above, so T fuzzy anti-continuous at f_0 .

4.10.Theorem

If $T:(F(X),N_1^*)\rightarrow(F(Y),N_2^*)$ and $\psi:(F(X),N_1^*)\rightarrow(F(Y),N_2^*)$ are two fuzzy sequentially anti-continuous functions then

(1)$T\psi$ is sequentially fuzzy anti-continuous over the same field F.

(2) If ψ(f)≠0, ∀ x∈X then $\frac{T}{\psi}$ is sequentially fuzzy anti-continuous over the same field F.

Proof :

let $\{f_n\}_n$ be sequence in $F(X)$ such that $f_n \to f$ in $(F(X), N_1^*)$.

thus for all t>0, we have

$$\lim_{n\to\infty} N_1^*(f_n-f,t)=0 \qquad \ldots\ldots(1)$$

Since T,ψ are sequentially fuzzy anti-continuous at f then from (1) we have

$$\lim_{n\to\infty} N_2^*(T(f_n)-T(f),t)=0 \qquad \forall t>0\,,$$

$$\lim_{n\to\infty} N_2^*(\psi(f_n)-\psi(f),t)=0 \quad \forall t>0\,.$$

Now, $N_2^*\big((T\psi)(f_n)-(T\psi)(f),t\big)=$

$$N_2^*\big((T(f_n)(\psi(f_n)-\psi(f))+\psi(f)(T(f_n)-T(f),t\big)=$$

$$N_2^*\Big((T(f_n)-T(f))(\psi(f_n)-\psi(f))+T(f)\big(\psi(f_n)-\psi(f)\big)+\psi(f)(T(f_n)-T(f),t)\Big) \le \mathrm{Max}\{N_2^*\left((T(f_n)-T(f))(\psi(f_n)-\psi(f),\frac{t}{3}\right)$$

$$,N_2^*\left(T(f)\big(\psi(f_n)-\psi(f)\big),\frac{t}{3}\right),N_2^*\left(\psi(f)\big(T(f_n)-T(f)\big),\frac{t}{3}\right)\}=$$

$$\mathrm{Max}\left\{\begin{matrix} N_2^*\left((T(f_n)-T(f),\frac{t}{3|\psi(f_n)-\psi(f)|}\right),\ N_2^*\left(\big(\psi(f_n)-\psi(f)\big),\frac{t}{3|T(f)|}\right) \\ ,N_2^*\left(\big(T(f_n)-T(f)\big),\frac{t}{3|\psi(f)|}\right)\end{matrix}\right\}$$

Taking the limit as $n\to\infty$ we have,

$$\lim_{n\to\infty} N_2^*\big((T\psi)(f_n)-(T\psi)(f),t\big)\le$$

$$\mathrm{Max}\left\{\begin{matrix} \lim_{n\to\infty} N_2^*\left((T(f_n)-T(f),\frac{t}{3|\psi(f_n)-\psi(f)|}\right),\ \lim_{n\to\infty} N_2^*\left(\big(\psi(f_n)-\psi(f)\big),\frac{t}{3|T(f)|}\right) \\ ,\lim_{n\to\infty} N_2^*\left(\big(T(f_n)-T(f)\big),\frac{t}{3|\psi(f)|}\right)\end{matrix}\right\}$$

$$= Max\,\{0,0,0\} = 0.$$

But $\lim_{n\to\infty} N_2^*\big((T\psi)(f_n)-(T\psi)(f),t\big)\ge 0$,

Then $\lim_{n\to\infty} N_2^*\big((T\psi)(f_n)-(T\psi)(f),t\big)=0$.

Hence the proof

Here, we demonstrate that $\frac{1}{\psi}$ is sequentially fuzzy anti-continuous at f if $\psi(f)\neq 0$ for all $f\in F(X)$.

$$N_2^*\left(\frac{1}{\psi}(f_n)-\frac{1}{\psi}(f),t\right)=N_2^*\left(\frac{\psi(f_n)-\psi(f)}{\psi(f_n)\psi(f)},t\right)=$$

$N_2^*\left(\frac{1}{\psi(f_n)\psi(f)}, \frac{t}{|\psi(f_n)-\psi(f)|}\right)$. *taking the limit as* $n \to \infty$ *we have*

$$\lim_{n\to\infty} N_2^*\left(\frac{1}{\psi}(f_n)-\frac{1}{\psi}(f), t\right) = N_2^*\left(\frac{1}{\psi(f_n)\psi(f)}, \lim_{n\to\infty}\frac{t}{|\psi(f_n)-\psi(f)|}\right) = N_2^*\left(\frac{1}{\psi(f_n)\psi(f)}, \infty\right) = 0.$$

Hence $\frac{1}{\psi}$ is sequentially fuzzy anti-continuous.

The proof is completed by considering the product of T *and* $\frac{1}{\psi}$.

4.11.Theorem

Suppose T is a linear function from the fuzzy anti-norm space.$(F(X),N_1^*)$ to $(F(Y),N_2^*)$. ***T*** is sequentially fuzzy anti-continuous on F(X) if it is sequentially fuzzy anti-continuous at any point.

Proof : let $\{f_n\}$ be a sequence in $F(X)$and suppose $f \in F(X)$ be an arbitrary point

such that $f_n \longrightarrow f$ then
$\lim_{n\to\infty} N_1^*(f_n-f,t)=0$ for all t>0

i.e. $\lim_{n\to\infty} N_1^*((f_n-f+f_0)-f_0,t)=0$ for all t>0

since T is fuzzy anti-continuous in f_0 we have

$$\lim_{n\to\infty} N_2^*(T(f_n - f + f_0) - T(f_0), t) = 0 \quad for\ all\ t > 0$$

i.e. $\lim_{n\to\infty} N_2^*(T(f_n)-T(f)+T(f_0)-T(f_0),t)=0$ for all t>0

i.e. $\lim_{n\to\infty} N_2^*(T(f_n)-T(f),t)=0$ for all t>0

thus $\lim_{n\to\infty} N_1^*(f_n-f,t)=0$ for all t>0$\Longrightarrow$

$$\lim_{n\to\infty} N_2^*(T(f_n)-T(f),t)=0 \quad \text{for all } t>0$$

Then T is sequentially fuzzy anti-continuous on $F(X)$.

Conclusion

The main goal of this paper is to introduce and define fuzzy anti-normed space and Our work introduces the notions of convergent sequence, the Cauchy sequence in fuzzy anti-normed linear space, and also introduces the concept of continuous and sequentially continuous in fuzzy anti-normed linear space.

REFERENCES

[1] T. Bag and T.K. Samanta, finite-dimensional fuzzy normed linear space, the Journal of fuzzy mathematics,11 (3)(2003) 687-705.
[2] T. Bag and T. K. Samanta, Fuzzy bounded linear operators, Fuzzy Sets and Systems, 151 (2005), 513- 547.
[3] T. Bag and T. K. Samanta, A comparative study of fuzzy norms on a linear space, Fuzzy Sets and Systems 159 (2008), 670-684.
[4] A.K. Katsaras, Fuzzy topological vector spaces, Fuzzy Sets and Systems, 12 (1984), 143-154.
[5] S. C. Cheng and J. N. Mordesen, Fuzzy linear operators and fuzzy normed linear spaces, Bull. Cal. Math. Soc., 86 (1994), 429-436.
[6] C. Felbin, Finite dimensional fuzzy normed linear spaces, Fuzzy Sets, and Systems, 48 (1992) 239-248.
[7] I. Kramosil, and J. Michalek, Fuzzy metric and statistical metric space, Kybernetica, 11 (1975), 326-334.
[8] O. Kaleva and S. Seikkala, On fuzzy metric spaces, Fuzzy Sets and Systems, 12 (1984), 215-229.
[9] Iqbal H. Jebril & Samanta, T. K. (2010). Fuzzy Anti-Normed Space, Journal of Mathematics Technology, February, 66-77.
[10] L. Kocinac, Some topological properties of fuzzy antinormed linear spaces, J. Math. (2018).
[11] Nadaban, S. 2015. Fuzzy continuous mapping in fuzzy normed linear spaces, I. J. Computer Communications and Control, 10(6): 834-842.
[12] Jehad R. Kider and Ahmed H. Ali. 2018. Properties of fuzzy absolute value on and properties finite dimensional fuzzy normed space. Iraqi Journal of Science, 59(2B): 909-916