Journal of Al-Qadisiyah for Computer Science and Mathematics VOL.16(1) 2024, pp Math. 84–92

Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

Sinc approximation for solving fourth-order pseudo-Poisson equations

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ARTICLE INFO

Article history:

Keywords:

Sinc approximation

Pseudo-Poisson problem

numerical approximation

Galerkin method

ABSTRACT

In this article, we will investigate the Galerkin sinc approximation to solve the single pseudo-Poisson problem and in this method we will reach a linear system. We will solve this system by carefully choosing the length of the steps and the number of nodal points and with two methods; the first method, which is the usual method, is theoretically flawed and not practical and computationally efficient. Therefore, to solve the mentioned system, we introduce the orthogonalization technique, which solves both theoretical and computational problems. Numerical approximation is obtained whose accuracy is exponential and of order $O(\exp{(-c\sqrt{N})}$ where N is a transaction parameter and c is a constant independent of N. In the final part, we give some numerical examples of individual pseudo-Poisson problems to demonstrate the efficiency of the method.

MSC..

https://doi.org/ 10.29304/jqcsm.2024.16.11452

1. Introduction

Boundary value problems have received considerable attention in various fields of science and engineering. These problems are difficult to solve because of the presence of singularities, so many schemes have been proposed to overcome this difficulty. However, among the existing techniques, sinc-Galerkin and sinc-collocation methods are suitable for handling singularities based on some transformations applied to boundary value problems [1, 2]. In the papers [3, 4, 5] a general framework for first order differential equations and numerical solution of special second order ODE-based problems has been introduced. A quadrature method is based on combining two rules of the same precision level for numerical solution of the double integral in physical sciences is presented [6]. Collocation methods

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for solving special kind of integral equations are obtained [7]. The methods of numerical solution of different types of ordinary and partial differential equations have also been studied in recent years [8-11]. Convergence rate estimation for functions having a singularity at an endpoint with poly-Sinc discontinuous Galerkin methods is discussed in [12]. A non-classical sinc-collocation method for the solution of singular boundary value problems arising in physiology is discussed in [13]. Numerical solution of third-order boundary value problems using non-classical sinc-collocation method is expressed in [14]. Numerical methods using sinc nystrom formula are proposed and analyzed for second kind Fredholm integral equations over infinite intervals [15].

2. Preliminaries

Consider the following relation,

$$S(k.h)(x) = \frac{\sin(\frac{\pi(x-kh)}{h}]}{\frac{\pi(x-kh)}{h}} \tag{1}$$

This is cardinal (Whittaker) function. It is used to introduce the sink function, where h is a positive number and k is an integer. *S(k.h)(x)* is called the eminence of the sinc function with step length. For the defined and bounded function *f∈(-∞.∞),* the cardinal function of function f is defined as follows:

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh)S(k.h)(x). \tag{2}$$

From the aspect of derivation, integration, etc., the cardinal function plays the same role for sinc methods as polynomials for most classical numerical methods. Polynomials, many operators on $C(f,h)$ are implicitly transferred and each of these results leads to one of the explicit approximation methods. The function $C(f,h)$ is equivalent to many approximation formulas such as the trapezoid rule, many techniques obtained from the field of signal processing, etc.

Definition 1. Suppose the function f is defined in real numbers and *h>0* is given. We define the following series,

$$C(f,h)(x) \equiv \sum_{k=-\infty}^{\infty} f(kh) sinc(\frac{x-kh}{h})$$

which we have from relation (1).

$$sinc\left(\frac{x-kh}{h}\right) \equiv \begin{cases} \frac{\sin(\pi(x-kh)/h)}{\pi(x-kh)/h)}, & x \neq kh \\ 1, & x = kh \end{cases}$$

Wherever the series (2) converges, it is called cardinal function f.

Theorem 1. Suppose $\phi' F \in B(D)$ and $h > 0$. Let ϕ be a one-to-one cosine mapping from the domain D to Ds. Suppose $\Gamma = (\mathbb{R})$, $w_k = \psi(kh)$, $\psi = \phi^{-1}$ then for each $\xi \in \Gamma$ the limit of the relation

$$\epsilon(\phi' F)(\xi) \equiv F(\xi) - \sum_{k=-\infty}^{\infty} F(w_k) sinc(\frac{\phi(\xi) - kh}{h})$$

as follows:

$$\|\epsilon(\phi' F)\|_{\infty} \leq \frac{N(\phi' F, D)}{2\pi d sinh(\pi d/h)}$$

In addition, suppose there are constant values β.α and C that

$$|F(\xi)| \leq C \begin{cases} \exp\left(-\alpha|\phi(\xi)|\right., & \xi \in \Gamma_\alpha \\ \exp\left(-\beta|\phi(\xi)|\right., & \xi \in \Gamma_b \end{cases}$$

If we consider the following choices.

$$N = \left[\left|\frac{\alpha}{\beta} M + 1\right|\right],$$

$$h = \left(\frac{\pi d}{\alpha M}\right)^{1/2} \leq \frac{2\pi d}{\ln(2)}$$

Then for every $\xi \in \Gamma$ we have:

$$\epsilon_{M,N}(\phi' F)(\xi) \equiv F(\xi) - C_{M.N}(F, h, \phi)(\xi)$$

$$C_{M,N}(F, h, \phi)(\xi) \equiv \sum_{k=-\infty}^{\infty} F(\omega_k) sinc(\frac{\phi(\xi) - kh}{h})$$

And also

$$\|\epsilon_{M.N}(\phi' F)\|_\infty \leq K_5 M^{1/2} exp\left(-(\pi d \alpha M)^{1/2}\right)$$

which K_5 is a constant dependent on d, F, ϕ and D.

3. Solving fourth-order pseudo-Poisson equations

By choosing appropriate weight functions with sinc approximation method as basis function for solving the single pseudo-Poisson equation was first used by F. Stenger and arrived at a simple linear system. This method has exponential convergence Here we solve the single pseudo-Poisson problem using the sinc Galerkin method.

We are now investigating the equation of the fourth-order pseudo-Poisson problem

$$\begin{cases} \dfrac{\partial^4 u}{\partial x^4} + \dfrac{\partial^4 u}{\partial y^4} = f(x, y), \quad (x, y) \in \Omega \equiv (a_1, b_1) \times (a_2, b_2), \\ u\Big|_{\partial\Omega} = u_x \Big|_{\partial\Omega} = u_y \Big|_{\partial\Omega} = 0 \end{cases} \tag{3}$$

In [6], Smith has discussed the one-dimensional state of this problem using the sinc-Galerkin method; and the numerical approximation accuracy of order $O(e^{-\sqrt{\pi d \alpha N}})$ has been achieved.

Here we solve (3) by the sinc collocation method and using the orthogonalization technique.

Suppose $v_N(x, y)$ is defined by below relation with the choice of $h_1 = h_2 = h$ and $N_1 = N_2 = N$,

$$v_N(x, y) = \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} \frac{v_{k_1 k_2}}{g_{k_1} \bar{\bar{g}}_{k_2}} S_{k_1}(x) \bar{S}_{k_2}(y).$$

which $v_{k_1 k_2}$ is determined by the following relation:

$$\sum_{k_1=-N}^{N} \frac{v_{k_1 i_2}}{g_{k_1}} S_{k_1}^{(4)}(x_{i_1}) + \sum_{k_2=-N}^{N} \frac{v_{i_1 k_2}}{\bar{\bar{g}}_{k_2}} \bar{S}_{k_2}^{(4)}(y_{i_2}) = f_{i_1 i_2}$$

The matrix representation of the above equation is as follows:

$$S_x D\left(\frac{1}{g(x)}\right) U + U D\left(\frac{1}{\bar{\bar{g}}(y)}\right) S_y^t = W ,$$

that S_x and S_y are $m \times m$ matrices whose *ik-th* rank is $S_k^{(4)}(x_i)$ and $\overline{S}_k^{(4)}(y_i)$, respectively. U and W matrices are also f_{ik} and v_{ik} with *(ik).*

From previous relations we have:

$$S_k^{(4)}(x_i) = g_i^{(4)}\delta_{ki}^{(0)} + \frac{1}{h}a_i^{(1)}\delta_{ki}^{(1)} + \frac{1}{h^2}a_i^{(2)}\delta_{ki}^{(2)} + \frac{1}{h^3}a_i^{(3)}\delta_{ki}^{(3)} + \frac{1}{h^4}g_i\phi_i'^{(4)}\delta_{ki}^{(4)},$$

Where

$$a^{(1)} = 4g'''\phi' + 6g''\phi'' + 4g'\phi''' + g\phi^{(4)},$$

$$a^{(2)} = 6g''\phi'^2 + 12g\phi'\phi'' + 3g\phi''^2 + 4\phi'\phi''',$$

$$a^{(3)} = 4g'\phi'^3 + 6g\phi'^2\phi'',$$

and $\delta_{ki}^{(j)}$ *(j=0,...,4)* are defined by following relations,

$$\delta_{jl}^{(0)} = \begin{cases} 1 & if \quad j = l, \\ 0 & if \quad j \neq l, \end{cases}$$

$$\delta_{jl}^{(1)} = \begin{cases} 0 & if \quad j = l \\ -\dfrac{(-1)^{j-l}}{j-l} & if \quad j \neq l \end{cases}$$

$$\delta_{jl}^{(2)} = \begin{cases} -\dfrac{\pi^2}{3} & if \quad j = l \\ -\dfrac{2(-1)^{j-l}}{(j-l)^2} & if \quad j \neq l \end{cases}$$

$$\delta_{jl}^{(3)} = \begin{cases} 0 & if \quad j = l \\ -\dfrac{(-1)^{j-l}}{(j-l)^3}\left[6 - \pi^2(j-l)^2\right] & if \quad j \neq l \end{cases}$$

$$\delta_{jl}^{(4)} = \begin{cases} \dfrac{\pi^4}{5} & if \quad j = l \\ -\dfrac{(-1)^{j-l}}{(j-l)^4}\left[24 - 4\pi^2(j-l)^2\right] & if \quad j \neq l \end{cases}$$

Put

$$g(x) = \left(\phi'(x)\right)^{-\frac{3}{2}} \tag{4}$$

Then, $a^{(3)}(x)=0$.

Using (4) and $\phi'^{-\frac{3}{2}}\left(\phi'^{-\frac{1}{2}}\right)''=-\frac{1}{4}$, we will have $a^{(1)}(x)=0$ and $a^{(2)}(x)=-\frac{5}{2}\phi'^{\frac{5}{2}}(x)$.

Then,

$$g^{(4)}=\frac{9}{16}\phi'^{\frac{5}{2}}$$

And from the above equality we have:

$$S_k^{(4)}(x_i)=\frac{1}{h^4}\phi_i'^{\frac{5}{2}}\left\{\delta_{ki}^{(4)}-\frac{5}{2}h^2\delta_{ki}^{(2)}+\frac{9}{16}h^4\delta_{ki}^{(0)}\right\}$$

Similarly we will have:

$$\overline{S}_k^{(4)}(y_i)=\frac{1}{h^4}\overline{\phi'}_i^{\frac{5}{2}}\left\{\delta_{ki}^{(4)}-\frac{5}{2}h^2\delta_{ki}^{(2)}+\frac{9}{16}h^4\delta_{ki}^{(0)}\right\}$$

by choosing

$$\overline{g}(y)=\left(\overline{\phi'}(y)\right)^{-\frac{3}{2}}$$

We will have,

$$A_xY+YA_y=h^4W'$$

Where

$$\begin{aligned}A_x&=D\left(\phi'^2(x)\right)AD\left(\phi'^2(x)\right),\\ A_y&=D\left(\overline{\phi'}^2(y)\right)AD\left(\overline{\phi'}^2(y)\right),\end{aligned} \tag{5}$$

$$\begin{aligned}Y&=D\left(\phi'^{-\frac{1}{2}}(x)\right)UD\left(\overline{\phi'}^{-\frac{1}{2}}(y)\right),\\ W'&=D\left(\phi'^{-\frac{1}{2}}(x)\right)WD\left(\overline{\phi'}^{-\frac{1}{2}}(y)\right),\end{aligned} \tag{6}$$

And

$$A=(a_{ik})=\left(\delta_{ki}^{(4)}-\frac{5}{2}h^2\delta_{ki}^{(2)}+\frac{9}{16}h^4\delta_{ki}^{(0)}\right)=I^{(4)}-\frac{5}{2}h^2I^{(2)}+\frac{9}{16}h^4I^{(0)}$$

Since the definite matrix $I^{(4)}$ is positive [6], we can show; with the same method as mentioned before, that both matrices A_x and A_y are symmetric and positive definite. And under the assumption that we have $h_1=h_2=h$ and

$N_1 = N_2 = N$, we have $A_y = k^4 A_x$. Therefore, a method similar to the orthogonalization technique mentioned earlier can also be used for the pseudo-Poisson problem; In this case, matrix Y' becomes as follows:

$$Y' = (y'_{ik}) = \left(\frac{h^4}{\lambda_i + k^4 \lambda_k} \right)$$

Also, the positive parameter α should be chosen so that,

$$|u(x,y)| \le C \left((x-a)(b-y)\right)^{\alpha+\frac{3}{2}}$$

4. Numerical results

In this section, we use the sinc approximation method to numerically solve single fourth-order pseudo-Poisson problems. In the examples of this section, the exact solution has undetermined fourth derivatives on the boundary.

The positive parameter α should be chosen so that the exact solution *u(x,y)* applies in the following condition:

$$|u(x,y)| \le C \left((x-a)(b-y)\right)^{\alpha+\frac{1}{2}} \qquad (7)$$

We take the parameter *h* equal to $\left(\frac{\pi d}{\alpha N}\right)^{\frac{1}{2}}$, where *d* is chosen equal to $\frac{\pi}{2}$ in the investigated problems.

The absolute value of the maximum error between the numerical approximation $(v_N(x,y))$ and the exact solution $(u(x,y))$ at nodal points sinc with $\|E\|_g$ and the absolute value of the maximum error in 100 equidistant points with $\|E\|_u$, which are equidistant points (x_i, y_j) and $x_i = a_1 + \frac{i}{100}(b_1 - a_1)$, $y_j = a_2 + \frac{j}{100}(b_2 - a_2)$ are chosen.

Example 1. Consider the following fourth-order pseudo-Poisson problem

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = f(x,y), \qquad u(x,y) = (1-x^2)^3 (y-1)^3 (4-y)^3,$$

that $\Omega = (-1,1) \times (1,4)$ and $\alpha = \frac{3}{2}$. The results are shown in Table 1.

Example 2. Consider the following fourth-order pseudo- Poisson problem

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = f(x,y), \qquad u(x,y) = y^3 \sin^3(x) \ln^3(y),$$

Where . $\Omega = (0,\pi) \times (0,1)$

The exact answer of $u(x,y)$ has a singular point of $y = 0$.

We select parameter α as $\frac{3}{2}$. The results are shown in Table 2.

Table 1 - The results of example 1

N	h	$\|E\|_g$	$\|E\|_u$
2	1.282550	9.5118D-1	9.5118D-1
4	0.906900	2.7328D-2	5.9729D-2
8	0.641275	3.0343D-3	3.7935D-3
16	0.453450	7.3851D-5	7.3851D-5
32	0.320637	9.3745D-6	9.4706D-6

Table 2 - The results of example 2

N	h	$\|E\|_g$	$\|E\|_u$
2	1.282550	5.8964D-3	8.3871D-3
4	0.906900	2.8201D-3	2.9605D-3
8	0.641275	5.4495D-4	5.4531D-4
16	0.453450	2.9301D-5	2.9605D-5
32	0.320637	2.5942D-7	2.6026D-7

Example 3. Consider the following fourth-order pseudo- Poisson problem

$$\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}=f(x,y), \qquad u(x,y)=\left(x(1-x)\right)^{\frac{7}{2}}\left(y(2-y)\right)^{\frac{10}{3}},$$

Where, $\Omega=(0,1)\times(0,2)$. This example presents a higher order of singularities, where u has undefined fifth derivatives on the boundary Ω. We select the parameter $\alpha=\frac{11}{6}$. The gained results are shown in Table 3.

Table 3 - The results of example 3

N	h	$\|E\|_g$	$\|E\|_u$
2	1.160110	8.5673D-4	8.5673D-4
4	0.820322	1.3176D-5	3.4646D-5
8	0.580055	1.0888D-6	1.4810D-6
16	0.410161	1.5501D-8	1.6013D-8
32	0.290027	3.4867D-10	3.4871D-10

4. Conclusion

This method is efficient in dealing with singularity and work well in contact with fourth-order pseudo-Poisson type of problems, because the singularity of the equation occurs at the end point of the interval. Numerical approximation is obtained whose accuracy is exponential and of order $O(\exp(-c\sqrt{N})$ where N is a transaction parameter and c is a constant independent of N.

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