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On Anti Fuzzy Relation And Anti Fuzzy Quotient Subgroups

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Abstract:
In this research, we define a special fuzzy relation on a group G by the anti fuzzy subgroup of G and study some properties of this relation.
Also, we define anti fuzzy quotient subgroup by using some special fuzzy relation which is defined in this study.

Keywords: Anti fuzzy subgroup , Anti fuzzy quotient subgroup , Anti similarity, Anti compatible, Anti congruence .

Introduction:
In 1965 Zadeh introduced the concepts of fuzzy set [3], in 1971 Rosenfeld formulated the term of fuzzy subgroup [1]. In 2004 Aktas define the fuzzy quotient subgroup by using some special fuzzy relation [2] . In this paper, we define some new special fuzzy anti congruence relations and drive some simple consequence . Then using those relations to define suitable anti fuzzy quotient subgroup of G/H .

1. Preliminary
In this section , we give the preliminary definitions and results that will be required in this paper.
Definition (1.1): Let X be a non-empty set. A fuzzy subset (set) of the set X is a function $\mu: X\rightarrow[0, 1]$. [3]
Definition (1.2): A fuzzy binary relation of a set X is a fuzzy subset of $X\times X$.
By a fuzzy relation we mean a fuzzy binary relation given by $\mu: X\times X\rightarrow[0, 1]$. [2]
Definition (1.3): If μ_1 and μ_2 are two relations of a set X, then their max-product composition denoted by $\mu_1 o \mu_2$ is defined as $\mu_1 o \mu_2(x,y)=\max_z \{ \mu_1(x,z) , \mu_2(z,y) \}$. [2]
Definition (1.4): Let G be any group. A fuzzy subset μ of a group G is called anti fuzzy subgroup of the group G if:
1- $\mu(xy) \le \max\{\mu(x), \mu(y)\}$ for every $x, y\in G$.
2. $\mu(x)=\mu(x^{-1})$ for every $x\in G$. [4]
Definition (1.5): An anti fuzzy subgroup μ of a group G is called an anti fuzzy normal subgroup of G if $\mu(xy) =\mu(yx)$ for all $x,y\in G$. [5]
Definition (1.6): A fuzzy relation μ of a set X is said to be anti reflexive if $\mu(x,x)=0$ for every $x\in X$ and said to be symmetric if $\mu(x,y)=\mu(y,x)$ for every $x, y\in X$. [2]
Definition (1.7): A fuzzy relation μ of a set X is said to be anti transitive if $\mu o \mu \ge\mu$. [2]

Definition (1.8): A fuzzy relation μ of a set X is said to be anti similarity if μ is anti reflexive, symmetric and anti transitive. [2]
Definition (1.9): A fuzzy anti compatible and anti similarity relation on a semi group S is called a fuzzy anti congruence. [2]

2. Main Results

In this section we shall define some special fuzzy relation and give some its results.
Definition (2.1): A fuzzy relation of a semi group S is said to be anti compatible iff $\max\{\mu(a, b), \mu(c, d)\} \geq \mu(ac, bd)$, for every a, b, c, d∈S.
Definition (2.2): Let G be a group and μ be anti fuzzy normal subgroup of a group G. We define the fuzzy relation α on G as follows :

$$\alpha(a,b) = \begin{cases} \max\{\mu(a), \mu(b)\} & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases}$$

Now, we give some properties of α.
Proposition (2.3): Let G be a group and μ be anti fuzzy subgroup of a group G. Then the relation α is an anti similarity relation on G.
Proof:
1) α(a, a) = 0 for every a∈G, therefore α is anti reflexive.
2) α(a, b) = max{μ(a), μ(b)} = max {μ(b), μ(a)} = α(b, a) for every a, b∈G.
therefore α is symmetric.
3) $(\alpha o \alpha)(a, c) = \max_{b\in G}\{\alpha(a,b), \alpha(b,c)\}$

$$= \max_{b\in G}\{\max\{\mu(a), \mu(b)\}, \max\{\mu(b), \mu(c)\}\}$$

(since max {μ(a), μ(b)} ≥ μ(a) and max {μ(b), μ(c)} ≥ μ(c) for all b∈G)

$$\geq \max\{\mu(a), \mu(c)\}$$

$$= \alpha(a, c)$$, for every a, c∈G

Therefore α is anti similarity relation
Lemma (2.4): $\alpha(a, b) = \alpha(a^{-1}, b^{-1}) = \alpha(b^{-1}, a^{-1})$ for every a, b∈G.
Proof:
$\alpha(a, b) = \max\{\mu(a), \mu(b)\} = \max\{\mu(a^{-1}), \mu(b^{-1})\} = \alpha(a^{-1}, b^{-1})$.
also $\alpha(a^{-1}, b^{-1}) = \alpha(b^{-1}, a^{-1})$ (since α is a symmetric relation).
Proposition (2.5): The fuzzy relation α is fuzzy anti compatible.
Proof:

$$\begin{aligned} \alpha(ac, bd) &= \max\{\mu(ac), \mu(bd)\} \\ &\leq \max\{\max\{\mu(a), \mu(c)\}, \max\{\mu(b), \mu(d)\}\} \\ &= \max\{\max\{\mu(a), \mu(b)\}, \max\{\mu(c), \mu(d)\}\} \\ &= \max\{\alpha(a, b), \alpha(c, d)\} \end{aligned}$$

therefore α is anti compatible. ■
Lemma (2.6): The fuzzy relation α is a fuzzy anti congruence.

Proof: By proposition (2.3), we have α is a fuzzy anti similarity relation and from **proposition (2.5)** α is anti compatible, therefore α is anti congruence. ■

Proposition (2.7): The fuzzy relation α is anti fuzzy subgroup of the group $G\times G$.

Proof:

1) Let $(x, y).(w, z) \in G\times G$

$$\begin{aligned}\alpha((x, y).(w, z)) = \alpha(xw, yz) &= \alpha(xw, yz)\\ &= \max\{\mu(xw), \mu(yz)\}\\ &\leq \max\{\max\{\mu(x), \mu(w)\}, \max\{\mu(y), \mu(z)\}\}\\ &= \max\{\max\{\mu(x), \mu(y), \mu(w), \mu(z)\}\}\\ &= \max\{\max\{\mu(x), \mu(y)\}, \max\{\mu(w), \mu(z)\}\}\\ &= \max\{\alpha(x, y), \alpha(w, z)\}\end{aligned}$$

2) Let $(x, y) \in G\times G$

$\alpha(x, y) = \alpha(x^{-1}, y^{-1})$ (by Lemma (2.4))

Definition (2.8): Let G be a group and H be a normal subgroup of G . $R:G/H \to [0, 1]$ defined by $R(xH) = \sup_{h\in H} \alpha(x, h)$ for all $xH\in G/H$.

Proposition (2.9):

The fuzzy set R is anti fuzzy normal quotient subgroup of G/H

Proof:

$$\begin{aligned}R(xHyH) = R(xyH) = \sup_{h\in H} (xy,h) &= \sup_{h\in H} \max\{\mu(xy), \mu(h)\}\\ &\leq \sup_{h\in H} \max\{\max\{\mu(x), \mu(y)\}, \mu(h)\}\\ &= \sup_{h\in H} \max\{\max\{\mu(x), \mu(h)\}, \{\max\{\mu(y), \mu(h)\}\}\\ &= \sup_{h\in H} \max\{\alpha(x, h)\ \alpha(y, h)\}\\ &= \max\{R(xH), R(yH)\}\end{aligned}$$

and

$$\begin{aligned}R(x^{-1}H) = \sup_{h\in H} \alpha(x^{-1}, h) &= \sup_{h\in H} \max\{\mu(x^{-1}), \mu(h)\}\\ &= \sup_{h\in H} \max\{\mu(x), \mu(h)\}\\ &= \sup_{h\in H} \alpha(x, h) = R(xH)\end{aligned}$$

Thus, R is anti fuzzy quotient subgroup of G/H

$$\begin{aligned}R(xHyH) = \alpha(xy,h) = \max\{\mu(xy), \mu(h)\} &= \max\{\mu(yx), \mu(h)\}\\ &= \alpha(yx, h) = R(yHxH)\end{aligned}$$

Hence , R is is anti fuzzy quotient normal subgroup of G/H ■

Proposition (2.10): Let H be a normal subgroup of a group G . Then R(xHyH)= R(yH) if and only if R(xH)= R(H) for all xH , yH $\in$ G/H .

Proof:

Let xH , yH $\in$ G/H , such that R(xHyH)= R(yH)

Now, choose yH=H then R(xH)= R(H)

Conversely, suppose R(xH)= R(H)

$$\begin{aligned} R(xHyH) &\le \max\{ R(xH), R(yH)\} \\ &= \max\{ R(H), R(yH)\} \\ &= \max\{ \sup_{h\in H} \alpha(e, h), \sup_{h\in H} \alpha(y, h)\} \\ &= \max\{ \sup_{h\in H} \{ \max\{\mu(e), \mu(h)\}, \sup_{h\in H} \{ \max\{\mu(y), \mu(h)\}\} \\ &= \max\{ \sup_{h\in H} \mu(h), \mu(y)\} \\ &= \sup_{h\in H} \{\max\{ \mu(h), \mu(y)\} \\ &= \sup_{h\in H} \alpha(y, h) \\ &= R(xH) \quad \blacksquare \end{aligned}$$

Definition (2.11): Let H be a normal subgroup of a group G , the fuzzy relation μ_R on G/H is defined by $\mu_R(xH, yH) = R(xy^{-1}H)$, for all (xH, yH) $\in$ G/H × G/H .

Proposition (2.12): The fuzzy relation μ_R is a fuzzy anti congruence on G/H.

Proof: Let xH , yH $\in$ G/H .

Then μ_R is fuzzy anti reflexive , since $\mu_R(xH, xH) = R(xx^{-1}H) = R(H) = 0$

μ_R is fuzzy symmetric , since

$$\begin{aligned} \mu_R(xH, yH) &= R(xx^{-1}H) = R((yx^{-1})^{-1} H) \\ &= R(yx^{-1}H) = R(yHx^{-1}H) \\ &= \mu_R(yH, xH) . \end{aligned}$$

$$\begin{aligned} \mu_R \, o\mu_R \,(xH,yH) &= \max_{zH\in G/H} \{\mu_R(xH, zH), \mu_R(zH, yH)\} \\ &= \max_{zH\in G/H} \{R(x z^{-1}H), R(z y^{-1}H)\} \\ &= \max_{z\in G}\{ \sup_{h\in H} \alpha(x z^{-1}, h), \sup_{h\in H} \alpha(z y^{-1}, h)\} \\ &= \max_{z\in G} \{ \sup_{h\in H} \{\max\{\mu(xz^{-1}), \mu(h)\}, \sup_{h\in H} \{ \max \\ &\quad \{\mu(zy^{-1}), \mu(h)\}\} \\ &\ge \max_{z\in G} \{ \sup_{h\in H} \{\max\{\max\{\mu(x), \mu(z^{-1})\}, \mu(h)\}\}. \\ &\quad \sup_{h\in H} \{ \max\{\max\{\mu(z), \mu(y^{-1})\}, \mu(h)\}\}\} \\ &= \max_{z\in G} \{ \sup_{h\in H} \{\max\{\mu(xy^{-1}), \mu(z)\}, \mu(h)\}\} \end{aligned}$$

$$\geq \sup_{h\in H} \{ \max\{\mu(xy^{-1}), \mu(h)\}\}$$
$$= \sup_{h\in H} \alpha(x y^{-1}, h)$$
$$= R(xy^{-1}H)$$
$$= \mu(xH, yH)$$

Hence , μ_R is a fuzzy anti transitive .

On other hand ,

Let xH , yH, zH and wH $\in$ G/H . Then

$$\max\{ \mu_R (xH, yH) , \mu_R (zH ,wH)\} = \max \{R(xy^{-1}H) , R(zw^{-1}H)\}$$
$$= \max \{ \sup_{h\in H} \alpha (xy^{-1} ,h) , \sup_{h\in H} \alpha (zw^{-1}, h)\}$$
$$= \max\{ \sup_{h\in H} \max \{\mu(xy^{-1}), \mu(h)\}$$
$$, \sup_{h\in H} \max\{\mu(zw^{-1}), \mu(h)\}$$
$$= \max \{ \sup_{h\in H} \max\{\mu(xy^{-1}), \mu(zy^{-1}) , \mu(h)\}$$
$$\geq \sup_{h\in H} \max\{\mu(xz)(yw)^{-1}, \mu(h)\}$$
$$= \sup_{h\in H} \alpha ((xz)(yw)^{-1}, h)$$
$$= R(xz (yw)^{-1}H)$$
$$= \mu_R (xzH , ywH)$$

Hence , μ_R is fuzzy anti compatible .

Thus , μ_R is fuzzy anti congruence on G/H. ■

Proposition (2.13): The fuzzy set $\bar{\mu} : {}^{G/H}\!/_{R} \rightarrow [0,1]$ defined by $\bar{\mu}(R_{aH})=R(aH)$ for all $R_{aH} \in {}^{G/H}\!/_{R}$ then $\bar{\mu}$ is an anti fuzzy normal subgroup of ${}^{G/H}\!/_{R}$

Proof:

1- Let R_{aH} , $R_{bH} \in {}^{G/H}\!/_{R}$

$$\bar{\mu}(R_{aH} \text{ o } R_{bH}) = R_{aHbH} = R(aHbH)$$
$$= R(abH)$$
$$= \sup_{h\in H} \{ \alpha (ab,h)\}$$
$$= \sup_{h\in H} \{\max\{\mu(ab), \mu(h)\}\}$$
$$\leq \sup_{h\in H} \max\{\max\{\mu(a), \mu(b)\}, \mu(h)\}$$
$$= \max\{ \sup_{h\in H} \max\{\mu(a), \mu(h)\}, \sup_{h\in H} \max\{\mu(b), \mu(h)\}\}$$
$$= \max\{ \alpha (a , h) , \alpha (b , h)\}$$

$= \max\{R(aH), R(bH)\}$

2- Let $R_{aH} \in G/H/R$

$\bar{\mu}(R^{-1}_{aH}) = \bar{\mu}(R_{a^{-1}H}) = R(a^{-1}H) = R(aH) = \bar{\mu}(R_{aH})$

3- Let $R_{aH}, R_{aH} \in G/H/R$. Then

$\bar{\mu}(R_{aH}\ R_{bH}) = \bar{\mu}(R_{abH})$

$= R(abH) = R(aHbH)$

$= R(bHaH) = R(baH)$

$= \bar{\mu}(R_{baH}) = \bar{\mu}(R_{bH}R_{aH})$

From 1,2 and 3 , then $\bar{\mu}$ is an anti fuzzy normal subgroup of $G/H/R$. ■

Remark : We call $\bar{\mu}$ the fuzzy quotient group determined by R

Theorem (2.14): $G/H/R$ is homomorphism image of G such that $R = \bar{\mu} o\theta$ and $\ker(\theta) = (G/H)_R$

Proof:

Define the map $\theta: G/H \rightarrow G/H/R$ by $\theta(aH) = R_{aH}$, $a \in G$

Clearly θ is well defined

Let $aH, bH \in G/H$

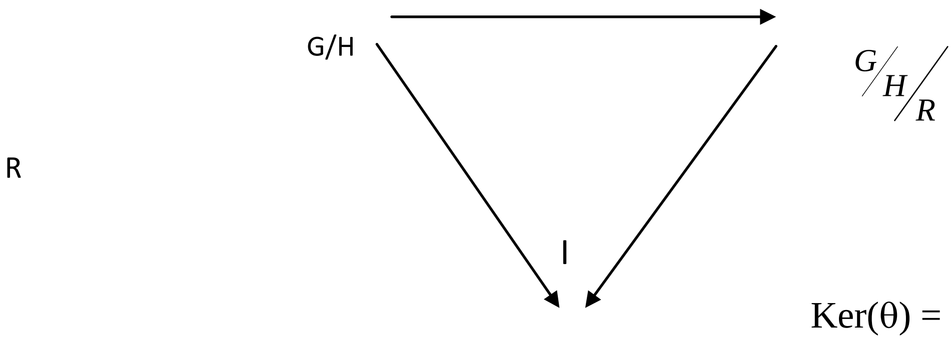
$\theta(aHbH) = \theta(abH) = R_{abH} = R(aH)(bH)$

$= R_{aH} R_{bH} = \theta(aH)\ \theta(bH)$

Let $R_{aH} \in G/H/R$, $aH \in G/H$

$\theta(aH) = R_{aH}$

Thus , θ is onto homomorphism



$\text{Ker}(\theta) = \{ aH \in G/H \mid \theta(aH) = R_H \}$

$= \{ aH \in G/H \mid R_{aH} = R_H \}$

$= \{ aH \in G/H \mid R(aH) = R(H) \}$

$= \{ aH \in G/H \mid aH \in (G/H)_R \}$

$= (G/H)_R$ ■

Corollary (2.15) : $\frac{G/H}{(G/H)_R} \cong \frac{G/H}{R}$

Proof:

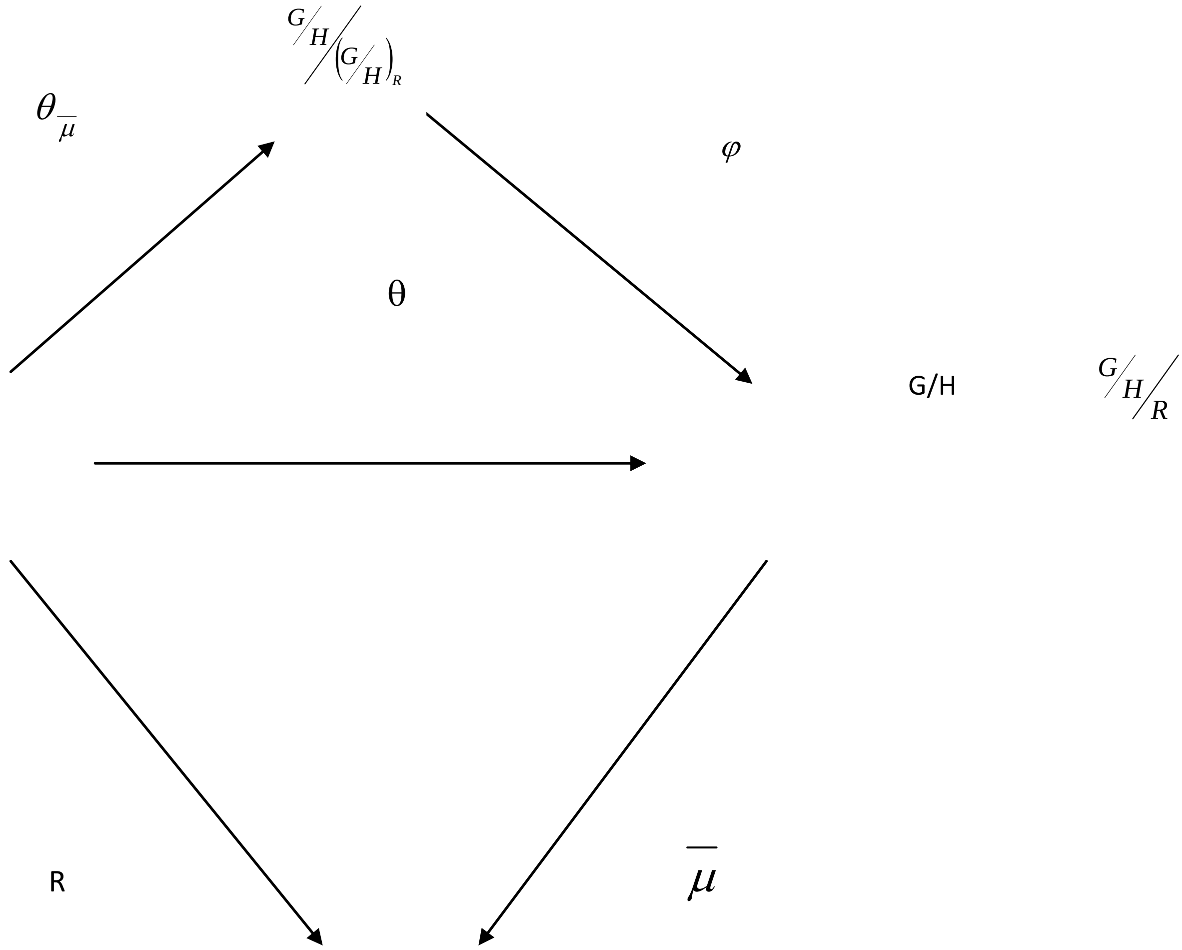
From proposition (2.13) the epimorphism θ: G/H → $\frac{G/H}{R}$, θ(aH)= R_{aH} with ker (θ)= $(G/H)_R$

Thus, by the first group isomorphic theorem

$$\frac{G/H}{\ker(\theta)} \cong \frac{G/H}{R}$$

$$\frac{G/H}{(G/H)_R} \cong \frac{G/H}{R} \quad . \blacksquare$$

Remark : We explain all these results in the following diagram :



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