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Page39-46

Construction A Topology On Graphs

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Abstract

In this paper, we have constructed a topology on graphs and a topology on subgraphs with examples. We proved that a graph is connected if, and only if, it is connected with respect to its topology. Also, we gave given a new topological concept on two topologies and we called it symmetric with examples.

Also, we proved that, if two graphs are isomorphic, then their topologies are symmetric and we have shown by example that the converse is not true in general.

Introduction:

In this paper, we constructed a topology on an undirected graph.In 1964, Ahlborn J. T.[7], defined a topology on a directed graph $G(V,E)$ by a subset A of V is an open set if there does not exist an edge from the set $V-A$ to the set A.In 1968, Bhargave N. J. and Ahlborn J. T. [6], defined a topology on a directed graph $G(V,E)$ where a subset A of V is an open set if for any pair of vertices (v,w) with w in A and v not in A, then $(v,w)\notin E$.Our definition is on an undirected graph on the set of edges. We discussed the connectedness of each of the graph and the topological space that induces by that graph. Throughout this paper, a graph means that a finite undirected graph.

1. Fundamental Concepts

Definition 1.1[3] A graph $G(V,E)$ consists of V, a non-empty set of vertices (or nodes) and E, a set of edges.
Each edge has two vertices associated with it, called its end points. An edge e and a vertex v are adjacent if v is an end point of e. An edge joining a vertex to itself is called a loop.

Example 1.2

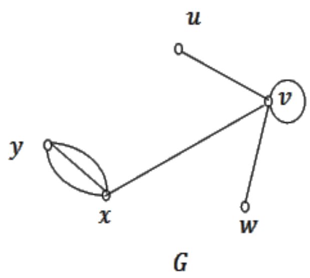


Figure (1.1) A graph $G(V,E)$ with $V(G) = \{x, y, u, v, w\}$ and $E(G) = \{\{u,v\}, \{v,w\}, \{x,v\}, \{x,y\}, \{y,x\}, \{v,v\}, \{x,y\}\}$ can be represented as in the following:

G Has five vertices and seven edges. The edge $\{v,v\}$ is loop. An edge $\{u,v\}$ is said to connect the vertices u and v, and this is sometime written as uv.

Definition 1.3[3]

A subgraph of a graph $G(V,E)$ is a graph $H(W,F)$ where $W \subseteq V$

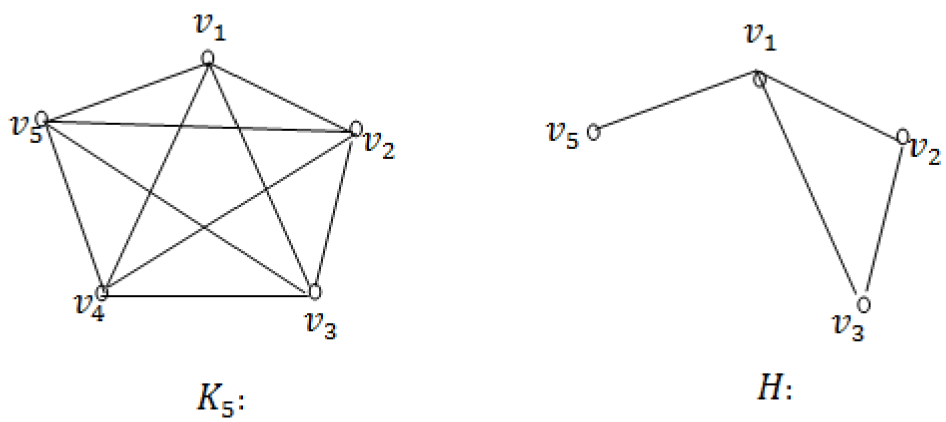


Figure (1.2): H is a subgraph of K_5.

and $F \subseteq E$. A subgraph H of G is proper subgraph of G if $H \neq G$.

Definition 1.4[5]

Let $G(V,E)$ be a graph. A walk in G is a list $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges such that for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. A trail is a walk with no repeated edge. A u,v-walk or u,v-trail has first vertex u and last vertex v; these are their endpoints. A u,v-path (a path between u and v) is a trail with $u \neq v$ and no repeated vertex.

Example 1.5

Consider the following graph:

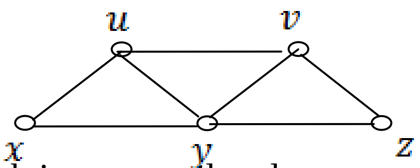


Figure (1.3)

$x\ y\ v\ y\ z$ is a walk which is not a trail and $x\ y\ v\ u\ y\ z$ is a trail which is not a path; $x\ y\ v\ u$ is a path but not a cycle and $y\ z\ v\ y$ is a cycle. Again $x\ u\ y\ v\ z\ y\ x$ is a closed trail but not a cycle.

Definition 1.6[5]

The degree of a vertex v in a graph is the number of incident edges.

Definition 1.7[5]

A cycle is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle.

Definition 1.8[1] A graph G is connected if for every pair of distinct vertices $u, v \in V(G)$, the graph G has a u, v-path. Otherwise, we say the graph is disconnected.Example 1.9

Consider the following graphs:

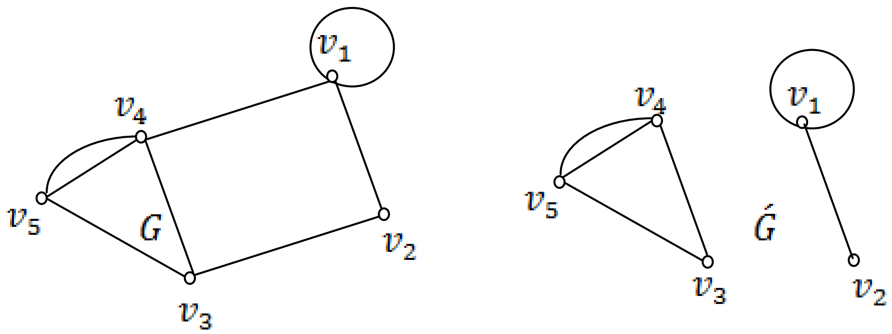


Figure (1.4): G is a connected graph, while $\acute{G}$ is disconnected.

Definition 1.10[1]

Let G be a graph and let $H_1, \dots, H_k$ be connected subgraphs of G whose vertices sets and edges sets are pairwise disjoint and such that they cover all the vertices and edges of G. That is,

$V(G) = V(H_1) \cup \dots \cup V(H_k)$,

$E(G) = E(H_1) \cup \dots \cup E(H_k)$,

Where $V(H_i) \cap V(H_j) = E(H_i) \cap E(H_j) = \emptyset$ for each distinct i and j.

Each of the subgraph H_i is called a component or a connected component of G.

In the Example (1.7) $\acute{G}$ has two a connected components.

Definition 1.11[4]

Two graphs G and G^1 are said to be isomorphic to each other if there is an one-to-one correspondence between their vertices and an one-to-one correspondence between their edges such that the incident relationship must be preserved.
In other words, two graphs $G(V,E)$ and $G^1(V^1,E^1)$ are said to be isomorphic, denoted by $G \cong G^1$, if there exists bijections
$f: V \longrightarrow V^1$ and $g: E \longrightarrow E^1$
Such that $g(v_i v_j) = f(v_i)f(v_j)$ for any edge $v_i\, v_j$ in G.

Example 1.12
Consider the following graphs:

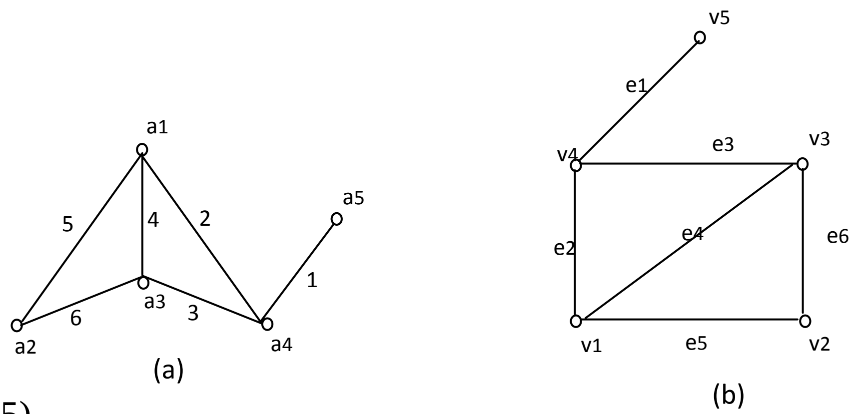


Figure (1.5)
Observe that these graphs are isomorphic. The correspondence between these two graphs is as follows:
$f(a_i) = v_i \qquad for\ 1 \le i \le 5$
$g(i) = e_i \qquad for\ 1 \le i \le 6$
and except the labeling of their vertices and edges of the isomorphic graphs , they are same, perhaps may be drawn differently.
Theorem 1.13[4]
If two graphs are isomorphic then the degrees of the symmetric vertices are equal.
The converse of this theorem is not true:
Consider the two graphs G_1 and G_2. These two graphs are not isomorphic.

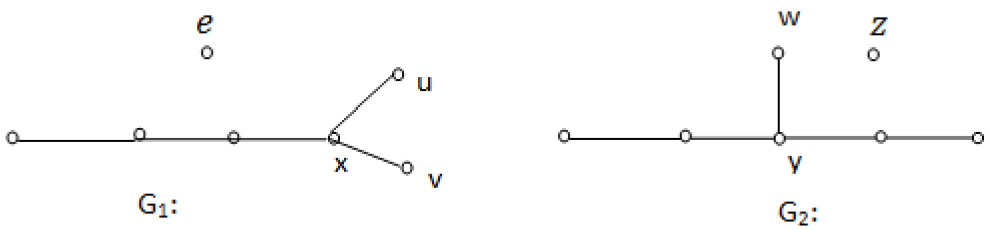


Figure (1.6)
Verification: In a contrary way, suppose that the graphs are isomorphic. then $\rho(x) = \rho(y) = 3$ and there is no vertex other then x and y whose degree is 3. So x and y be

associated. In this case, the number of pendent (end) vertices adjacent to x must be equal to the number of pendent vertices adjacent to y.

Observing the graphs, we can conclude that there are two pendent vertices adjacent to x, and there is only one pendent vertex adjacent to y, a contradiction. So the given to two graphs are not isomorphic.

Definition 1.14[2]

A topological space $(X, \mathcal{T})$ is called connected if there does not exist nonempty disjoint open subsets of X whose union is X.

2. Main Results

Definition 2.1

Let $G(V_G, E_G)$ be a graph. We considered that each an isolated vertex as an edge and we called it an isolated edge. Put $E = E_G \cup S$, where S denoted the set of isolated edges. We have defined $\mathcal{T}_E$ on E as follows:

$A \in \mathcal{T}_E$ if, and only if, $A \in P(E), P(E)$ the power set of E, such that there is no a vertex $x_0 \notin e_i, \forall\, e_i \in A$ such that $x_0 \in e_j$ for some $e_j \in A$.

Theorem 2.2

$\mathcal{T}_E$ is a topology on E.

Proof.

[O$_1$] $\emptyset$ and $E \in \mathcal{T}_E$.

[O$_2$] Now, Let $A_1, A_2 \in \mathcal{T}_E$.

We have there is no a vertex $x_1 \notin e_i, \forall\, e_i \in A_1$ such that $x_1 \in e$, $e \in A_1$.

Also there is no a vertex $x_2 \notin e_j, \forall\, e_j \in A_2$ such that $x_2 \in e^*$, $e^* \in A_2$.

Implies that there is no a vertex $x \notin e_k, \forall\, e_k \in A_1 \cup A_2$ such that $x \in e_0$, $e_0 \in A_1 \cup A_2$.

[O$_3$] Let $A_1, A_2 \in \mathcal{T}_E$;

$$A_1 \cap A_2 = \begin{cases} \emptyset \\ A_3 \end{cases}$$

Where A_3 is a set of isolated vertices, $\emptyset \in \mathcal{T}_E$ and $A_3 \in \mathcal{T}_E$ by [O$_2$].

Hence $\mathcal{T}_E$ is a topology on E.

Example 2.3

Consider the following graph $G(V_G, E_G)$ of seven vertices and four edges:

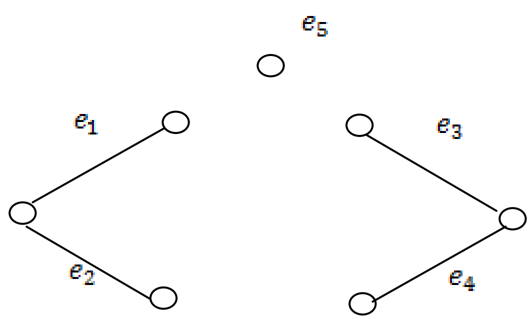


Figure (1.7)

Then $E = \{e_1, e_2, e_3, e_4, e_5\}$ and
$\mathcal{T}_E = \{E, \emptyset, \{e_1, e_2\}, \{e_3, e_4\}, \{e_5\}, \{e_1, e_2, e_5\}, \{e_3, e_4, e_5\}, \{e_1, e_2, e_3, e_4\}\}$.

Example 2.4

Consider the following graph $G(V_G, E_G)$ of six vertices and three edges:

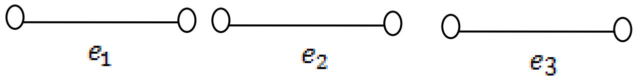


Figure (1.8)

Then $\mathcal{T}_E$ is the discrete topology on E.

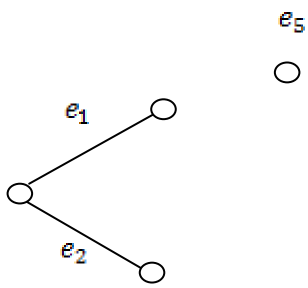
Definition 2.5

Let $G(V_G, E_G)$ be a graph and let $S(V_s, E_s)$ be a subgraph of G. Then the topology $\mathcal{T}_{E^*}$ on E_s that obtaining from the same topology on it as a subspace of the space $(E, \mathcal{T}_E)$.

Example 2.6

Consider the subgraph $S(V_S, E_S)$ from the Figure (1.7) has four vertices and two edges:

Figure (1.9)



$E = \{e_1, e_2, e_3, e_4, e_5\}$ $\mathcal{T}_E = \{E, \emptyset, \{e_1, e_2\}, \{e_3, e_4\}, \{e_5\}, \{e_1, e_2, e_5\}, \{e_3, e_4, e_5\}$, $\{e_1, e_2, e_3, e_4\}\}$. Let $E_s = \{e_1, e_2, e_5\}$ be subspace of E with $\mathcal{T}_{E^*} = \{E_s, \emptyset, \{e_1, e_2\}, \{e_5\}\}$ is topology on E_s

Theorem 2.7

A graph $G(V_G, E_G)$ is connected if, and only if, $(E, \mathcal{T}_E)$ is connected.

Proof.

First, Suppose that $(E, \mathcal{T}_E)$ is disconnected. This mean there exists nonempty disjoint open sets A and A^c of E such that $A \cup A^c = E$.

Implies that there is no a vertex $x_0 \notin e, \forall\, e \in A$ such that $x_0 \in e_j$ for some $e_j \in A$.

Therefore there is no a path between a vertex $v \in A$ and a vertex $w \in A^c$.

A contradiction with the connectedness of G.

Then $(E, \mathcal{T}_E)$ must be connected.

Now, Suppose that $G(V_G, E_G)$ is disconnected.

Implies that there are vertices v and w of G such that there is no a path connected between them. This implies that there are two disjoint open sets containing them such that their union is E. A contradiction with the connectedness of $(E, \mathcal{T}_E)$.

Then $G(V_G, E_G)$ must be connected. ■

Definition 2.8

Let $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ be two finite spaces. We say that $\mathcal{T}_1$ and $\mathcal{T}_2$ are symmetric, denoted by $\mathcal{T}_1 \sim \mathcal{T}_2$ if, and only if, they satisfy the following conditions:

$|\mathcal{T}_1| = |\mathcal{T}_2|$,and

If $W_1 \in \mathcal{T}_1$, then $\exists\, W_2 \in \mathcal{T}_2$ such that $|W_1| = |W_2|$, and conversely.

Example 2.9

Consider the following topologies on $X = \{a, b, c, d, e\}$,

$\mathcal{T}_1 = \{X, \emptyset, \{a\}, \{b, c, d\}, \{a, b, c, d\}\}$ and $\mathcal{T}_2 = \{X, \emptyset, \{a\}, \{c, d, e\}, \{a, c, d, e\}\}$

Observe that $\mathcal{T}_1$ and $\mathcal{T}_2$ are symmetric.

Example 2.10

Consider the following topologies on $X = \{a, b, c, d, e\}$:

$\mathcal{T}_1 = \{X, \emptyset, \{a, b\}, \{b, c, d\}, \{a, b, c, d\}, \{b\}\}$, and $\mathcal{T}_2 = \{X, \emptyset, \{a\}, \{b, c, d\}, \{a, b, c, d\}\}$.

They are not symmetric for $|\mathcal{T}_1| \neq |\mathcal{T}_2|$.

Theorem 2.11

Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs and $\mathcal{T}_1 \, and \, \mathcal{T}_2$ be their topologies. If $G_1 \cong G_2$, then $\mathcal{T}_1 \sim \mathcal{T}_2$.

Proof.

Since $G_1 \cong G_2$ this means that there exist bijections $f : V_1 \longrightarrow V_2$ and $g : E_1 \longrightarrow E_2$ such that $g(v_i v_j) = f(v_i) f(v_j)$ for any edge $v_i v_j$ in G. Then $|\mathcal{T}_1| = |\mathcal{T}_2|$.

Let $W \in \mathcal{T}_1$. Either W is isolated, then $f(W)$ is isolated and $|W| = |f(W)|$.

or W is not isolated this means that there is no $v \notin W$ such that $v \in e, \forall e \in W$.

Implies that there is no $v^* \notin f(W)$ such that $v^* \in e^*, \forall\, e^* \in f(W)$ for if $\exists\, v^* \notin f(W)$ such that $v^* \in e^*, \forall\, e^* \in f(W)$ implies that $\exists\, v_0 \in W$ such that $\rho(v_0) \neq \rho(f(v_0))$. A Contradiction for $G_1 \cong G_2$.

Then $|W| = |f(W)|$.

Let $W^* \in \mathcal{T}_2$. This means there exist $W \in \mathcal{T}_1$ such that $f(W) = W^*$.

Either W^* is isolated then $|W| = |W^*|$.

or W^* is not isolated this means that there is no $x \notin W^*$ such that

$x \in e^*, \forall\, e^* \in W^*$ ……….(∗)

If (∗) holds, then there is no $y \notin W$ such that $y \in e, \forall\, e \in W$.

For if $\exists\, y \notin W$ such that $y \in e, \forall\, e \in W$, implies that $\exists\, V \in W$ such that $\rho(V) \neq \rho(f(V))$, a contradiction.

Then $|W| = |W^*|$.

<u>Remark 2.12</u>

The converse of Theorem (2.11) is not true in general.

Consider the following graphs G_1 and G_2

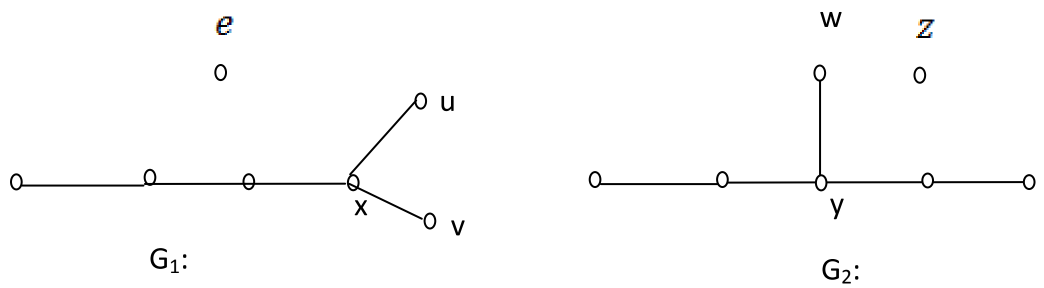


Figure (1.10)
These graphs have symmetric topologies but they are not isomorphic.

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