On generalization of a subclass of p-valent harmonic functions

[1]Waggas Galib Atshan & [2]Asraa Abdul Jaleel Husien

Department of Mathematics College of Computer Science and Mathematics
University of Al-Qadisiya ,   Diwaniya – Iraq
E-mail: [1]waggashnd@gmail.com, waggas_hnd@yahoo.com
[2] asraalsade1@gmail.com, asra.alsade@yahoo.com

Abstract. In the present paper, we introduce a generalization of subclass of p-valent harmonic functions in the open unit disk *U*, we obtain various properties for this subclass, like, coefficient bounds, extreme points, distortion theorem, convex set, convolution property and convex combinations.

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Introduction: A continuous complex-valued function $f = u + iv$ defined in a simple connected   complex domain D is said to be harmonic in D, if both u and v are real harmonic in D.   Let $f = h + \bar{g}$ be defined in any simply connected domain, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D. (See Clunie and Sheil-Small [6]).

Denote by *H(p)* the class of functions $f = h + \bar{g}$ that are harmonic p- valent and sense-preserving in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For $f = h + \bar{g} \in H(p)$, we may express the analytic functions h and g as

$$h(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n , \qquad g(z) = \sum_{n=p}^{\infty} b_n z^n , \quad |b_p| < 1. \qquad (1.1)$$

Let *A(p)* denote the subclass of *H(p)* consisting of functions $f = h + \bar{g}$, where h and g are given by

$$h(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n , g(z) = - \sum_{n=p}^{\infty} b_n z^n , (a_n \geq 0, b_n \geq 0, |b_p| < 1). \qquad (1.2)$$

We introduce here a class $H_q^{\beta}(p, \alpha)$ of harmonic functions of the form (1.1) that satisfy the inequality

$$Re\left\{(1-\beta)\frac{f^{(q)}(z)}{\delta(p,q)z^{p-q}}+\beta\frac{f^{(q+1)}(z)}{\delta(p,q+1)z^{p-(q+1)}}\right\}$$

$$\geq \frac{\alpha}{\delta(p,q+1)}, \qquad (1.3)$$

where

$$f^{(q)}(z) = \delta(p,q)z^{p-q} + \sum_{n=p+1}^{\infty} \delta(n,q)\,a_n\,z^{n-q}\,,$$

$$\delta(i,j) = \frac{i!}{(i-j)!} = \begin{cases} 1 & j=0 \\ i(i-1)\dots(i-j+1) & j\neq 0\,, \end{cases}$$

$0 \leq \alpha < p, p \in N, q \in N_0 = N \cup \{0\}, p > q$ and $\beta \geq 0$.

We deem it worthwhile to point here the relevance of the function class $H_q^{\beta}(p,\alpha)$ with those classes of functions which have been studied recently. Indeed, we observe that:

(i) $H_0^1(1,0) \equiv S_H^*$ (Silverman[10]); H_{λ}^k(Darus and Al Shaqsi [7]) and $H(\lambda)$(Yalçin and Öztürk][11]).

(ii) $H_0^1(1,\alpha) \equiv N_H(\alpha)$ (Ahuja and Jahangiri [1]);

(iii) $H_0^1(p,\alpha) \equiv H_{\lambda}^k(p,\alpha)$ (Al Shaqsi and Darus [2]);

(iv) $H_0^{\beta}(p,\alpha) \equiv H(p,\alpha,\beta)$(Atshan et al. [4]).

Also, we note that the analytic part of the class $H_0^1(p,\alpha)$ was introduced and studied by Goel and Sohi [8].

Also, several authors studied some same properties of various other classes, like, Aouf et al. [3], Joshi and Sangle [9] and Atshan and Wanas [5].

We further denote by $A_q^{\beta}(p,\alpha)$ the subclass of $H_q^{\beta}(p,\alpha)$ that satisfies the relation

$$A_q^{\beta}(p,\alpha) = A(p) \cap H_q^{\beta}(p,\alpha). \qquad (1.4)$$

Coefficient bounds

First, we determine the sufficient condition for $f = h + \bar{g}$ to be in the class $H_q^{\beta}(p,\alpha)$.
Theorem 1. Let $f = h + \bar{g}$ (h and g being given by (1.1)). If

$$\sum_{n=p+1}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]\,|a_n| + \sum_{n=p}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]\,|b_n|$$
$$\leq \delta(p,q+1)-\alpha,$$
(2.1)

where $\beta \geq 0$, $0 \leq \alpha < p$, $p \in N$, $q \in N_0$ and $p > q$, then f is harmonic p-valent sense-preserving in U and $f \in H_q^\beta(p,\alpha)$.

Proof. Let $w(z) = (1-\beta)\dfrac{f^{(q)}(z)}{\delta(p,q)z^{p-q}} + \beta\dfrac{f^{(q+1)}(z)}{\delta(p,q+1)z^{p-(q+1)}}$. To prove that $Re\{w(z)\}$ $\geq \dfrac{\alpha}{\delta(p,q+1)}$, it is sufficient to show equivalently that

$$|(p-\alpha)+\delta(p,q+1)w(z)| \geq |(p+\alpha)-\delta(p,q+1)w(z)|.$$

Substituting for $w(z)$ and resorting to simple calculations, we find that

$$|(p-\alpha)+\delta(p,q+1)w(z)|$$

$$= \Bigg| (p-\alpha)$$

$$+\delta(p,q$$

$$+1)\Bigg[(1-\beta)\frac{\left(\delta(p,q)z^{p-q} + \sum\limits_{n=p+1}^{\infty}\delta(n,q)\,a_n\,z^{n-q} + \sum\limits_{n=p}^{\infty}\delta(n,q)\,b_n\,(\overline{z})^{n-q}\right)}{\delta(p,q)z^{p-q}}$$

$$+\beta\frac{\left(\delta(p,q+1)z^{p-(q+1)} + \sum\limits_{n=p+1}^{\infty}\delta(n,q+1)a_n z^{n-(q+1)} + \sum\limits_{n=p}^{\infty}\delta(n,q+1)b_n(\overline{z})^{n-(q+1)}\right)}{\delta(p,q+1)z^{p-(q+1)}}\Bigg]\Bigg|$$

$$\geq (p+\delta(p,q+1)-\alpha) - \sum_{n=p+1}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]\,|a_n||z|^{n-p}$$

$$- \sum_{n=p}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]\,|b_n|\,|z|^{n-p} \quad (2.2)$$

and
$$|(p+\alpha)-\delta(p,q+1)w(z)|$$

$$= \left| (p+\alpha) \right.$$

$$- \delta(p,q+1)\left[(1-\beta)\frac{\left(\delta(p,q)z^{p-q} + \sum_{n=p+1}^{\infty} \delta(n,q)\, a_n\, z^{n-q} + \sum_{n=p}^{\infty} \delta(n,q)\, b_n\, (\overline{z})^{n-q} \right)}{\delta(p,q)z^{p-q}} \right.$$

$$\left. \left. + \beta\frac{\left(\delta(p,q+1)z^{p-(q+1)} + \sum_{n=p+1}^{\infty} \delta(n,q+1)a_n z^{n-(q+1)} + \sum_{n=p}^{\infty} \delta(n,q+1)b_n(\overline{z})^{n-(q+1)} \right)}{\delta(p,q+1)z^{p-(q+1)}} \right] \right|$$

$$\leq (p-\delta(p,q+1)+\alpha) + \sum_{n=p+1}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]\,|a_n|\,|z|^{n-p}$$

$$+ \sum_{n=p}^{\infty} \delta(n,q)[(p-q)$$

$$+\beta(n-p)]\,|b_n||z|^{n-p}\,, \qquad (2.3)$$

from (2.2) and (2.3), we have

$$|(p-\alpha)+\delta(p,q+1)w(z)| - |(p+\alpha)-\delta(p,q+1)w(z)|$$

$$> 2\left\{ (\delta(p,q+1)-\alpha) - \sum_{n=p+1}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]\,|a_n| \right.$$

$$\left. - \sum_{n=p}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]\,|b_n| \right\} \geq 0.$$

The harmonic functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{x_n}{\delta(n,q)[(p-q)+\beta(n-p)]}\, z^n$$

$$+ \sum_{n=p}^{\infty} \frac{\overline{y_n}}{\delta(n,q)[(p-q)+\beta(n-p)]}\, (\overline{z})^n \qquad (2.4)$$

$\left(\sum\limits_{n=p+1}^{\infty} |x_n| + \sum\limits_{n=p}^{\infty} |y_n| = \delta(p, q+1) - \alpha \right)$ show that the coefficient bound given by (2.1)is sharp.

The functions of the form (2.4) are in $H_q^\beta(p, \alpha)$ because in view of (2.1), we have

$$\sum_{n=p+1}^{\infty} \delta(n, q)[(p-q)+\beta(n-p)] |a_n| + \sum_{n=p}^{\infty} \delta(n, q)[(p-q)+\beta(n-p)] |b_n|$$

$$= \sum_{n=p+1}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]. \frac{|x_n|}{\delta(n,q)[(p-q)+\beta(n-p)]}$$

$$+ \sum_{n=p}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]. \frac{|y_n|}{\delta(n,q)[(p-q)+\beta(n-p)]}$$

$$= \sum_{n=p+1}^{\infty} |x_n| + \sum_{n=p}^{\infty} |y_n| = \delta(p, q+1) - \alpha.$$

Now, we need to prove that the condition (2.1) is also necessary for functions of (1.2) to be in the class $A_q^\beta(p, \alpha)$.

Theorem 2. Let $= h + \bar{g}$, where h and g are given by (1.2). Then $f \in A_q^\beta(p, \alpha)$ if and only if

$$\sum_{n=p+1}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)] \, a_n + \sum_{n=p}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)] \, b_n$$

$\leq \delta(p, q+1) - \alpha,$ (2.5)
where $\beta \geq 0, 0 \leq \alpha < p, p \in N, q \in N_0$ and $p > q$

Proof. Since $A_q^\beta(p, \alpha) \subset H_q^\beta(p, \alpha),$ we only need to prove the "only if" part of this theorem.

Let $f(z) \in A_q^\beta(p, \alpha).$ Using (1.3), we get

$$Re\left\{ \delta(p, q+1) \left[(1-\beta) \left(\frac{h^{(q)}(z) + \overline{g^{(q)}(z)}}{\delta(p,q)z^{p-q}} \right) + \beta \left(\frac{h^{(q+1)}(z) + \overline{g^{(q+1)}(z)}}{\delta(p, q+1)z^{p-(q+1)}} \right) \right] \right\}$$

$$= Re \left\{ \delta(p, q+1)(1 \right.$$

$$\left. -\beta\ ) \left(\frac{\left(\delta(p,q)z^{p-q} - \sum_{n=p+1}^{\infty} \delta(n,q)a_n z^{n-q} - \sum_{n=p}^{\infty} \delta(n,q)b_n (\overline{z})^{n-q} \right)}{\delta(p,q)z^{p-q}} \right) \right.$$

$$+\delta\ (p,q$$

$$+1)\beta \left(\frac{\left(\delta(p,q+1)z^{p-(q+1)} - \sum_{n=p+1}^{\infty} \delta(n,q+1)a_n z^{n-(q+1)} - \sum_{n=p}^{\infty} \delta(n,q+1)b_n (\overline{z})^{n-(q+1)} \right)}{\delta(p,q+1)z^{p-(q+1)}} \right) \right\}$$

$$= Re \left\{ \delta(p,q+1) - \sum_{n=p+1}^{\infty} \delta(n,q)[(p-q) + \beta(n-p)]\, a_n\, z^{n-p} \right.$$

$$\left. - \sum_{n=p}^{\infty} \delta(n,q)[(p-q) + \beta(n-p)]\, b_n\, (\overline{z})^{n-p} \right\} \geq \alpha.$$

If we choose z to be real and $z \rightarrow 1^-$, we get

$$\delta(p,q+1) - \sum_{n=p+1}^{\infty} \delta(n,q)[(p-q) + \beta(n-p)]\, a_n$$

$$- \sum_{n=p}^{\infty} \delta(n,q)[(p-q) + \beta(n-p)]\, b_n \geq \alpha.$$

Hence

$$\sum_{n=p+1}^{\infty} \delta(n,q)[(p-q) + \beta(n-p)]\, a_n + \sum_{n=p}^{\infty} \delta(n,q)[(p-q) + \beta(n-p)]\, b_n$$

$$\leq \delta(p,q+1) - \alpha,$$

which completes the proof of Theorem 2.

Some properties for the class $A_q^{\beta}(p,\alpha)$

Extreme points for the class $A_q^{\beta}(p,\alpha)$ are given in the following Theorem.

 Theorem 3. Let the function $f(z)$ be given by (1.2). Then $f(z) \in A_q^{\beta}(p,\alpha)$ if and only if

$$f(z) = \sum_{n=p}^{\infty} (\lambda_n h_n(z) + \mathcal{E}_n g_n(z)), \qquad (3.1)$$

where $z \in U$, $h_p(z) = z^p$,

$$h_n(z) = z^p$$
$$- \frac{\delta(p, q+1) - \alpha}{\delta(n, q)[(p-q) + \beta(n-p)]} z^n, \qquad (n = p+1, p+2, \dots) \quad (3.2)$$

and

$$g_n(z) = z^p - \frac{\delta(p, q+1) - \alpha}{\delta(n, q)[(p-q) + \beta(n-p)]} (\bar{z})^n, \qquad (n = p, p+1, \dots), \quad (3.3)$$

$$\sum_{n=p}^{\infty} (\lambda_n + \mathcal{E}_n) = 1 \ (\lambda_n \geq 0, \mathcal{E}_n \geq 0).$$

In particular, the extreme points of $A_q^\beta(p, \alpha)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. Suppose $f(z)$ is of the form (3.1). Using (3.2) and (3.3), we get

$$f(z) = \sum_{n=p}^{\infty} (\lambda_n h_n(z) + \mathcal{E}_n g_n(z))$$

$$= \sum_{n=p}^{\infty} (\lambda_n + \mathcal{E}_n) z^p - \sum_{n=p+1}^{\infty} \frac{\delta(p, q+1) - \alpha}{\delta(n, q)[(p-q) + \beta(n-p)]} \lambda_n z^n$$

$$- \sum_{n=p}^{\infty} \frac{\delta(p, q+1) - \alpha}{\delta(n, q)[(p-q) + \beta(n-p)]} \mathcal{E}_n (\bar{z})^n$$

$$= z^p - \sum_{n=p}^{\infty} \frac{\delta(p, q+1) - \alpha}{\delta(n, q)[(p-q) + \beta(n-p)]} \lambda_n (z)^n$$

$$- \sum_{n=p}^{\infty} \frac{\delta(p, q+1) - \alpha}{\delta(n, q)[(p-q) + \beta(n-p)]} \mathcal{E}_n (\bar{z})^n,$$

then

$$\sum_{n=p+1}^{\infty} \delta(n, q)[(p-q) + \beta(n-p)] \frac{\delta(p, q+1) - \alpha}{\delta(n, q)[(p-q) + \beta(n-p)]} \lambda_n$$

$$+ \sum_{n=p}^{\infty} \delta(n, q)[(p-q) + \beta(n-p)] \frac{\delta(p, q+1) - \alpha}{\delta(n, q)[(p-q) + \beta(n-p)]} \mathcal{E}_n$$

$$= (\delta(p, q+1) - \alpha)\left[\sum_{n=p}^{\infty}(\lambda_n + \mathcal{E}_n) - \lambda_p\right] = (\delta(p, q+1) - \alpha)[1 - \lambda_p]$$

$$\leq \delta(p, q+1) - \alpha$$

and so $f(z) \in A_q^\beta(p, \alpha)$.

Conversely, assume that $f(z) \in A_q^\beta(p, \alpha)$. Then

$$a_n \leq \frac{\delta(p, q+1) - \alpha}{\delta(n, q)[(p-q) + \beta(n-p)]}\lambda_n,$$

and

$$b_n \leq \frac{\delta(p, q+1) - \alpha}{\delta(n, q)[(p-q) + \beta(n-p)]}\mathcal{E}_n.$$

Set

$$\lambda_n = \frac{\delta(n, q)[(p-q) + \beta(n-p)]}{\delta(p, q+1) - \alpha}a_n \quad (n = p+1, p+2, \ldots),$$

$$\mathcal{E}_n = \frac{\delta(n, q)[(p-q) + \beta(n-p)]}{\delta(p, q+1) - \alpha}b_n \quad (n = p, p+1, \ldots),$$

where $\sum_{n=p}^{\infty}(\lambda_n + \mathcal{E}_n) = 1$, we have

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n - \sum_{n=p}^{\infty} b_n (\bar{z})^n$$

$$= z^p - \sum_{n=p+1}^{\infty}\frac{\delta(p, q+1) - \alpha}{\delta(n, q)[(p-q) + \beta(n-p)]}\lambda_n z^n$$

$$- \sum_{n=p}^{\infty}\frac{\delta(p, q+1) - \alpha}{\delta(n, q)[(p-q) + \beta(n-p)]}\mathcal{E}_n(\bar{z})^n.$$

$$= z^p - \sum_{n=p+1}^{\infty}(z^p - h_n(z))\lambda_n - \sum_{n=p}^{\infty}(z^p - g_n(z))\mathcal{E}_n$$

$$= \left[1 - \left(\sum_{n=p+1}^{\infty}\lambda_n + \sum_{n=p}^{\infty}\mathcal{E}_n\right)\right]z^p + \sum_{n=p+1}^{\infty}\lambda_n h_n(z) + \sum_{n=p}^{\infty}\mathcal{E}_n g_n(z)$$

$$= \lambda_p \, h_p(z) + \sum_{n=p+1}^{\infty} \lambda_n \, h_n(z) + \sum_{n=p}^{\infty} \varepsilon_n \, g_n(z)$$

$$= \sum_{n=p}^{\infty} (\lambda_n \, h_n(z) + \varepsilon_n \, g_n(z)).$$

This completes the proof of Theorem 3.

The next, we get the distortion Theorem.

Theorem 4. Let the function $f(z)$ defined by (1.2) be in the class $A_q^\beta(p,\alpha)$. Then for $|z| = r < 1$, we have

$$|f(z)| \le (1 + b_p)r^p + \left| \frac{\delta(p,q+1) - \alpha}{\delta(p+1,q)[p-q+\beta]} - \frac{\delta(p,q)(p-q)b_p}{\delta(p+1,q)[p-q+\beta]} \right| r^{p+1},$$

(3.4)

and

$$|f(z)| \ge (1 - b_p)r^p - \left| \frac{\delta(p,q+1) - \alpha}{\delta(p+1,q)[p-q+\beta]} - \frac{\delta(p,q)(p-q)b_p}{\delta(p+1,q)[p-q+\beta]} \right| r^{p+1}.$$

(3.5)

The equalities in (3.4)and (3.5) are attained for the functions $f(z)$ given by

$$f(z) = z^p + b_p(\bar{z})^p + \left| \frac{\delta(p,q+1) - \alpha}{\delta(p+1,q)[p-q+\beta]} - \frac{\delta(p,q)(p-q)b_p}{\delta(p+1,q)[p-q+\beta]} \right| (\bar{z})^{p+1},$$

and

$$f(z) = z^p - b_p(\bar{z})^p - \left| \frac{\delta(p,q+1) - \alpha}{\delta(p+1,q)[p-q+\beta]} - \frac{\delta(p,q)(p-q)b_p}{\delta(p+1,q)[p-q+\beta]} \right| (\bar{z})^{p+1}.$$

Proof. Suppose $(z) \in A_q^\beta(p,\alpha)$. Using (1.1)and (2.1) of Theorem 1, we find that

$$|f(z)| \le (1 + b_p)r^p + \sum_{n=p+1}^{\infty} (a_n + b_n)\, r^n \le (1 + b_p)r^p + \sum_{n=p+1}^{\infty} (a_n + b_n)\, r^{p+1}$$

$$= (1 + b_p)r^p + \frac{\delta(p,q+1) - \alpha}{\delta(p+1,q)[p-q+\beta]} \times$$

$$\sum_{n=p+1}^{\infty} \frac{\delta(p+1,q)[p-q+\beta]}{\delta(p,q+1) - \alpha}(a_n + b_n)\, r^{p+1}$$

$$\leq (1+b_p)r^p + \frac{\delta(p,q+1)-\alpha}{\delta(p+1,q)[(p-q)+\beta]} \times$$

$$\sum_{n=p+1}^{\infty} \frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha}(a_n+b_n)\,r^{p+1}$$

$$\leq (1+b_p)r^p + \frac{\delta(p,q+1)-\alpha}{\delta(p+1,q)[p-q+\beta]}\left[1-\frac{\delta(p,q)(p-q)b_p}{\delta(p,q+1)-\alpha}\right]r^{p+1}$$

$$= (1+b_p)r^p + \left[\frac{\delta(p,q+1)-\alpha}{\delta(p+1,q)[p-q+\beta]} - \frac{\delta(p,q)(p-q)b_p}{\delta(p+1,q)[p-q+\beta]}\right]r^{p+1}.$$

Similarly, we can prove the left- hand inequality.
This completes the proof of Theorem 4.

Theorem 5. The class $A_q^\beta(p,\alpha)$ is convex set.

   Proof. Let the function $f_j(z)$ $(j=1,2)$ be in the class $A_q^\beta(p,\alpha)$. It is sufficient to show that the function h defined by

$$h(z) = (1-t)\,f_1(z) + t\,f_2(z),\,(0 \leq t < 1)$$

is in the class $A_q^\beta(p,\alpha)$,
where

$$f_j(z) = z^p - \sum_{n=p+1}^{\infty} a_{n,j}\,z^n - \sum_{n=p}^{\infty} b_{n,j}\,(\bar{z})^n,\ (j=1,2)$$

since for $0 \leq t \leq 1$,

$$h(z) = z^p - \sum_{n=p+1}^{\infty}\left((1-t)a_{n,1} - ta_{n,2}\right)z^n - \sum_{n=p}^{\infty}\left((1-t)b_{n,1} - tb_{n,2}\right)(\bar{z})^n.$$

In view of Theorem 2, we have

$$\sum_{n=p+1}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]\left((1-t)a_{n,1}-ta_{n,2}\right)$$

$$+\sum_{n=p}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]\left((1-t)b_{n,1}-tb_{n,2}\right)$$

$$= (1-t)\left(\sum_{n=p+1}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]a_{n,1}\right.$$

$$+ \sum_{n=p}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]b_{n,1}\bigg)$$

$$+t\left(\sum_{n=p+1}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]a_{n,2}\right.$$

$$+ \sum_{n=p}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]b_{n,2}\bigg)$$

$$\leq (1-t)[\delta(p,q+1)-\alpha] + t\,[\delta(p,q+1)-\alpha] = \delta(p,q+1)-\alpha.$$

Hence $h(z) \in A_q^{\beta}(p,\alpha)$.

We define the convolution two harmonic functions $f(z)$ and $F(z)$ by

$$(f * F)(z) = z^p - \sum_{n=p+1}^{\infty} a_n A_n\, z^n - \sum_{n=p}^{\infty} b_n B_n\,(\overline{z})^n, \qquad (3.6)$$

where

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n\, z^n - \sum_{n=p}^{\infty} b_n\,(\overline{z})^n,$$

and

$$F(z) = z^p - \sum_{n=p+1}^{\infty} A_n\, z^n - \sum_{n=p}^{\infty} B_n\,(\overline{z})^n.$$

Theorem 6. For $0 \leq \sigma \leq \alpha < p$, let $f(z) \in A_q^{\beta}(p,\alpha)$ and $F(z) \in A_q^{\beta}(p,\sigma)$. Then

$$(f * F)(z) \in A_q^{\beta}(p,\alpha) \subset A_q^{\beta}(p,\sigma).$$

Proof. Let the convolution $(f * F)(z)$ be of the form (3.6), Then we want to prove that the coefficient of $(f * F)(z)$ satisfy the condition of the Theorem 2.

Since $f(z) \in A_q^{\beta}(p,\alpha)$ and $(z) \in A_q^{\beta}(p,\sigma)$, Then by Theorem 2, we have

$$\sum_{n=p+1}^{\infty} \frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha}\,a_n + \sum_{n=p}^{\infty} \frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha}\,b_n$$

$$\leq 1, \quad (3.7)$$

and

$$\sum_{n=p+1}^{\infty} \frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\sigma} A_n + \sum_{n=p}^{\infty} \frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\sigma} B_n$$

$$\leq 1. \quad (3.8)$$

From (3.8), we obtain the following inequalities

$$A_n < \frac{\delta(p,q+1)-\sigma}{\delta(n,q)[(p-q)+\beta(n-p)]} \quad (n=p+1, p+2, \dots),$$

$$B_n < \frac{\delta(p,q+1)-\sigma}{\delta(n,q)[(p-q)+\beta(n-p)]} \quad (n=p, p+1, p+2, \dots).$$

Therefore,

$$\sum_{n=p+1}^{\infty} \frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha} a_n A_n$$

$$+ \sum_{n=p}^{\infty} \frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha} b_n B_n$$

$$= \frac{\delta(p,q)(p-q)}{\delta(p,q+1)-\alpha} b_p B_p + \sum_{n=p+1}^{\infty} \frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha} a_n A_n$$

$$+ \sum_{n=p+1}^{\infty} \frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha} b_n B_n$$

$$< \frac{[\delta(p,q)(p-q)][\delta(p,q+1)-\sigma]}{[\delta(p,q+1)-\alpha][\delta(p,q)(p-q)]} b_p$$

$$+ \sum_{n=p+1}^{\infty} \frac{\delta(n,q)[(p-q)+\beta(n-p)][\delta(p,q+1)-\sigma]}{[\delta(p,q+1)-\alpha](\delta(n,q)[(p-q)+\beta(n-p)])} a_n$$

$$+ \sum_{n=p+1}^{\infty} \frac{\delta(n,q)[(p-q)+\beta(n-p)][\delta(p,q+1)-\sigma]}{[\delta(p,q+1)-\alpha](\delta(n,q)[(p-q)+\beta(n-p)])} b_n$$

$$= \sum_{n=p+1}^{\infty} \frac{\delta(n,q)[(p-q)+\beta(n-p)][\delta(p,q+1)-\sigma]}{[\delta(p,q+1)-\alpha](\delta(n,q)[(p-q)+\beta(n-p)])} a_n$$

$$+\sum_{n=p}^{\infty}\frac{\delta(n,q)[(p-q)+\beta(n-p)][\delta(p,q+1)-\sigma]}{[\delta(p,q+1)-\alpha](\delta(n,q)[(p-q)+\beta(n-p)])}b_n$$

$$\leq\sum_{n=p+1}^{\infty}\frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha}a_n+\sum_{n=p}^{\infty}\frac{\delta(n,q)[(p-q)+\beta(n-p)]}{\delta(p,q+1)-\alpha}b_n\leq 1.$$

Then

$(f*F)(z)\in A_q^{\beta}(p,\alpha)\subset A_q^{\beta}(p,\sigma)$, and the proof is complete.

Next, theorem to prove the class $A_q^{\beta}(p,\alpha)$ is closed under convex combination.

Theorem 7. Let $0\leq c_i<1$ for $i=1,2,\dots$ and $\sum_{i=1}^{\infty}c_i=1$, if the functions $f(z)$ defined by

$$f_i(z)=z^p-\sum_{n=p+1}^{\infty}a_{n,i}\,z^n-\sum_{n=p}^{\infty}b_{n,i}\,(\bar{z})^n,\ \ (z\in U,i=1,2,\dots),$$

are in the class $A_q^{\beta}(p,\alpha)$, for every $i=1,2,\dots$, then $\sum_{i=1}^{\infty}c_i\,f_i(z)$ of the form

$$\sum_{i=1}^{\infty}c_i\,f_i(z)=z^p-\sum_{n=p+1}^{\infty}\left(\sum_{i=1}^{\infty}c_i\,a_{n,i}\right)z^n-\sum_{n=p}^{\infty}\left(\sum_{i=1}^{\infty}c_i\,b_{n,i}\right)(\bar{z})^n,$$

is in the class $A_q^{\beta}(p,\alpha)$.

Proof. Since $f_i(z)\in A_q^{\beta}(p,\alpha)$, it follows from Theorem 2 that

$$\sum_{n=p+1}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]a_{n,i}$$

$$+\sum_{n=p}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]b_{n,i}\leq\delta(p,q+1)-\alpha,$$

for every $i=1,2,\dots$. Hence

$$\sum_{n=p+1}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]\left(\sum_{i=1}^{\infty}c_i\,a_{n,i}\right)$$

$$+\sum_{n=p}^{\infty}\delta(n,q)[(p-q)+\beta(n-p)]\left(\sum_{i=1}^{\infty}c_i\,b_{n,i}\right)$$

$$= \sum_{i=1}^{\infty} c_i \left(\sum_{n=p+1}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]a_{n,i} + \sum_{n=p}^{\infty} \delta(n,q)[(p-q)+\beta(n-p)]b_{n,i} \right)$$

$$\leq \sum_{i=1}^{\infty} c_i \left(\delta(p,q+1)-\alpha \right) = \delta(p,q+1)-\alpha.$$

Thus $\sum_{i=1}^{\infty} c_i f(z) \in A_q^{\beta}(p,\alpha)$ and the proof is complete.

Reference
[1] O. Ahuja and J. Jahangiri, Noshiro-type harmonic univalent functions, Sci. Math. Japan, 65 (2002), 293–299.
[2] K. Al Shaqsi and M. Darus, A new class of multivalent harmonic functions, General Math., 14 (2006), 37–46.
[3] M. K. Aouf, A. A. Shamandy, A. O. Mostafa and A. K. Wagdy, A new class of harmonic
p-valent functions of complex order, proceedings of Pakistan Academy of Sciences, 49 (3)(2012), 219-225.
[4] W. G. Atshan, S. R. Kulkaini and R. K. Raina, A class of multivalent harmonic functions involving a generalized Ruscheweyh type operator, Matematicki vesink, 60(2008), 207-213.
[5] W. G. Atshan and A. K. Wanas, On a new class of harmonic univalent functions, Matematicki vesink, 65(2013), 1-10.
[6] J. Clunie and T. Shell-Small, Harmonic univalent functions, Ann. Acad. Aci. Fenn. Ser. AI Math., 9 (1984), 3–25.
[7] M. Darus and K. Al Shaqsi, On harmonic univalent functions defined by a generaliz-zed Ruscheweyh derivatives operator, Lobachevskii J. Math., 22 (2006), 19–26.
[8] R. Goel and N. Sohi, New criteria for p-valent, Indian J. Pure Appl. Math., 7(2004), 55–61.
[9] S. B. Joshi and N. D. Sangle, New subclass of Goodman-type p-valent harmonic fun ctions, Facully of Sciences and Math., Unversity of Niš, Serbia, 22(1)(2008), 193-204.
[10] H. Silverman, Harmonic univalent functions with negative coefficients, Proc. Amer. Math. Soc., 51 (1998), 283–289.
[11] S. Yalçin and M. Öztürk, A new subclass of complex harmonic functions, Math. Inequal. Appl., 7 (2004), 55–61.