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On Fuzzy Group Spaces

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Abstract
The main aim of this work is to fuzzify the concept of group action to create a new concept (to the best of our knowledge) namely fuzzy group spaces. And investigate the properties of these spaces.

Introduction
One of very important concepts in geometrical topology is the concept of group actions. In this work, we fuzzify this concept as a natural transition from the corresponding crisp structure to create the concept of fuzzy action, which considered as basis of our main definition and clarified the properties of them.In 1965, Zadeh introduced the concept of Fuzzy set [15]. Chang used this concept to introduce the first definition of fuzzy topological spaces [2]. Later, Lowen introduced new definition for these spaces by using the concept of constant fuzzy set [6].The difference between two definitions led to difference definitions for the fuzzy topological group.After that, Foster David also introduced his definition of fuzzy topological group based on Lowen’s definition [4], while Whilst, Yu Chun Hai and Ma Ji Liang introduced their definitions based on Chang's definition [12, 13, 14]. In 2009, Ail tazid introduced a concept of fuzzy topological transformation group by using the Lowen and Foster’s definitions [1].In this paper, the Chang and Yu Chun’s definitions used to construct a definition of fuzzy action and fuzzy group spaces and study the properties of these concepts.

1. Preliminaries

In this section, we recall some preliminary definitions and results to be used in the sequel.

Throughout this paper, the symbol I will denote the unite interval $[0,1]$. Let X be a set,a fuzzy subset A of X is a mapping$A : X \to [0,1]$. Any fuzzy set taking values 0 and 1 only, said to be crisp.The support of a fuzzy set A is a set defined as follows: $supp(A) = \{x \in X : A(x) > 0\}$. [15, 5]

Let A, B be fuzzy sets in X then:

$A \leq B \Leftrightarrow A(x) \leq B(x)$, for all $x \in X$.

$C = A \vee B \Leftrightarrow C(x) = max\{A(x),B(x)\}$, for all $x \in X$(union).

$D = A \wedge B \Leftrightarrow D(x) = min\{A(x),B(x)\}$, for all $x \in X$ (intersection).

Let f be a mapping from a set X to a set Y. Then The inverse image of a fuzzy set Bin Y, written $f^{-1}(B)$ is a fuzzy set in X, such that:

$(f^{-1}(B))(x) = B(f(x))$for all $x \in X$;

The image of a fuzzy set Ain X, written $f(A)$, is a fuzzy set in Y, such that:

$$(f(A))(x) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} A(x) & \text{if} f^{-1}(y) \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

For all $y \in Y$, where $f^{-1}(y) = \{x \in X: f(x) = y\}$[4].

The constant fuzzy set C_α in X defined as follows:$C_\alpha(x) = \alpha$ for all $x \in X$, thus $0 = C_0$ and $1 = C_1$. A fuzzy point $x_\alpha$in X is a fuzzy set defined as follows:

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x; \\ 0 & \text{if } y \neq x. \end{cases}$$

Where $0 < \alpha \leq 1$; α is called its value and x is support for x_α. The set of all fuzzy points in X will be denoted $FP(X)$. The fuzzy point x_α is said to be contained in a fuzzy set A, or belong to A, denoted by $x_\alpha \in A$, if and only if $\alpha \leq A(x)$.A fuzzy point x_λ is said to be $\text{qusai} - \text{coincedent}$ with A, denoted by $x_\lambda qA$ if and only if $\lambda + A(x) > 1$. A fuzzy set A is said to be quasi-coincedent with B, denoted by AqB, if and only if there exists $x \in X$ such that $A(x) > B^c(x)$, or $A(x) + B(x) > 1$[12].

Let D be a direct set, X be an set, and FP be the set of all the fuzzy point in X. The mapping $x: D \longrightarrow FP$ is called a fuzzy net in X. A fuzzy net is often denoted by $\{x_\alpha\}_{\alpha \in D}$.A fuzzy net $\{t_\beta\}_{\beta \in E}$in X is called a fuzzy subnet of a fuzzy net $\{x_\alpha\}_{\alpha \in D}$ in X if and only if there is a mapping $N: E \longrightarrow D$ such that:

$t = x \circ N$, that is, for each $i \in E, t_i = x_{N(i)}$;

For each $\alpha \in D$, there exists some $m \in E$ such that, if $E \ni p \leq m, N(p) \leq \alpha$.

Let $x = \{x_\alpha\}_{\alpha \in D}$be a fuzzy net in X, then x is said to be:Quasi-coincident with $A \in I^X$ if and only if for each $\alpha \in D$, x_α is quasi-coincident with A; Eventually quasi-coincident with A if and only if for each $\alpha \in D$ there exist an m in D such that $x_\alpha$is quasi-coincident with A for all $\alpha \geq m$; Frequently quasi-coincident with A if and only if for each $\alpha \in D$ there exist an n in D such that $n \geq \alpha$ and x_n is quasi-coincident with A; In A, if and only if for each $\alpha \in D$, $x_\alpha \in A$ [5].

A fuzzy topology is a family $\mathcal{T}$ of fuzzy sets in X which satisfies the following conditions:

$0_X, 1_X \in \mathcal{T}$;

If $A, B \in \mathcal{T}$, then $A \wedge B \in \mathcal{T}$;

iii. If $A_i \in \mathcal{T}$ for each $i \in I$, then $\bigvee_{i \in I} A_i \in \mathcal{T}$.

The pair $(X, \mathcal{T})$called fuzzy topological space, for short fts X. Every member of $\mathcal{T}$ is called an open fuzzy set, a fuzzy set is a closed if and only if its complement is open[2]. A fully-stratified topology on X is a fuzzy topology on X contain all constant fuzzy set [6].A fuzzy set U in a fts $(X, \mathcal{T})$ is a neighborhood, at short nbhd, of a fuzzy point x_r if and only if there is $B \in \mathcal{T}$, such that $x_r \in B \leq U$. The nbhd system of a fuzzy point is the family of all nbhd's of this point. A fuzzy set U in a fts X is a neighborhood, at short nbhd, of a fuzzy set A, if and only if there exist an open fuzzy set B in X such that $A \leq B \leq U$. A fuzzy set U in a fts $(X,\mathcal{T})$ is called a $\mathrm{Q-neighborhood}$ of x_λ if and only if there exists a $B \in \mathcal{T}$ such that $x_\lambda q B < A$. The family consisting of all the $\mathrm{Q-neighborhoods}$ of x_λ is called the system of $\mathrm{Q-neighborhoods}$ of x_λ.The interior and closure of a fuzzy set A in X defined respectively:$A^\circ = \sup\{B : B \in \mathcal{T}, B \leq A\}$; $\bar{A} = \inf\{B : B^c \in \mathcal{T}, A \leq B\}$ [7].

Let $\{x_\alpha\}_{\alpha \in D}$ be a fuzzy net in a fuzzy topological space X, then $\{x_\alpha\}_{\alpha \in D}$ is:Converge to z if $\{x_\alpha\}_{\alpha \in D}$ is eventually quasi – coincident with each $\mathrm{Q-neighborhood}$ of z, written $x_\alpha \longrightarrow z$;Has z as a cluster point if $\{x_\alpha\}_{\alpha \in D}$ is frequently quasi – coincident with each $\mathrm{Q-neighborhood}$ of z, written $x_\alpha \propto z$.In a fts $(X, \mathcal{T})$, a fuzzy point $z \in \bar{A}$ if and only if there is a fuzzy net $\{s_\alpha\}_{\alpha \in D}$ in A such $\{s_\alpha\}_{\alpha \in D}$ is converge to z [7].

A mapping f from a fts $(X, \mathcal{T})$ to a fts $(Y, \mathcal{U})$ is fuzzy continuous, at short $\mathrm{F-continuous}$,if and only if the inverse image of eachopen fuzzy set in Y is open fuzzy set in X.

Let f be a mapping from a fts X into a fts Y, then: f is called fuzzy open (fuzzy closed) mapping, for short $\mathrm{F-open}$ ($\mathrm{F-closed}$), if $f(A)$ is an open (closed) fuzzy set in Y for every open (closed) fuzzy set in X [2].

Theorem [8]: Let f be a mapping from a fts X into a fts Y, then the following are equivalent

f is $\mathrm{F-continuous}$;

For each fuzzy net $\{x_\alpha\}_{\alpha \in D}$ converge to z, then $\{f(x_\alpha)\}_{\alpha \in D}$ is a fuzzy net in Y converge to $f(z)$.

Remarks:

It is not necessary the constant mapping from a fuzzy topological space X to a fuzzy topological space Y be $\mathrm{F-continuous}$.

If X, Y are fully stratified spaces, then the constant map from X into Y is $\mathrm{F-continuous}$.

The identity map is $\mathrm{F-continuous}$, $\mathrm{F-open}$, $\mathrm{F-closed}$.

The composition of $F-$continuous mappings is $F-$continuous.

The composition of $F-$open($F-$closed) mappings is $F-$open ($F-$closed).

Definition [8]: Let $(X,\mathcal{T})$ be a fts, and $A \leq X$. The induced fuzzy topology $\mathcal{T}_A$ on A is $\{A \wedge B : B \in \mathcal{T}\}$.The pair $(A,\mathcal{T}_A)$ is called a fuzzy subspace of $(X,\mathcal{T})$, for short subspace of $(X,\mathcal{T})$.

Proposition [8]: Let $(A,\mathcal{T}_A)$ be a subspace of $(X,\mathcal{T})$, then B closed fuzzy set in A if and only if there is a closed fuzzy set C in X such that $B = C \wedge A$.

Proposition: Let $(X,\mathcal{T})$ be a fts and A be a non-empty fuzzy subset of X then the inclusion map $i_A : (A,\mathcal{T}_A) \longrightarrow (X,\mathcal{T})$ is:

$F-$continuous;

$F-$closed if A closed;

$F-$open if A open.

<u>Proof:</u>Clear

Proposition [8]:A bijective mapping $f : X \longrightarrow Y$ is fuzzy open (fuzzy closed) if and only if $f^{-1}$is $F-$continuous.

Definition [8]: Let $(X,\mathcal{T})$ be a fts.

A sub family $\mathfrak{B}$ of $\mathcal{T}$ is called a base for $\mathcal{T}$ if and only if for each $A \in \mathcal{T}$, there exist $\mathfrak{B}_A \subseteq \mathfrak{B}$ such that $A = \bigvee_{B \in \mathfrak{B}_A} B$;

A sub family $\mathcal{P}$ of $\mathcal{T}$ is called a subbase for $\mathcal{T}$ if and only if

$$\mathfrak{B} = \left\{ \bigwedge_{D \in \mathcal{F}} D : \text{where } \mathcal{F} \text{ is finite subset of } \mathcal{P} \right\} \text{is a base of } \mathcal{T}.$$

Proposition: Let f be a mapping from a fts $(X,\mathcal{T})$ into a fts $(Y,\mathcal{U})$ and $\mathfrak{B}$ is a base for $\mathcal{U}$, then f is $F-$continuous if and only if $f^{-1}(A) \in \mathcal{T},\ \ \forall A \in \mathfrak{B}$.

<u>Proof:</u>Clear

Proposition: Let f be a mapping from a fts $(X,\mathcal{T})$ into a fts $(Y,\mathcal{U})$ and $\mathfrak{B}$ is a base for $\mathcal{T}$, then f is $F-$open if and only if $f(A) \in \mathcal{U},\ \ \forall A \in \mathfrak{B}$.

<u>Proof:</u>Clear

Proposition[5]:A mapping $f : X \longrightarrow Y$ is $F-$closed if and only if$\overline{f(A)} \leq f(\bar{A})$, for all $A \in I^X$.

Proposition[5]:A mapping $f : X \longrightarrow Y$ is $F-$open if and only if$f(A^\circ) \leq (f(A))^\circ$, for all $A \in I^X$.

Definition [5]:A fuzzy homeomorphism is a bijective $F-$continuous mapping has inverse is also $F-$continuous. If there exist a fuzzy homeomorphism of one fuzzy space onto another, the two fuzzy spaces are said to be $F-$homeomorphic , denoted by $\cong$, and each is a fuzzy homeomorphic of the other.

Definition [9]: Let $\{(X_j,\mathcal{T}_j), j \in J\}$ be a family of fuzzy topological space. The product fuzzy topology $\mathcal{T}$ on the set $X = \prod_{j \in J} X_j$is the coarsest fuzzy topology on X making all the projection mapping $Pr_j : X \longrightarrow X_j$ fuzzy continuous.

Theorem [9]: Let $\{(X_j, \mathcal{T}_j), j \in J\}$ be a family of fuzzy topological space, A sub base for the product fuzzy topology is given by $\mathcal{S} = \{Pr_j^{-1}(A_j): A_j \in T_j, j \in J\}$ so that a base can be taken to be $\mathfrak{B} = \{\wedge_{K=1}^{n} Pr_{jk}^{-1}(A_{jk}): A_{jk} \in \mathcal{T}_{jk}\}$.

Remark [9]: Basic open fuzzy sets of product fuzzy topology on $X_1 \times X_2$ are of the form $Pr_1^{-1}(A_1) \wedge Pr_2^{-1}(A_2)$ which is equal to $A_1 \times A_2$.

Proposition [4]: Let $\{(X_j, \mathcal{T}_j)\}_{j \in J}$ and $\{(Y_i, \mathcal{U}_i)\}_{j \in J}$, be two families of fuzzy topological spaces. For each $j \in J$, let $f_j: (X_j, \mathcal{T}_j) \longrightarrow (Y_j, \mathcal{U}_j)$. Then the product mapping

$$f = \prod_{j \in J} f_j : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{U}), (x_j) \longmapsto \left(f_j(x_j)\right) \text{ is F} - \text{continuous if } f_j \text{ is F} - \text{continuous for each } j \in J.$$

Theorem [11]: Let $(Y, \mathcal{U})$ and $(X_j, \mathcal{T}_j)$, where $j \in J$, be a fuzzy topological spaces, then $f: (Y, \mathcal{U}) \longrightarrow \prod_{j \in J} (X_j, \mathcal{T}_j)$ is F − continuous if and only if $Pr_j \circ f$ is F − continuous.

Proposition: Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be fully stratified spaces, then for each $a \in X_1$ the mapping $f: (X_2, \mathcal{T}_2) \longrightarrow (X_1, \mathcal{T}_1) \times (X_2, \mathcal{T}_2)$, $x \longmapsto (a, x)$ is fuzzy continuous.

<u>Proof</u>: Clear

Proposition: Let $(X, \mathcal{T})$ be fully stratified space and $(Y, \mathcal{U})$ be any fts then there is a F − homeomorphism from X into $\{y\} \times X$, i.e. $X \cong \{y\} \times X$ where $y \in Y$.

<u>Proof:</u> Clear

Proposition [3]: Let $(X, \mathcal{T})$ be the product fuzzy topological space of a finite family of fuzzy topological spaces $\{(X_j, \mathcal{T}_j): j = 1, 2, \dots, n\}, n \geq 2$ then the following properties are known to hold in topology:

i. $$\overline{\left(\prod_{j=1}^{n} W_j\right)} = \prod_{j=1}^{n} \overline{W_j};$$

ii. $$\left(\prod_{j=1}^{n} W_j\right)^{\circ} = \prod_{j=1}^{n} (W_j)^{\circ};$$

But they fail in fuzzy topology.

Proposition: Let $(X, \mathcal{T})$ be a fts. Then the diagonal map $\Delta: X \longrightarrow X \times X, x \longmapsto (x, x)$, is F − continuous.

<u>Proof</u>: Clear

Proposition[8]: A fuzzy net $\{x_\beta\}_{\beta \in D}$ in the product space $(X, \mathcal{T})$ converge to a fuzzy point z if and only if for each $p_\alpha: X \longrightarrow X_\alpha$ the fuzzy net $Pr_\alpha \circ x = \{Pr_\alpha(x_\beta)\}_{\beta \in D}$ in X_α converges to the fuzzy point $p_\alpha(z)$.

Proposition [11]: Let $(X, \mathcal{T})$ be a fts, Y a set, and $q: X \longrightarrow Y$ a surjection map. Then $\mathcal{T}_q = \{B: q^{-1}(B) \in \mathcal{T}\}$ is fuzzy topology for Y making q − continuous, called quotient fyzzy topology.

Proposition: Let $(X,\mathcal{T}),(Y,\mathcal{U})$ be a fuzzy topological spaces, R be an a equivalence relation on X, and $f:(X/_R,\mathcal{T}_q)\longrightarrow(Y,\mathcal{U})$. Then f is $\mathrm{F-continuous}$ if and only if $f\circ q$ is $\mathrm{F-continuous}$. Where $\mathcal{T}_q$ is quotient fuzzy topology on $X/_R$ defined by $q:X\longrightarrow X/_R$.

Proof: $(\Longrightarrow)$Let $f$be $\mathrm{F-continuous}$,

$f\circ q$is$\mathrm{F-continuous}$.(composition of $\mathrm{F-continuous}$ mappings)

$(\Longleftarrow)$ Let $f\circ q$ be $\mathrm{F-continuous}$, and Let $V\in\mathcal{U}$ then

$(f\circ q)^{-1}(V)=q^{-1}(f^{-1}(V))$is open fuzzy set in X. Hence $f^{-1}(V)$ is open fuzzy set in $X/_R$. Therefor f is $\mathrm{F-continuous}$.

Proposition: Let $(X,\mathcal{T})$, $(Z,\mathcal{U})$ be a fuzzy topological spaces, R be an equivalence relation on X and $f:X\longrightarrow Z$ is $\mathrm{F-continuous}$ then there is unique $\mathrm{F-continuous}$ map $f^*:X/_R\longrightarrow Z$ such that $f^*\circ q=f$.

<u>Proof:</u>By the universal property of the projection map in set, there is a unique map $f^*:X/_R\rightarrow Z$such that $f^*\circ q=f$.And by proposition (1.21) the proof is completed.

Definition: The property in proposition 1.25 called the universal property of fuzzy quotient map.

Fuzzy Topological Groups

This section will contain the definition of fuzzy topological group and its remarks, properties and propositions. In addition, we review the concepts of fuzzy topological group and quotient fuzzy topological group.

Definition [12,14]: Let G be a group and $(G,\mathcal{T})$ is fts. $(G,\mathcal{T})$ is called fuzzy topological group, for short ftg, if and only if**:**

For all $a,b\in X$ and any $Q-\mathrm{neighborhood}\,W$ of the fuzzy point $(ab)_\lambda$ there are $Q-\mathrm{neighborhood}\,U$of a_λ and V of $b_\lambda$such that $UV\leq W$;

For all $a\in X$ and any $Q-\mathrm{neighborhood}V$ of a_λ^{-1}, there exist a $Q-\mathrm{neighborhood}U$ of a_λ such that $U^{-1}\leq V$.

Proposition: Let $(G,\mathcal{T})$ be fully stratified space, then $(G,\mathcal{T})$ is a fuzzy topological group if and only if it is satisfies the following conditions:

The mapping $\mu:G\times G\longrightarrow G,(x,y)\longmapsto xy$, is fuzzy continuous;

The mapping $\mathcal{N}:G\longrightarrow G,x\longmapsto x^{-1}$, is fuzzy continuous.

<u>Proof:</u> Clear

Proposition: Let $(G,\mathcal{T})$ be a ftg. If $(G,\mathcal{T})$ is a fully stratified space and $g\in G$, then the following mappings are $\mathrm{F-homeomorphism}$.

$\mathcal{N}:G\longrightarrow G,x\longmapsto x^{-1}$;

$\mathcal{L}_g:G\longrightarrow G,\ x\longmapsto gx$;

$\mathcal{R}_g:G\longrightarrow G,x\longmapsto xg$.

<u>Proof:</u>

$\mathcal{N}$ is bijective $F-\mathrm{continuous}$ and its inverse $\mathcal{N}^{-1}=\mathcal{N}$. So$\mathcal{N}$is $F-\mathrm{homeomorphism}$.

$\mathcal{L}_g$ is bijective and its inverse $(\mathcal{L}_g)^{-1}=\mathcal{L}_{g^{-1}}$ and$\mathcal{L}_g$=G$\xrightarrow{\Delta}G\times G\xrightarrow{C_g\times I_G}G\times G\xrightarrow{\mu}G$

Therefor $\mathcal{L}_x=\mu\circ C_g\times I_G\circ\Delta$is $\mathrm{F-continuous}$.

In similar way as ii

Proposition: Let $(G,\mathcal{T})$be a ftg, if G is a fully stratified space, then:

If A is closed (open) fuzzy set in G and $a \in G$, then aA, Aa, A^{-1} are closed (open) fuzzy sets.

If A is open fuzzy set in $(G,\mathcal{T})$ and B is crisp set.then AB, BA are open fuzzy sets in $(G,\mathcal{T})$.

<u>Proof:</u>

Let Abe closed (open) fuzzy set in G. Since $\mathcal{L}_g, \mathcal{R}_g, \mathcal{N}$ are $\mathrm{F-closed}$ $(\mathrm{F-open})$ map. (proposition 2.3)Then $\mathcal{L}_a(A) = aA, \mathcal{R}_a(A) = Aa, \mathcal{N}(A) = A^{-1}$are closed (open) fuzzy sets.

ii. Since $BA = \bigvee_{b \in B} bA$ and $AB = \bigvee_{b \in B} A\,b$

therefor BA ,AB are open fuzzy sets in G.

Definition [10]: Let G be a group and A a fuzzy set in G, then A is a fuzzy sub group of G if and only if the following conditions are satisfied:

$A(ab) \geq \min\{A(a), A(b)\}, \forall a, b \in G;$

$A(a^{-1}) \geq A(a), \forall a \in G.$

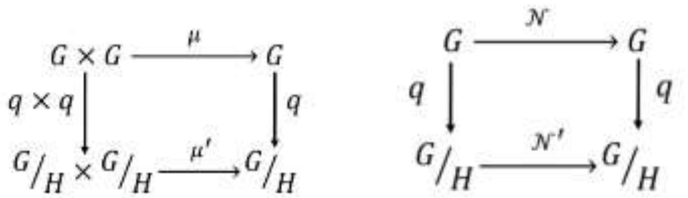
Definition [10]: Let H be a fuzzy subgroup of the group G. If $aHa^{-1} \leq H$ for all $a \in G$, then H is called a normal fuzzy subgroup of G.

Definition [12]: Let $(G,\mathcal{T})$ be a ftg and H a fuzzy subgroup of G. Then $(H,\mathcal{T}_H)$ is called a subgroup of ftg $(G,\mathcal{T})$. If H is normal fuzzy subgroup, then $(H,\mathcal{T}_H)$ is called normal fuzzy subgroup of ftg $(G,\mathcal{T})$. Thereafter $(H,\mathcal{T}_H)$ is denoted by H for short.

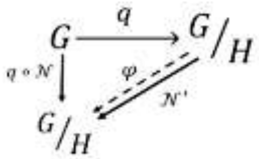
Proposition: Let G be a ftg, H is a crisp normal subgroup of G, then ${}^{G}/_{H}$ is fuzzy topological group.

<u>Proof:</u>Let G be a ftg and H be a crisp normal subgroup of G, and ${}^{G}/_{H} = \{aH : a \in H\}$, the set of left fuzzy cosets of H. The map $: G \longrightarrow {}^{G}/_{H}, a \mapsto aH$, define a quotient fuzzy topology on ${}^{G}/_{H}$Also, the quotient map q is fuzzy open; for if A is open fuzzy set in G, then $q^{-1}(q(A)) = AH$ is open fuzzy set. (proposition2.4).And it follows that $q(A)$ is open fuzzy set in ${}^{G}/_{H}$by the definition of the quotient fuzzy topology.

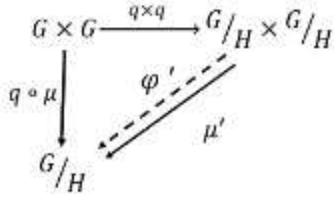
Now, if H is normal fuzzy subgroup then ${}^{G}/_{H}$ has a canonical fuzzy group structure (quotient fuzzy group).If $\mu, \mu'$are the multiplication in $G$and ${}^{G}/_{H}$, and $\mathcal{N}, \mathcal{N}'$ are the inversions in G and ${}^{G}/_{H}$ respectively.Then $\mu', \mathcal{N}'$are uniquely defined by the following commutative diagrams:



To prove that $\mu', \mathcal{N}'$ are $\mathrm{F-continuous}$. By using the following commutative diagram:



$q \circ \mathcal{N}$ is $\mathrm{F-continuous}$ and q is quotient map so by the universal property of q there exist a unique $\mathrm{F-continuous}$ map $\varphi: {}^{G}/_{H} \to {}^{G}/_{H}$ making $\varphi \circ q = q \circ \mathcal{N}$.But $\mathcal{N}'$satisfies this condition $\mathcal{N}' \circ q = q \circ \mathcal{N}$ so $\mathcal{N}' = \varphi$. So $\mathcal{N}'$is $\mathrm{F-continuous}$.Also by using the following commutative diagram:

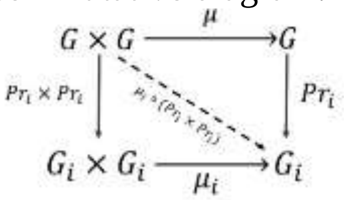


Since $q \times q$is $\mathrm{F-open, F-continuous\ and\ surjection\ map}$, hence $q \times q$ is quotient map.Since $q \circ \mu$is $\mathrm{F-continuous}$, so by universal property of $q \times q$ there exist a unique $\mathrm{F-continuous}$ map $\varphi' \circ (q \times q) = q \circ \mu$. But μ' satisfies the condition$\mu' \circ (q \times q) = q \circ \mu$.
So $\mu' = \varphi'$, and so μ' is $\mathrm{F-continuous}$.

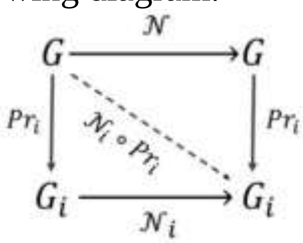
Definition: The fuzzy topological group ${}^{G}/_{H}$called quotient fuzzy topological group.

Proposition: The product of fully stratified fuzzy topological groups is fuzzy topological groups.

Proof: Let$\{G_i\}_{i\in I}$ be a family of fuzzy topological groups, their product $G = \prod G_i$has a natural group structure (product of groups) with multiplication μ and inverse $\mathcal{N}$.And a natural fuzzy topology (product of fuzzy topological space).By using the following commutative diagram:



Then$\mu_i \circ (Pr_i \times Pr_i)$ is $\mathrm{F-continuous}$ since$\mu_i$and $Pr_i$are $\mathrm{F-continuous}$. Therefor $Pr_i \circ \mu = \mu_i \circ (Pr_i \times Pr_i)$ is $\mathrm{F-continuous}$. Therefore μ is $\mathrm{F-continuous}$. (Proposition 1.17) Now, by using the following diagram:



Then $\mathcal{N}_i \circ Pr_i$ is $\mathrm{F-continuous}$ Since $\mathcal{N}_i$and Pr_i are $\mathrm{F-continuous}$.Therefor $Pr_i \circ \mathcal{N} = \mathcal{N}_i \circ Pr_i$ is $\mathrm{F-continuous}$.Therefore $\mathcal{N}$is $\mathrm{F-continuous}$. (proposition 1.17) .

So $G = \prod_{i \in I} G_i$ be a fuzzy topological group.

2.11 Definition: The fuzzy topological group $G = \prod_{i \in I} G_i$ called product fuzzy topological groups.

Fuzzy group spaces

In this section, we introduce the definition of fuzzy group space and given some examples. In addition, we review the definition of fuzzy orbit, fuzzy stabilizer and fuzzy action and study some its properties.

Definition: Let G be a fuzzy topological group and X be a fuzzy topological space. A left fuzzy action of G on X is a $F-\text{continuous}$ map $\theta: G \times X \longrightarrow X$ such that:

$\theta(e, x) = x$, for all $x \in X$ where e is the identity element in G;

$\theta(g_1, \theta(g_2, x)) = \theta(g_1 g_2, x)$, for all $x \in X$ and $g_1, g_2 \in G$.

The space X together with fuzzy action θ is called fuzzy group space and denoted by $FG-\text{space}$, more precisely (left $FG-\text{space}$). In similar way one can define a right $FG-\text{space}$.

Notation: For simplicity we denote $\theta(g, x)$ by $g.x$.

Definition: Let (X, θ) be $FG-\text{space}$, then:

The fuzzy orbit of $x \in X$ under the fuzzy subgroup A of G is defined to be the fuzzy set $Ax = \{\theta(a, x): a \in A\}$, where: $Ax(y) = \sup\{A(g): \theta(g, x) = y\}$. And the set of all fuzzy orbit denoted by $X/_A$ and called it fuzzy orbit space under A.

The fuzzy stabilizer of $x \in X$ under the fuzzy subgroup A of G is defined to be the fuzzy set $S_x^A = \{a \in A: supp(\theta(a, x)) = x\}$.

The fuzzy kernel of the fuzzy action θ under the fuzzy subgroup A of G is defined to be the fuzzy set: $(ker\theta)_A = \{a \in A: supp(\theta(a, x)) = x \text{ for all } x \in X\}$.

Remarks: The fuzzy action θ of G on X defines the following mappings:

$\theta_g: X \longrightarrow X$, defined by $\theta_g(x) = \theta(g, x), \forall g \in G$;

$\theta_x: G \longrightarrow X$, defined by $\theta_x(g) = \theta(g, x), \forall x \in X$;

$q: X \longrightarrow X/_G$ defined by $q(x) = G_x$.

Proposition: let (X, θ) be a $FG-\text{space}$, Then:

θ_e is the identity mapping of X;

$\theta_g \circ \theta_h = \theta_{gh}$;

$(\theta_g)^{-1} = \theta_{g^{-1}}$.

Proof:

Let $x \in X$ then $\theta_e(x) = \theta(e, x) = x$. Therefor $\theta_e(x) = I_X$.

$\theta_{g_1} \circ \theta_{g_2}(x) = \theta_{g_1}(\theta_{g_2}(x)) = \theta_{g_1}(\theta(g_2, x)) = \theta(g_1, \theta(g_2, x)) = \theta(g_1 g_2, x) = \theta_{g_1 g_2}(x)$.

Since $\theta_g \circ \theta_{g^{-1}}(x) = \theta_g(\theta(g^{-1}, x)) = \theta(\theta(gg^{-1}, x) = \theta(e, x)$.

Hence $\theta_g \circ \theta_{g^{-1}} = I_X$ Similarly $\theta_{g^{-1}} \circ \theta_g = I_X$

Therefore $(\theta_g)^{-1} = \theta_{g^{-1}}$.

Remark: As a consequence of proposition 3.1.9 the set $\{\theta_g : g \in G\}$ with the composition law is group.

Proposition:
Let (X,θ) be a fully stratified $FG-\text{space}$ and G is fully stratified ftg. Then:
$\theta_g : X \longrightarrow X$ is $\text{F}-\text{homeomorphism}$
$\theta_x : G \longrightarrow X$ is $\text{F}-\text{continuous}$ map
Proof:
θ_g is a bijective and its inverse is $(\theta_g)^{-1} = \theta_{g^{-1}}$ and $\theta_g : X \xrightarrow{\cong} \{g\} \times X \xrightarrow{\theta} X$
So $\theta_g = \theta \circ \cong$, Hence θ_g and $\theta_{g^{-1}}$ are $\text{F}-\text{continuous}$.
Therefore θ_g is $\text{F}-\text{homeomorphism}$.
$\theta_x = G \xrightarrow{\cong} G \times \{x\} \xrightarrow{\theta} X$. So $\theta_x = \theta \circ \cong$, where $\cong : G \longrightarrow G \times \{x\}, g \longrightarrow (g,x)$ which is $\text{F}-\text{continuous}$ by proposition 2.15, therefor θ_x is $\text{F}-\text{continuous}$.
Proposition: Let (X,θ) be $FG-\text{space}$, then:
The fuzzy stabilizer of $x \in X$ under the fuzzy subgroup A of G is a fuzzy subgroup of A;
ii. $(ker\theta)_A = \bigwedge_{x \in X} S_x^A$.
$(ker\theta)_A$ is a normal fuzzy subgroup of A;
<u>Proof:</u>
Let $g_1, g_2 \in S_x^A$ then
$$supp(\theta(g_1 g_2, x)) = supp\left(\theta(g_1, \theta(g_2, x))\right) = supp\left(\theta(g_1, x_{A(g_2)})\right)$$
$$= supp(x_{min\{A(g_1), A(g_2)\}}) = x$$
$S_x^A(g_1 g_2) = min\{A(g_1), A(g_2)\} \geq min\{S_x^A(g_1), S_x^A(g_2)\}$, because $S_x^A \leq A$.
Hence $g_1 g_2 \in S_x^A$. Now let $g \in S_x^A$
$$\theta(g^{-1}, x) = \theta(g^{-1}, g.x) = \theta(g^{-1}g, x) = \theta\left(e_{min\{A(g^{-1}), A(g)\}}, x\right) = x_{min\{A(g^{-1}), A(g)\}} = x_{A(g)}.$$
Hence $supp(\theta(g^{-1}, x)) = x$. Since $S_x^A(g^{-1}) = A(g) \geq S_x^A(g)$
Therefor $g^{-1} \in S_x^A$. So S_x^A is fuzzy subgroup of A.
$g_\alpha \in (ker\theta)_A \Leftrightarrow \alpha \leq A(g)$ and $supp(\theta(g,x))$ for all $x \in X$.
$\Leftrightarrow g_\alpha \in S_x^A$, for all $x \in X$.
$\Leftrightarrow g_\alpha \in \bigwedge_{x \in X} S_x^A$.
From (ii) $(ker\theta)_A$ subgroup of A.
Let $h \in A$ and $g \in (ker\theta)_A$. $\theta(hgh^{-1}, x) = h(h^{-1}.x) = x_{min\{A(h), A(h^{-1}), A(g)\}}$, for all $x \in X$. So $supp(\theta(hgh^{-1}, x)) = x$, for all $x \in X$. And $hgh^{-1} \in A$, because A is fuzzy subgroup hence $hgh^{-1} \in (ker\theta)_A$ and $h(ker\theta)_A h^{-1} \leq (ker\theta)_A$.
Therefore $(ker\theta)_A$ is normal fuzzy subgroup.
Proposition: Let (X,θ) be a fully stratified $FG-\text{space}$. If $A \leq X$ and $B \leq G$, then the following are satisfies:

$(gA)^{\circ} = gA^{\circ}$, for all $g \in G$;
$\overline{gA} = g\bar{A}$, for all $g \in G$;

Proof:
Let $g \in G$ and $A \leq X$.
Since θ_g is $\mathrm{F-homomorphism}$ (Proposition3.7)then $\theta_g(A^{\circ}) = (\theta_g(A))^{\circ}$, i.e. $\theta(g, A^{\circ}) = (\theta(g,A))^{\circ}$. And so $(gA)^{\circ} = gA^{\circ}$, for all $g \in G$.
In similar way as i.
Proposition: Let (X, θ) be a fully stratified FG − space. If $A \leq X$ and $B \leq G$, such that $\overline{(B \times A)} = \bar{B} \times \bar{A}$, then the following are satisfies:
$\theta(\bar{B} \times \bar{A}) \leq \overline{\theta(B \times A)}$;
$(\theta(B \times A))^{\circ} \leq \theta(B^{\circ} \times A^{\circ})$;
$(\theta(B^{\circ} \times A))^{\circ} = (\theta(B \times A^{\circ}))^{\circ} = (\theta(B \times A))^{\circ}$;
$\overline{\theta(\bar{B} \times A)} = \overline{\theta(B \times \bar{A})} = \overline{\theta(B \times A)}$.
Proof: Let $A \leq X, B \leq G$.
$\theta(\bar{B} \times \bar{A}) = \theta(\overline{B \times A}) \leq \overline{\theta(B \times A)}$, (θ is F − continuous).
In similar way as i.
$(\theta(B \times A))^{\circ} \leq \theta((B \times A)^{\circ}) = \theta(B^{\circ} \times A^{\circ}) \leq \theta(B^{\circ} \times A)$ since θ is F − continuous. and $\left((\theta(B \times A))^{\circ}\right)^{\circ} \leq \left(\theta(B^{\circ} \times A)\right)^{\circ}$
Therefore $(\theta(B \times A))^{\circ} \leq \left(\theta(B^{\circ} \times A)\right)^{\circ}$, ... (1)
Since $\theta(B^{\circ} \times A) \leq \theta(B \times A)$
Hence $\left(\theta(B^{\circ} \times A)\right)^{\circ} \leq (\theta(B \times A))^{\circ}$, ... (2)
From (1) and (2) we have $\left(\theta(B^{\circ} \times A)\right)^{\circ} = (\theta(B \times A))^{\circ}$.
Since $\theta(B \times A) \leq \theta(\bar{B} \times A)$
Hence $\overline{\theta(B \times A)} \leq \overline{\theta(\bar{B} \times A)}$, ... (1)
Since $\theta(\bar{B} \times A) \leq \theta(\bar{B} \times \bar{A}) \leq \overline{\theta(B \times A)}$
Therefor $\overline{\theta(\bar{B} \times A)} \leq \overline{\left(\overline{\theta(B \times A)}\right)} = \overline{\theta(B \times A)}$, ... (2)
From (1) and (2) we have:
$\overline{\theta(\bar{B} \times A)} = \overline{\theta(B \times A)}$.
Proposition: Let (X, θ) be a fully stratified $FG - \mathrm{space}$ and G be a fully stratified fuzzy topological group. Let A be open fuzzy set, and C_α be a constant fuzzy subset of G, then $\theta(C_\alpha \times A)$ is open fuzzy set in X.
Proof: Let $y \in X$

$$\begin{aligned}\theta(C_\alpha \times A)(y) &= sup\{(C_\alpha \times A)(g,x):\theta(g,x) = y\}\\ &= sup\{C_\alpha(g) \wedge A(x):\theta(g,x) = y\}\\ &= sup\{\alpha \wedge A(x):\theta(g,x) = y\}\\ &= \alpha \wedge sup\{A(x):\theta_g(x) = y\}\end{aligned}$$

$= \alpha \wedge sup\{A(\theta_{g^{-1}}(y)) : \theta_{g^{-1}}(y) = x\}$

$= \alpha \wedge sup\left\{\left(\theta_g(A)\right)(y)\right\}$, Where $\theta_{g^{-1}}(y) = x$

$= \left\{\alpha \wedge \left\{\bigvee (\theta_g(A))\right\}\right\}(y)$, Where $\theta_{g^{-1}}(y) = x$

Therefor $\theta(C_\alpha, A) = C_\alpha \wedge \left\{\bigvee \left(\theta_g(A)\right)\right\}$.

Since θ_g is $\mathrm{F}-$ open, (proposition 3.6) And A is open fuzzy set in X.

Hence $\theta_g(A)$is open fuzzy set So $\theta(C_\alpha \times A)$ is open fuzzy set in X.

Proposition: Let (X, θ) be a fully stratified $FG-$ space and G be a fully stratified ftg. Then θ is $\mathrm{F}-$ open.

<u>Proof:</u> Let $W = U \times V$ be basic open fuzzy set in $G \times X$, where U open fuzzy subset of G and V open fuzzy subset of X.

Since $\theta(U \times V) = \theta\left(\bigvee_{g \in supp\,(U)} (g_\alpha \times V)\right)$, Where $\alpha = U(g)$.

$$= \bigvee_{g \in supp\,(U)} \theta(g_\alpha \times V).$$

Now, we must prove that $\theta(\{g_\alpha\} \times V)$ is open fuzzy subset in X.

$\theta(g_\alpha \times V)(x) = sup\{(g_\alpha \times V)(t, y) : \theta(t, y) = x\}$

$= sup\{g_\alpha(t) \wedge V(y) : \theta(t, y) = x\}$

$= \alpha \wedge V(y)$, where $\theta(g, y) = x$

$= \alpha \wedge V(y)$, where $\theta_g(y) = x$

$= \alpha \wedge V(\theta_{g^{-1}}(x))$

$= \alpha \wedge \theta_g(V(x))$

$= (\alpha \wedge \theta_g(V))(x)$

Hence $\theta(g_\alpha \times V) = C_\alpha \wedge \theta_g(V)$

Since $\theta_g$is $\mathrm{F}-$ homemorphism, (proposition 3.6)then $\theta(g_\alpha \times V)$ is open fuzzy set in X, hence$\theta(U \times V)$ is open fuzzy set in X. Therefore θ is$\mathrm{F}-$ openmap.

Proposition: Let (X, θ) be a $FG-$ space. If A is compact fuzzy set in G and B is compact fuzzy set in X then AB is compact fuzzy set in X.

Proof: Clear

Definition: Let X and Y be fuzzy topological spaces. We called the product fuzzy space $X \times Y$ is $\mathrm{C}-$ product fuzzy topological space if for all closed fuzzy set D in $X \times Y$ can written as follow:

$D = \bigvee_{i=1}^{n} V_i \times U_i$, where:

V_i is closed fuzzy set in X and $U_i$is closed fuzzy set in Y,for all $i = 1, 2, \dots, n$ where $n \in N$.

Proposition: let (X, θ) be a fully stratified $FG-$ space, G be a finite fully stratified ftg and $G \times X$ be $\mathrm{C}-$ product fuzzy topological space then the fuzzy action $\theta$is closed.

Proof:

Let V be a closed fuzzy set in $G\times X$. Since $G\times X$ is C − product fuzzy topological space then $V=\bigvee_{i=1}^{n} V_i\times U_i$, where:

V_i is closed fuzzy set in G and U_i is closed fuzzy set in X, for all $i=1,2,\dots,n$ where $n\in N$. since

$$\theta(V)=\theta\Big(\bigvee_{i=1}^{n}(V_i\times U_i)\Big)=\bigvee_{i=1}^{n}\theta(V_i\times U_i)$$

Therefor $\theta(V_i\times U_i)=\theta\left(\bigvee_{g\in supp(V_i)}(g_\alpha\times U_i)\right)=\bigvee_{g\in supp(V_i)}(\theta(g_\alpha\times U_i))$,

where $\alpha=V_i(g)$. Now let $x\in X$

$$\begin{aligned}\theta(g_\alpha\times U_i)(x)&=sup\{(g_\alpha\times U_i)(t,y):\theta(t,y)=x\}, \text{ Where } \alpha=V_i(g).\\&=sup\{(g_\alpha(t)\wedge U_i(y)):\theta(t,y)=x\}, \text{ Where } \alpha=V_i(g).\\&=\alpha\wedge U_i(y): where\ (\theta(g,y))=x\\&=\alpha\wedge U_i(y): where\ \theta_g(y)=x\\&=\alpha\wedge U_i(\theta_{g^{-1}}(x))\\&=\alpha\wedge\theta_g(U_i(x)), \text{ Because } U_i(\theta_{g^{-1}})=\theta_g(U_i)\\&=(\alpha\wedge\theta_g(U_i))(x)\end{aligned}$$

Therefor $\theta(g_\alpha\times U_i)=\alpha\wedge\theta_g(U_i)$.

Now since θ_g is F − homeomorphism. Then $\theta_g(U_i)$ is closed fuzzy subset in X.

So $\theta(g_\alpha\times U_i)$ is closed fuzzy set in X because it is intersection of two closed fuzzy sets in X. Since G finite and $V_i\le G$

So $\theta(V_i\times U_i)=\bigvee_{g\in supp(V_i)}(\theta(g_\alpha\times U_i))$

is closed fuzzy set. (finite union of closed fuzzy sets).

Since $\theta(V)=\bigvee_{i=1}^{n}\theta(V_i\times U_i)$

Therefor $\theta(V)$ is closed fuzzy set in X

Hence θ is F − closed map.

Definition: Let (X,θ) be a FG − space, then a fuzzy action θ of G on X is called:

Transitive under a fuzzy subgroup A of G if and only if $supp(Ax)=X$, for all $x\in X$.

Trivial under a fuzzy subgroup A of G if and only if $supp((ker\theta)_A)=supp(A)$.

Effective under a fuzzy subgroup A of G if and only if $supp((ker\theta)_A)=\{e\}$.

Free under a fuzzy subgroup A of G if $supp(S_x^A)=\{e\}$, for all $x\in X$.

Definition: Let (X,θ) be a FG − space. Then (X,θ) is called:

Free FG − space under the fuzzy subgroup A of G if the fuzzy action of G on X is free under the fuzzy subgroup A;

Effective $FG-$ space under the fuzzy subgroup A of G if the fuzzy action of G on X is effective under the fuzzy subgroup A;

Transitive $FG-$ space under the fuzzy subgroup A of G if the fuzzy action of G on X is transitive under the fuzzy subgroup A.

Trivial $FG-$ space under the fuzzy subgroup A of G if the fuzzy action of G on X is trivial under the fuzzy subgroup A.

Proposition: Let (X,θ_1) and (Y,θ_2) be $FG-$ spaces, If X and G are fully stratified. Then $(X\times Y,\theta)$ is $FG-$ space. Where $\theta: G\times(X\times Y)\longrightarrow(X\times Y), (g,(x,y))\mapsto(\theta_1(g,x),\theta_2(g,y))$.

Proof:

$\theta(e,(x,y))=(\theta_1(e,x),\theta_2(e,y))=(x,y)$

$$\theta\left(g_1,\theta(g_2,(x,y))\right)=\theta\left(g_1,(\theta_1(g_2,x),\theta_2(g_2,y))\right)$$
$$=\left(\theta_1(g_1,\theta_1(g_2,x)),\theta_2(g_1,\theta_2(g_2,y))\right)$$
$$=(\theta_1(g_1g_2,x),\theta_2(g_1g_2,y))$$
$$=\theta(g_1g_2,(x,y))$$

Since $\theta: G\times(X\times Y)\xrightarrow{\Delta\times I_{X\times Y}} G\times G\times(X,Y)\xrightarrow{I_G\times\cong\times I_Y} G\times X\times G\times Y \xrightarrow{\theta_1\times\theta_2} X\times Y$.

Hence $\theta=(\theta_1\times\theta_2)\circ(I_G\times\cong\times I_Y)\circ(\Delta\times I_{X\times Y})$. So θ is $\mathrm{F}-$ continuous.

And so from (i),(ii) and (iii), θ is fuzzy action. Therefore $(X\times Y,\theta)$ be $FG-$ space.

Corollary: Let (X,θ_1) and (Y,θ_2) be $FG-$ spaces, Where X and G are fully stratified. Then:

If (X,θ_1) is free $FG-$ space under the fuzzy subgroup A of G and (Y,θ_2) is free $FG-$ space under the fuzzy subgroup A of G. Then $(X\times Y,\theta)$ is free $FG-$ space under the fuzzy subgroup A of G.

If (X,θ_1) is transitive $FG-$ space under the fuzzy subgroup A of G and (Y,θ_2) is transitive $FG-$ space under the fuzzy subgroup A of G. Then $(X\times Y,\theta)$ is transitive $FG-$ space under the fuzzy subgroup A of G.

If (X,θ_1) is effective $FG-$ space under the fuzzy subgroup A of G and (Y,θ_2) is effective $FG-$ space under the fuzzy subgroup A of G. Then $(X\times Y,\theta)$ is effective $FG-$ space under the fuzzy subgroup A of G.

If (X,θ_1) is trivial $FG-$ space under the fuzzy subgroup A of G and (Y,θ_2) is trivial $FG-$ space under the fuzzy subgroup A of G. Then $(X\times Y,\theta)$ is trivial $FG-$ space under the fuzzy subgroup A of G.

Proof:

let $(x,y)\in X\times Y$

$$S^A_{(x,y)}=\{g\in A: supp(\theta(g,(x,y))=(x,y)\}$$
$$=\{g\in A: supp(\theta_1(g,x),\theta_2(g,y))=(x,y)\}$$
$$=\left\{g\in A:\left(supp(\theta_1(g,x)),supp(\theta_2(g,y))\right)=(x,y)\right\}$$
$$=\{g\in A: supp(\theta_1(g,x))=x \text{ and } supp(\theta_2(g,y))=\mathrm{y}\}$$

$= \{g \in A: supp(\theta_1(g,x)) = x\} \bigwedge \{g \in A: supp(\theta_2(g,y)) = y\}$

$= S_x^A \bigwedge S_y^A = \{e\} \bigwedge \{e\} = \{e\}$

So $S_{(x,y)}^A = \{e\}$ therefor $(X \times Y, \theta)$ is free $FG-$space under the fuzzy subgroup A of G.

Let $(x,y) \in X \times Y$

$A(x,y) = \{\theta(a,(x,y)): a \in A\} = \{(\theta_1(a,x), \theta_2(a,y)): a \in A\}$

$= (Ax, Ay) = (X, Y)$

Therefore $(X \times Y, \theta)$ is transitive $FG-$space under the fuzzy subgroup A of G.

$(ker\theta)_A = \{a \in A: supp\left(\theta(a,(x,y))\right) = (x,y) \text{ for all } (x,y) \in X \times Y\}$

$= \{a \in A: supp\left((\theta_1(a,x), \theta_2(a,y))\right) = (x,y) \text{ for all } (x,y) \in X \times Y\}$

$= \{a \in A: \left(supp(\theta_1(a,x)), supp(\theta_2(a,y))\right) = (x,y), \forall (x,y) \in X \times Y\}$

$= \{a \in A: supp(\theta_1(a,x)) = x \text{ and } supp(\theta_2(a,y)) = y, \forall\ (x,y) \in X \times Y\}$

$= \{a \in A: supp(\theta_1(a,x)) = x, \forall\ x \in X\} \bigwedge B$

Where $B = \{a \in A: supp(\theta_2(a,y)) = y, \forall\ y \in Y\}$

$= (ker\theta_1)_A \bigwedge (ker\theta_2)_A = \{e\} \bigwedge \{e\} = \{e\}$

Therefore $(X \times Y, \theta)$ is effective $FG-$space under the fuzzy subgroup A of G.

$(ker\theta)_A = \{a \in A: supp\left(\theta(a,(x,y))\right) = (x,y) \text{ for all } (x,y) \in X \times Y\}$

$= \{a \in A: supp\left((\theta_1(a,x), \theta_2(a,y))\right) = (x,y) \text{ for all } (x,y) \in X \times Y\}$

$= \left\{a \in A: \left(supp(\theta_1(a,x)), supp(\theta_2(a,y))\right) = (x,y), \forall (x,y) \in X \times Y\right\}$

$= \left\{a \in A: supp(\theta_1(a,x)) = x \text{ and } supp(\theta_2(a,y)) = y, \forall\ (x,y) \in X \times Y\right\}$

$= \{a \in A: supp(\theta_1(a,x)) = x, \forall\ x \in X\}$

$\bigwedge \{a \in A: supp(\theta_2(a,y)) = y, \forall\ y \in Y\}$

$= (ker\theta_1)_A \bigwedge (ker\theta_2)_A = A \bigwedge A = A$

Therefore $(X \times Y, \theta)$ is trivial $FG-$space under the fuzzy subgroup A of G.

Proposition: Let (X, θ) be $FG-$space. And let A be a fuzzy subgroup of G, then $(X/_A, \vartheta)$ is $FG-$space. Where $X/_A = \{Ax: x \in X\}$ and $\vartheta: G \times X/_A \longrightarrow X/_A, (g, Ax) \longmapsto \pi(\theta(g,x))$,
and $\pi: X \longrightarrow X/_A, x \longmapsto Ax$.

<u>Proof:</u> Let (X, θ) be $FG-$space. Then

$\vartheta(g_1, \vartheta(g_2, Ax)) = \vartheta\left(g_1, \pi(\theta(g_1, x))\right) = \vartheta(g_1, Ag_2x) = \pi(\theta(g_1, g_2.x))$

$= \pi(\theta(g_1g_2, x)) = \vartheta(g_1g_2, Ax)$

$\vartheta(e, Ax) = \pi(\theta(e,x)) = \pi(x) = Ax$

Since θ and π are $F-$continuous mappings then ϑ is $F-$continuous.

From (i),(ii) and (iii), $(X/_A, \vartheta)$ $FG-$space.

Proposition: Let (X,θ_1) be FG_1-space, and (Y,θ_2) is FG_2-space. Where G_2 and X_1 are fully stratified spaces. Then $X\times Y$ is $F(G_1\times G_2)-$space.

<u>Proof:</u> Define $\theta:(G_1\times G_2)\times(X\times Y)\longrightarrow(X\times Y)$ as follows $\theta((g,h),(x,y))=(\theta_1(g,x),\theta_2(h,y))$

$$\theta\Big((g_1,h_1),\theta((g_2,h_2),(x,y))\Big)=\theta\Big((g_1,h_1),(\theta_1(g_2,x),\theta_2(h_2,y))\Big)$$
$$=\Big(\theta_1(g_1,\theta_1(g_2,x)),\theta_2\big(h_1,\theta_2(h_2,y)\big)\Big)$$
$$=(\theta_1(g_1g_2,x),\theta_2(h_1h_2,y))$$
$$=\theta((g_1g_2,h_1h_2),(x,y))$$
$$=\theta((g_1,h_1)(g_2,h_2),(x,y)).$$

$\theta((e_1,e_2),(x,y))=(\theta_1(e_1,x),\theta_2(e_2,y))=(x,y)$

Since $\theta=(G_1\times G_2)\times(X\times Y)\xrightarrow{I_{G_1}\times\cong\times I_X}(G_1\times X)\times(G_2\times Y)\xrightarrow{\theta_1\times\theta_2}(X\times Y)$.

Since $I_{G_1}, I_X, \cong, \theta_1$ and θ_2 are fuzzy continuous mappings. So θ is $F-$continuous.

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