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Numerical Solution For Fornbrg-Whitham Equation

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Abstract

In this paper, homotopy perturbation method and variational iteration method are implemented to find numerical solution for Fornbrg-Whitham equation. To illustrate the simplicity and reliability of the methods, we will find the numerical results with the use of homotopy perturbation method and variational iteration method and compare them with the exact solution.

1-Introduction: The homotopy perturbation method (HPM) was proposed by Ji-Huan He[11,12,13]. Many authers try to improve this method to solve various nonlinear problems [6,10,14,16,17,19]. HPM yields a very rapid convergence of the solution series and sometimes one iteration leads to high accuracy of the solution. The variational iteration method (VIM) was proposed by Ji-Huan He in (1999). It was successfully applied to solve some classes of nonlinear problems. For linear problems, its exact solution can be obtained by only one iteration step due to the fact that Lagrange multiplier can be exactly identified. Recently, the application of the VIM in linear and nonlinear problems has been applied by scientists and engineers, for example, Burger's and coupled Burger's equation [15], nonlinear Burger's equation [9], Riccati differential equations [5], Troesch's problem [20], nonlinear singular boundary value problems [4], nth-order integro-differential equations [21], Cauchy problems [22], Riesz fractional partial differential equations [2], general linear Fredholm integro-differential equations [18], neutral fractional-differential equation with proportional delays [23], nonlinear wave equations [3], singular perturbation initial value problems [24].

The general form Fornbrg-Whitham equation [7] is given by:

$$u_t - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_x u_{xx} \quad , \qquad (1)$$

subject to the initial condition of

$$u(x,0) = \exp\left(\frac{1}{2}x\right) \qquad (2)$$

Then, the exact solution is given by:

$$u(x,t) = \exp\left(\frac{1}{2}x - \frac{2}{3}t\right) \qquad (3)$$

Abidi and Omrani[7] solved this equation by using Adomian decomposition method and homotopy analysis method and found the numerical results.

The paper is organized as follows: in the next section, the homotopy perturbation method is introduced. In section 3, variational iteration method is introduced. The application of homotopy perturbation method for solving Fornbrg-Whitham equation is introduced in section 4. The application of variational iteration method for solving Fornbrg-Whitham equation is introduced in section 5. The numerical solutions of the problem are obtained in section 6.Section 7 ends this paper with conclusion.

2-Homotopy perturbation method(HPM) [10,11]

To illustrate the HPM, Ji-Huan He considered the following nonlinear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega \qquad (4)$$

with boundary conditions

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma \qquad (5)$$

where is a general differential operator, is a boundary operator, is a known analytic function, is a boundary of the domain .

The operator A can be generally divided into two parts L and N,where L is linear, and N is nonlinear, therefore equation(4) can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0. \qquad (6)$$

The homotopy technique which is constructed by He[11], $v(r,p): \Omega \times [0,1] \to \mathbb{R}$ satisfied:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \qquad (7a)$$

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \qquad (7b)$$

where $r \in \Omega$ and $p \in [0,1]$ is called the homotopy parameter, and u_0 is an initial approximation of (4), which satisfies the boundary conditions. Obviously, from equation(7),we have:

$$H(v,0) = L(v) - L(u_0) = 0, \qquad (8)$$

$$H(v,1) = A(v) - f(r) = 0, \qquad (9)$$

and the changing process of p from 0 to 1, is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In the topology, this is called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic.

The embedding parameter $p \in [0,1]$ as a "small parameter" is used then assume that the solution of equation(6) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \qquad (10)$$

Setting $p = 1$ results in the approximate solution of equation(4):

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \qquad (11)$$

The series in(11) converges in most cases, however, the convergent rate depends upon the nonlinear operator (the following opinions are suggested by He[11])

(1)The second derivative of f with respect to must be small because the parameter may be relatively large.

(2)The norm of $L^{-1}(\partial N/\partial v)$ must be less than one so that the series to be converge.

Theorem[10]

Suppose that and be Banach space and is a contraction nonlinear mapping, that is

$\forall\, v, \tilde{v} \in X: \ \|N(v) - N(\tilde{v})\| \le \gamma \|v - \tilde{v}\|, \quad 0 < \gamma < 1$

Which according to Banach's fixed point theorem, having the fixed point , for which .

The sequence generated by the homotopy perturbation method will be regarded as

$$V_n = N(V_{n-1}), \quad V_{n-1} = \sum_{i=0}^{n-1} u_i, \quad n = 1,2,3 \ldots.$$

assume that where then the following statements hold:

(i)

(ii) .

(iii)

Proof: (i) By induction method on , for we have

$\|V_1 - u\| = \|N(V_0) - N(u)\| \le \gamma \|v_0 - u\|.$

Assume that as an induction hypothesis, then

$\|V_n - u\| = \|N(V_{n-1}) - N(u)\| \le \gamma \|V_{n-1} - u\| \le \gamma\gamma^{n-1}\|v_0 - u\| = \gamma^n \|v_0 - u\|.$

(ii) Using (i), we have

$\|V_n - u\| \le \gamma^n \|v_0 - u\| \le \gamma^n r < r \Rightarrow V_n \in B_r(u).$

(iii) Since , and , we deduce , that is .

3-Variational iteration method(VIM)[9]

Consider the following functional equation:

$N(u) + L(u) = g(x,t),$

where L is a linear operator, N is a nonlinear operator and $g(x,t)$ a known analytical function. According to variational iteration method, we can construct a correction functional as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\left(Lu_n(x,\tau) + N\tilde{u}_n(x,\tau) - g(x,t)\right) d\tau \qquad (12)$$

where λ is a general Lagrange multiplier, which can be identified optimally variational theory, and $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$. In this method, we determine first the Lagrange multiplier λ that will be identified optimally integration by parts. The successive approximations $u_{n+1}, n \ge 0$, of the solution u will be readily obtained upon using the determined Lagrange multiplier and using the initial approximation u_0. Consequently, the solution will be given as:

$u(x,t) = \lim_{n\to\infty} u_n(x,t).$

4- Application of homotopy perturbation method

Firstly, construct a homotopy perturbation method for equation(1) as follows:

$$(1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial v}{\partial t} - \frac{\partial^3 v}{\partial x^2 \partial t} + \frac{\partial v}{\partial x} - v\frac{\partial^3 v}{\partial x^3} + v\frac{\partial v}{\partial x} - 3\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\right) = 0 \qquad (13)$$

or

$$\left(\frac{\partial v}{\partial t}-\frac{\partial u_0}{\partial t}\right)+p\left(\frac{\partial u_0}{\partial t}-\frac{\partial^3 v}{\partial x^2\partial t}+\frac{\partial v}{\partial x}-v\frac{\partial^3 v}{\partial x^3}+v\frac{\partial v}{\partial x}-3\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2}\right)=0 \quad (14)$$

By substituting (10) into (12) and equating the coefficients of like terms with the identical powers of p, given:

$$\boldsymbol{p^0}: \frac{\partial v_0}{\partial t}=\frac{\partial u_0}{\partial t} \quad (15)$$

$$\boldsymbol{p^1}: \frac{\partial v_1}{\partial t}=-\frac{\partial u_0}{\partial t}+\frac{\partial^3 v_0}{\partial x^2\partial t}-\frac{\partial v_0}{\partial x}+v_0\frac{\partial^3 v_0}{\partial x^3}-v_0\frac{\partial v_0}{\partial x}+3\frac{\partial v_0}{\partial x}\frac{\partial^2 v_0}{\partial x^2} \quad (16)$$

$$\boldsymbol{p^2}: \frac{\partial v_2}{\partial t}=\frac{\partial^3 v_1}{\partial x^2\partial t}-\frac{\partial v_1}{\partial x}+v_1\frac{\partial^3 v_1}{\partial x^3}-v_1\frac{\partial v_1}{\partial x}+3\frac{\partial v_1}{\partial x}\frac{\partial^2 v_1}{\partial x^2} \quad (17)$$

$$\boldsymbol{p^3}: \frac{\partial v_3}{\partial t}=\frac{\partial^3 v_2}{\partial x^2\partial t}-\frac{\partial v_2}{\partial x}+v_2\frac{\partial^3 v_2}{\partial x^3}-v_2\frac{\partial v_2}{\partial x}+3\frac{\partial v_2}{\partial x}\frac{\partial^2 v_2}{\partial x^2} \quad (18)$$

$\vdots$

$$\boldsymbol{p^j}: \frac{\partial v_j}{\partial t}=\frac{\partial^3 v_{j-1}}{\partial x^2\partial t}-\frac{\partial v_{j-1}}{\partial x}+v_{j-1}\frac{\partial^3 v_{j-1}}{\partial x^3}-v_{j-1}\frac{\partial v_{j-1}}{\partial x}+3\frac{\partial v_{j-1}}{\partial x}\frac{\partial^2 v_{j-1}}{\partial x^2} \quad (19)$$

Integrating both sides of the above equations ,we obtain the following multiple solutions:

$$v_0=u_0 \quad (20)$$

$$v_j=\int_0^t\int_0^t\left(\frac{\partial^3 v_{j-1}}{\partial x^2\partial t}-\frac{\partial v_{j-1}}{\partial x}+\sum_{k=0}^{j-1}v_k\frac{\partial^2 v_{k-j-1}}{\partial x^2}-\sum_{k=0}^{j-1}v_k\frac{\partial v_{k-j-1}}{\partial x}+3\sum_{k=0}^{j-1}\frac{\partial v_k}{\partial x}\frac{\partial^2 v_{k-j-1}}{\partial x^2}\right)dtdt \;,$$

$$j=1,2,3,\ldots \quad (21)$$

Such that $u(x,0)=u_0(x,t)$, hence $\frac{\partial u_0}{\partial t}=0$.

The approximate solution is:

$$u=v_0+v_1+v_2+\cdots \quad (22)$$

5- Application of variational iteration method

For equation(1), the VIM formula takes the form

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t\lambda(\tau)\left(\frac{\partial u_n(x,\tau)}{\partial\tau}-\frac{\partial^3 u_n(x,\tau)}{\partial x^2\partial\tau}+\frac{\partial u_n(x,\tau)}{\partial x}-u_n(x,\tau)\frac{\partial^3 u_n(x,\tau)}{\partial x^3}\right.$$
$$\left.+u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial x}-3\frac{\partial u_n(x,\tau)}{\partial x}\frac{\partial^2 u_n(x,\tau)}{\partial x^2}\right)d\tau \quad (23)$$

Making the correction functional stationary, we have

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t\lambda(\tau)\left(\frac{\partial u_n(x,\tau)}{\partial\tau}-\frac{\partial^3 u_n(x,\tau)}{\partial x^2\partial\tau}\right)d\tau \quad (24)$$

Or equivalently,

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\left(\lambda(\tau)u_n(x,\tau)|_{\tau=t}-\int_0^t\lambda'(\tau)\,u_n(x,\tau)d\tau\right)$$

$$-\delta\left(\lambda(\tau)\frac{\partial^2 u_n(x,\tau)}{\partial x^2}\bigg|_{\tau=t} - \int_0^t \lambda'(\tau)\frac{\partial^2 u_n(x,\tau)}{\partial x^2}d\tau\right) \qquad (25)$$

Such that

$$\left(\lambda(\tau)\frac{\partial^2 u_n(x,\tau)}{\partial x^2}\bigg|_{\tau=t} - \int_0^t \lambda'(\tau)\frac{\partial^2 u_n(x,\tau)}{\partial x^2}d\tau\right) = 0$$

So

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\left(\lambda(\tau)u_n(x,\tau)|_{\tau=t} - \int_0^t \lambda'(\tau)\, u_n(x,\tau)d\tau\right) \qquad (26)$$

This yields the stationary condition

$$1 + \lambda(t) = 0 \qquad (27)$$

which leads to

$$\lambda(t) = -1 \qquad (28)$$

6-Numerical applications

For some values of and , we will compute the absolute errors for the differences between the exact solution(3) and the approximate solution (20) obtained by the 7-order HPM in table(1). While in table(2), we compute the absolut errors for the differences between the exact solution(3) and the approximate solution (9) obtained by the 7-order VIM.

	$x_i\backslash t_i$ 0.2	0.4	0.6	0.8
	0.11844 238375	0.10365 552439	0.09071 555501	0.07939 305409
	0.32195 977948	0.28176 492838	0.24659 044473	0.21581 269624
	0.87517 741805	0.76591 648472	0.67030 232500	0.58663 973055
	2.37897 887217	2.08197 686254	1.82207 062962	1.59465 211942
	6.46673 503853	5.65939 987271	4.95290 148267	4.33471 387894

Table(1) :represents the absolute errors for the differences between the exact solution and the approximate solution obtained by the 7-order HPM.

	$x_i \backslash t_i$ 0.2	0.4	0.6	0.8
	0.11844238375	0.10365552439	0.09071555501	0.07939305409
	0.32195977948	0.28176492839	0.24659044475	0.21581269626
	0.87517741806	0.76591648475	0.67030232504	0.58663973061
	2.37897887222	2.08197686264	1.82207062981	1.59465211970
	6.46673503870	5.65939987323	4.95290148371	4.33471388068

Table(2):represents the absolute errors for the differences between the exact solution and the approximate solution obtained by the 7-order VIM.

In table(2), some values of and , we find the exact solution and the approximate solution obtained by the HPM and VIM.

x_i	u_{exact}	u_{HPM}	u_{VIM}	$\lvert u_{\text{exact}} - u_{\text{HPM}} \rvert$	$\lvert u_{\text{exact}} - u_{\text{V}}$
-4	0.0048279499	0.00500752289	0.0050075230	1.79573012E-4	1.79573012E
-2	0.0131237287	0.00136118584	0.0136118589	4.88129756E-4	4.88130213E
0	0.0356739933	0.03700086759	0.0370008701	1.32687424E-3	1.32687672E
2	0.0969719679	0.10057878602	0.1005788037	3.60681815E-3	3.60683593E
4	0.2635971382	0.27340148636	0.2734016750	9.80434824E-3	9.80453690E

Table(3):represents the exact solution u_E, numerical solution u_{HPM} by 7-order HPM and u_{VIM} the numerical solution by 7-order VIM.

The error between and is less than the error between and , that means, HPM is more accurate than VIM.

In table(3), some values of x and $t = 5$, we found the exact solution and the approximate solution obtained by the HPM and VIM and compute the absolut errors for the differences between them.

We can find the approximate solution by HPM of order greater than 40 by the results in table(4) and we see the errors are very small. So, we can say that HPM is more accurate than VIM because it was very simple in application and does not take a long time in computing the results.

$x_i \backslash t_i$	0.2	0.4	0.6	0.8	1
-4	1.12259E-27	1.75923E-27	1.65715E-27	2.03616E-27	4.12750E-27
-2	3.05155E-27	4.78209E-27	4.50463E-27	5.53490E-27	1.12197E-26
0	8.29504E-27	1.29990E-26	1.22448E-26	1.50450E-27	3.04983E-26
2	2.25481E-26	3.53352E-26	3.32850E-26	1.50456E-26	8.29032E-26
4	6.12926E-27	9.60511E-26	9.95303E-26	4.08976E-26	2.25353E-25

Table(4):represents the absolute errors for the differences between the exact solution and the approximate solution obtained by HPM of order greater than 7 order.

7-Conclusion

We solved the Fornbrg-Whitham equation by homotopy perturbation method and variational iteration method. The exact and numerical results are presented and compared with each other in table 3 for some values of x and $t = 5$. Notice from the tables that the HPM and VIM are much accurate methods since the error is small and a few terms are required to obtain an accurate solution. So that the HPM and VIM are remarkably effective for solving the Fornbrg-Whitham equation. In our work, we used the Maple13 to calculate the results which are obtained by use of the iteration methods HPM and VIM.

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الملخص
في هذا البحث، طريقة اضطراب الهوموتوبي و طريقة التباين التكرارية طبقت لإيجاد الحلول العددية لمعادلة فورنبرك-وتهام. لتوضيح بساطة وقابلية الطريقتان، سنقوم بإيجاد النتائج العددية باستخدام طريقة الاضطراب الهوموتوبي و طريقة التغاير التكرارية ونقارنها مع النتائج العددية للحل التام.