Mine-Prime Submodules

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**Abstract**

Let \(O\) be commutative rings with identity, and all modules are (left) unitary of an \(O\) – module. A proper submodule \(P\) of an \(O\) – module \(G\) is called prime submodule, if for any \(r \in \text{Ann}(P)\) and \(m \in G\), implies that either \(mr \notin P\) or \(PG \subseteq P\). As strong from prime submodule we introduce in that paper the concept of Mine-Prime submodules and gave same basic properties, examples and characterizations of this concept. Moreover we study the behavior of Mine-Prime submodules in class of multiplication modules, furthermore we prove that by examples the residual of Mine-Prime submodules not to be Mine-Prime ideal of \(O\) so we gave under certain conditions several characterizations of Mine-Prime submodules.

**MSC.**

Introduction

Famous concept to start with in this paper was prime submodule this concept was first introduce by Dauns \([1]\). Many research interesting generalized prime submodules such as (semiprime, quasi prime) submodules see \([2, 3]\).

I recent time this concept was generalized by (nearly prime, nearly semiprime, nearly quasi prime) submodules by see \([4, 5, 6]\).

As strong from prime submodule the introduce the concept of (restrict nearly semiprime, restrict nearly prime) submodules see \([7, 8]\).

In this paper we introduce new strong from of prime submodule which we called Mine-Prime submodule we study this concept extensively.

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This concept consist three parts, part one deal with reeling well-know definition, propositions that we need in the sequel. Part two corned with introduce the definition of Mine-Prime submodules and gave several importation, characterization, basic proposition and example with of this concept.

Finally part three devoted to gave many characterization of Mine-Prime submodule in some types of module such as (multiplication, projective, faithful, content, ... modules)

1. **Basic Concepts and Prelminaes**

This part deal with reeling well-know definition, propositions that we need in the sequel.

"Recall that a proper ideal $S$ of a ring $O$ is called a prime ideal, if whenever $xy \in S$, for $x, y \in O$ implies that either $x \in S$ or $y \in S$[9]."

"A proper submodule $P$ of an $O$-module $G$ is called prime submodule if $\alpha x \in P$ for $\alpha \in O$, $x \in G$ implies that either $x \in P$ or $\alpha \in [P:O G]$ [1]."

"Recall that the residual of a submodule $P$ of an $O$-module $G$ denoted by $[P:O G]$ is an ideal of $O$ defined by $[P:O G] = \{\alpha \in O: \alpha G \subseteq P\}[10]$."

"A submodule $P$ of an $O$-module $G$ is called maximal submodule if $P \subseteq D \subseteq G$, then $D = G$ [11]."

"the Jacobson radical of $O$-module $G$ denoted by $J(G)$ is the intersection of all maximal submodule of $G$ [11]".

"A proper submodule $P$ of an $O$-module $G$ is called nearly prime submodule, if whenever $\alpha x \in P$ for $\alpha \in O$, $x \in G$ implies that either $x \in P$ or $\alpha \in [P:O G]$ [5]."

"Recall that for any submodule $S$ of a ring $O$ is multipilcatively closed subset of $O$ if $1eS$ and $xyeS$ for every $x, yeS$. And if $P$ is a submodule of an $O$-module $G$ and $S$ is multipilcatively closed subset of $O$, then $P_S = \{m \in G: \exists t \in S$ such that $tm \in P\}$ is a submodule of $G$ and $P \subseteq P_S$ [12]".

**Proposition 1.1** [9, Th. (5.1)]

"Let $S$ be a proper ideal of a ring $O$. Then $S$ is maximal ideal if and only if $S + (a) = O$ for any $a \notin S$."

"Recall that a submodule $P$ of an $O$-module $G$ is called small if $P + D = G$ implies that $D = G$ for any proper submodule $D$ of $G$ [13]".

**Proposition 1.2**[14, Coro. (9.1.5)(a)]

"If $U: G \rightarrow G'$ be $O$-epimorphism and $KerU$ is a small submodule of $G$, then $U(J(G)) = J(G')$ and $U^{-1}(J(G')) = J(G)$".

"We say a non-zero $O$-module $G$ is called hollow if every proper submodule of $G$ is small [13]".

"Recall that a submodule $[P:O G] = \{x \in G: \exists t \in S \subseteq P\}$, where $S$ is an ideal of $O$ and $P$ is a submodule of $G$ such that $P \subseteq [P:O G]$ and $[P:O G] = P, [S:O] = S$." [15, p.16]

"Recall that $G$ is an multiplication $O$-module is, if every submodule $P$ of $G$ is of the form $P = SG$ for some ideal $S$ of $O$, $G$ is a multiplication $O$-module if $P = [P:O G]$ [16]".

"Recall that for any submodule $P$ and $D$ of a multiplication $O$-module $G$ with $P = SG$ and $D = JG$ for some ideals $S$ and $J$ of $O$. The product $PD = SG.JG = SG$, that is $PD = SD$. In particular $PG = SGG = SG = P$. Also for any $x \in G$ we have $P = Sx$ and $x \in Oc$ as a submodule of $G$ [17]".

"Recall that $O$-module $G$ is projective if every $O$-epimorphism $f$ from $O$-module $G'$ into $O$-module $G''$ and for any $O$-homomorphism $g$ from $O$-module $G$ into $O$-$module G''$ there exists an $O$-homomorphism $h$ from $O$-module $G$ into $O$-module $G'$ such that $f \circ h = g$ [14]".
Proposition 1.3 [11, Pr. (17.10)]
"If $G$ is a projective $O$-module, then $J(O)G = J(G)$".

"Recall that $G$ is an faithful $O$-module $G$ if \( \text{ann}_O(G) = (0) \), where \( \text{ann}_O(G) = \{ r \in O : rG = (0) \} \), and \( [0 :_OG] = \text{ann}_O(G) \) [14]."

Proposition 1.4 [18, Re. p14]
"If $G$ is multiplication faithful $O$-module, then $J(O)G = J(G)$".

"Recall that an $O$-module $G$ is called content module if \( (\cap_{i \in I} S_i)G = \cap_{i \in I} S_i G \) for each family of ideals $S_i$ in $O$ [19]."

Proposition 1.5 [18, Pr. (1.11)]
"If $G$ is content module, then $J(O)G = J(G)$".

"Recall that a ring $O$ is called a good ring if $J(G) = J(O)G$ for any $O$-module $G$ [14]."

"Recall that an $O$-module $G$ is finitely generated if $G = Ov_1 + Ov_2 + \cdots + Ov_n$ where $v_1, v_2, \ldots, v_n \in G$ [14]."

Proposition 1.6 [19, Co. (S)]
"Let $G$ be multiplication finitely generated $O$-module with $SG \neq G$ for all maximal ideal $S$ of $O$, then $J(G) = J(O)G$".

Proposition 1.7 [20, Co. of Th. (9)]
"Let $G$ be generated multiplication finitely $O$-module and $S, J$ are ideals of $O$. Then $SG \subseteq JG$ if and only if $S \subseteq J + \text{ann}_O(G)$".

2. Basic Properties Mine-Prime

In this part of this research we introduce the definition of Mine-Prime submodule and we give some properties, characterizations of this concept.

Definition 2.1

A proper submodule $P$ for an $O -$ module $G$ is called Mine-Prime (for short $MP$) submodule, if for any $rmeP$, for $r \in O$, $meG$, implies that either $meP \cap J(G)$ or $rG \subseteq P \cap J(G)$.

And we called an ideal $S$ of a ring $O$ is $MP$ ideal of if $S$ is $MP O$-submodule for an $O -$ module $O$.

Remarks, and examples 2.2

1. Let $O = Z, G = Z_4$, the submodule $P = (2)$ is $MP$ submodule of $Z_4$. Thus for each $seZ, meZ_4$, if $smeP$, impales that either $meP \cap J(G) = (2) \cap (2) = (2) \cap (2) = (2)$ or $s \in [P \cap J(Z_4) \cap Z_4] = [(2) \cap Z_4] = 2Z$.

2. Every $MP$ submodule for an $O -$ module $V$ is Prime submodule for $G$, but the opposite is not true.

Proof: it is clear that every $MP$ submodule for an $O -$ module $G$ is prime submodule.

For the converse consider the example:

Let $G = Z_{12}, O = Z$, the submodule $P = (2)$ it is clearly that $P$ is prime submodule. But $P$ isnot $MP$ submodule of $Z_{12}$, since $2.2eP$, for $2eZ, 2eZ_{12}$ but $2 \notin P \cap J(G) = (\bar{6})$ and $2Z_{12} \not\subset (\bar{6})$

3- Every $MP$ submodule for $O -$ module $G$ is nearly prime submodule for $G$, but contrariwise isn't true.
Proof: it is clear that every MP submodule for an $O -$module $G$ is nearly prime submodule.

For the converse consider the example:

Let $G = Z_{12}, O = Z,$ and the submodule $P = \langle 2 \rangle$ it's nearly prime submodule of $Z_{12}$. Thus for each $s \in Z, x \in Z_{12},$ if $s x e P$, implies that either $x e + J(G) = \langle 2 \rangle + \langle 6 \rangle = \langle 2 \rangle$ or $s e [P + J(G); z G] = \langle 2 \rangle; z G]$ = 2Z. But $P$ is not MP submodule of $G$ (by Remarks and Examples 2.2 (2))

4. If $P$ and $D$ are proper submodules of an $O$-module $G$ with $D \not\subseteq P$, and $P$ is MP submodule for $G$, then $D$ is not to is MP submodule for $G$. The following example explain that:

Let $G = Z_4, O = Z$, the submodule $P = \langle 2 \rangle$ is MP submodule of $G$ (by Remarks and Examples 2.2 (1)) and $D = \langle 0 \rangle$ is submodule of $G$ such that $D \not\subseteq P$, but $D$ is not MP submodule of $G$, since $2 \not\in D \cap J(G) = \langle 0 \rangle$ and $2Z_4 \not\subseteq \langle 0 \rangle$.

**Proposition 2.3**

Let $G$ is an $O -$module, and $P$ is submodule for $G$. Then $P$ is MP submodule for $M$ if and only if for any submodule $D$ for $G$ and any ideal $S$ of $O$ with $SD \subseteq P$, implies that either $D \subseteq P \cap (G)$ or $S \subseteq [P \cap (G); \phi]$. 

**Proof**

$(\Rightarrow)$ Suppose $SD \subseteq P$, for $D$ is submodule for $G$ and $S$ is an ideal of $O$, with $D \not\subseteq P \cap (G)$, then $\exists x e D$ and $x \not\in P \cap (G)$. Since $SD \subseteq P$ then for any $a e S, axeP$. But $P$ is MP submodule for $G$ and $x \not\in P \cap (G)$ then $ae [P \cap (G); \phi]$, hence $S \subseteq [P \cap (G); \phi]$. 

$(\Leftarrow)$ Suppose $r e P$, for $reO, xeG$, then $(r)(x) \subseteq P$, so by hypothesis either $(x) \subseteq P \cap (G)$ or $(r) \subseteq [P \cap (G); \phi]$. That is either $xeP \cap (G)$ or $r e [P \cap (G); \phi]$. Hence $P$ is MP submodule for $G$.

As direct application of Proposition 2.3 we gave the following corollaries.

**Corollary 2.4**

Let $G$ is an $O -$module, and $P$ is submodule for $G$. Then $P$ is MP submodule for $G$ if and only if for any submodule $D$ for $G$ and any $s e O$ with $s D \subseteq P$, implies that either $D \subseteq P \cap (G)$ or $se [P \cap (G); \phi]$. 

**Corollary 2.5**

Let $G$ is an $O -$module, and $P$ is submodule for $M$. Then $P$ is MP submodule for $G$ if and only if for any $se O$ with $s G \subseteq P$, implies that either $G \subseteq P \cap (G)$ or $se [P \cap (G); \phi]$. 

**Corollary 2.6**

Let $G$ is an $O -$module, and $P$ is submodule for $G$. Then $P$ is MP submodule for $G$ if and only if for any ideal $S$ of $O, m \in G$ with $Sm \subseteq P$, implies that either $m \in P \cap (G)$ or $S \subseteq [P \cap (G); \phi]$. 

**Proposition 2.7**

Let $P$ is a proper submodule for an $O -$module $G$, and $[P \cap (G); \phi] G$ is a prime ideal of $O$. Then $P$ is MP submodule for $G$ if and only if $P(S) \subseteq P \cap (G)$ for each multiplicatively closed subset $S$ of $O$ such that $S \cap [P \cap (G); \phi] G = \phi$. 

**Proof**

$(\Rightarrow)$ Suppose $P$ is MP submodule for $G$, and let $x e P(S)$, then there exists $se S$ such that $s x e P$. But $P$ is MP submodule for $G$, so either $xeP \cap (G)$ or $se [P \cap (G); \phi]$. But if $se [P \cap (G); \phi]$, implies that $se S \subseteq [P \cap (G); \phi] = \phi$, which is a contradiction. Thus $xeP \cap (G)$ and hence $P(S) \subseteq P \cap (G)$.
Suppose \( rxeP \), for \( r \in O, x \in G \), such that \( x \notin P \cap (G) \) and \( r \notin [P \cap (G);_O G] \). But \( S \) is a multiplicatively closed subset for \( O \), then \( S = \{1, r, r^2, r^3, \ldots\} \), and since \( [P \cap (G);_O G] \) is a prime ideal of \( O \), then \( S \cap [P \cap (G);_O G] = \emptyset \). But \( x \notin P \cap (G) \), implies that \( x \notin P \) and hence \( rx \notin P \) which is a contradiction. Thus, either \( xeP \cap (G) \) or \( re[P \cap (G);_O G] \), therefore \( P \) is MP submodule for \( G \).

**Proposition 2.8**

Let \( G \) be an \( O - \)module, and \( P \) is submodule for \( G \) with \( [P \cap (G);_O G] \) is a maximal ideal for \( O \). Then \( P \) is MP submodule for \( G \).

**Proof**

Let \( saeP \), for \( saO, xeG \), with \( s \notin [P \cap (G);_O G] \). Since \( [P \cap (G);_O G] \) is a maximal ideal of \( O \), by Proposition 1.1 \( O = \langle s \rangle + [P \cap (G);_O G] \), wherever \( \langle s \rangle \) is ideal of \( O \) generated by \( s \), we obtain \( \exists \ a \in O \) and \( \exists \ b \in [P \cap (G);_O G] \) such that \( 1 = as + b \), hence \( x = asx + bxeP \cap (G) \). Hence \( P \) is an MP submodule for \( G \).

**Proposition 2.9**

Let \( G \) be an \( O - \)module, and \( P \) is a proper submodule for \( G \), with \( [D;_O G] \notin [P \cap (G);_O G] \), and \( P \cap (G) \) is a proper submodule of \( D \) for each submodule \( D \) for \( M \) such that \( [P \cap (G);_O G] \) is a prime ideal of \( O \). Then \( P \) is MP submodule for \( G \).

**Proof**

Suppose \( rxeP \), for \( reO, xeG \), and \( x \notin P \cap (G) \). Then \( P \cap (G) = P \cap (G) + \langle x \rangle = D \) and so \( [D;_O G] \notin [P \cap (G);_O G] \), then there exists \( ae[D;_G G] \) and \( a \notin [P \cap (G);_O G] \). That is \( aG \subseteq D \) and \( aG \notin P \cap (G) \). Thus \( aG \subseteq D \), implies that \( raG \subseteq r(P \cap (G) + \langle x \rangle) \subseteq P \cap (G) \). It follows that \( ra \in [P \cap (G);_O G] \). But \( [P \cap (G);_O G] \) is a prime ideal of \( O \) and \( a \notin [P \cap (G);_O G] \) then \( re[P \cap (G);_O G] \). Hence \( P \) is an MP submodule for \( G \).

**Proposition 2.10**

Let \( G \) be an \( O - \)module, and \( P \) be a submodule of \( G \) with \( j(G) \subseteq P \). Then \( P \) is an MP submodule of \( G \) if and only if \( [P;_G S] \) is an MP submodule of \( G \), for every nonzero ideal \( S \) of \( O \).

**Proof**

\((\Rightarrow)\) Suppose that \( P \) is MP submodule of \( G \), and let \( rm \in [P;_G S] \), for \( r \in O, m \in G \), and \( S \) is an ideal of \( O \), then \( r(m) \subseteq P \). But \( P \) is MP submodule of \( G \), then by Corollary 2.4 either \( (m) \subseteq P \cap (G) \) or \( rG \subseteq P \cap (G) \). But \( rG \subseteq P \cap (G) \), implies that \( P \cap (G) \subseteq P \). Thus \( \langle m \rangle \subseteq P \cap (G) \), it follows that either \( m \notin [P;_G S] \) or \( rG \subseteq P \). That is either \( m \notin [P;_G S] \cap (G) \) or \( rG \subseteq P \subseteq [P;_G S] \cap (G) \). Hence \( [P;_G S] \) is MP submodule of \( G \).

\((\Leftarrow)\) Suppose \( [P;_G S] \) is MP submodule of \( G \), for every nonzero ideal \( S \) of \( O \), hence \( \cup S = O \), we \( [P;_G O] = P \) is an MP submodule of \( G \).

**Proposition 2.11**

Let \( O : G \rightarrow G \) be an \( O - \)epimorphism and \( KerU \) is small submodule of \( G \), and \( P \) be an MP submodule of \( G \). Then \( U^{-1}(P) \) is a MP submodule of \( G \).

**Proof**

\((\Rightarrow)\) Suppose that \( P \) is an MP submodule of \( G \), and let \( rm \in [P;_G S], \) for \( r \in O, m \in G \), and \( S \) is an ideal of \( O \), then \( r(m) \subseteq P \). But \( P \) is MP submodule of \( G \), then by Proposition 1.2, either \( (m) \subseteq P \cap (G) \) or \( rG \subseteq P \cap (G) \). But \( rG \subseteq P \), implies that \( P \cap (G) \subseteq P \). Thus \( \langle m \rangle \subseteq P \cap (G) \), it follows that either \( m \notin [P;_G S] \) or \( rG \subseteq P \). That is either \( m \notin [P;_G S] \cap (G) \) or \( rG \subseteq P \subseteq [P;_G S] \cap (G) \). Hence \( [P;_G S] \) is MP submodule of \( G \).

\((\Leftarrow)\) Suppose \( [P;_G S] \) is MP submodule of \( G \), for every nonzero ideal \( S \) of \( O \), hence \( \cup S = O \), we \( [P;_G O] = P \) is an MP submodule of \( G \).

**Proposition 2.12**

Let \( O : G \rightarrow G' \) be an \( O - \)epimorphism and \( KerU \) is small submodule of \( G \). If \( P \) is a MP submodule of \( G \) with \( KerU \subseteq P \). Then \( U(P) \) is a MP submodule of \( G' \).

**Proof**
$U(P)$ is proper submodule of $G'$, if not $U(P) = G'$, that is for each $x \in G, U(x) \in G' = U(P)$, it follows that $\exists b \in P$ s.t $U(b) = U(x)$, hence $U(b-x) = 0$, then $b - x \in Ker U \subseteq P$, hence $x \in P$, that is $G \subseteq P$, but $P \subseteq G$, it follows $P = G$ contradiction since $P$ is a proper submodule of $G$.

Let $sx' \in U(P)$, for $s \in O, x' \in G'$. Since $U$ is epimorphism there exist none zero $x \in G$ such that $U(x) = x'$, so $sx' = sx \in U(P)$, then there exist none zero $b' \in P$ s.t $U(b') = U(b)$, implies that $U(rx - b') = 0$, hence $rx - b' \in Ker U \subseteq P$, implies that $x \in P$. But $P$ is a MP submodule of $G$, then either $x \in P \cap (G)$ or $sG \subseteq P \cap (G)$, it follows that either $x' = U(x) \in U(P) \cap U((G))$ or $sU(G) \subseteq U(P) \cap U((G))$. Hence by Proposition 1.2 , we have either $x' \in U(P) \cap (G')$ or $sG' \subseteq U(P) \cap (G')$. That is $U(P)$ is a MP submodule of $G'$.

The following corollaries are a direct application of Proposition 1.2.

**Corollary 2.13**

Let $G$ be a hollow $O$-module and $U : G \rightarrow G'$ be a $O$-epimorphism, and $P$ is a MP submodule of $G$ with $Ker U \subseteq P$. Then $U(P)$ is a MP submodule of $G'$.

**Corollary 2.14**

Let $P$ be a submodule of a hollow $O$-module $G$ and $C$ be a submodule of $G$ with $C \subseteq P$. If $P$ is $O$-module of $G$, then $P/C$ is a $O$-module of $G$.

**Proof**

Follow from Corollary 2.13 by setting $y: G \rightarrow G/C$ be an epimorphism with $Ker y = C \subseteq P$.

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**3: Characterizations of Mine-Prime submodules in some types of module.**

We start this part by following characterization of Mine-Prime submodule in class of multiplication module.

**Proposition 3.1**

Let $G$ be a multiplication $O$-module, and $P$ is proper submodule of $G$. Then $P$ is MP submodule of $G$ if and only if whenever $LD \subseteq P$ for $L, D$ are submodules of $G$, implies that either $D \subseteq P \cap (G)$ or $L \subseteq P \cap (J(M))$.

**Proof**

($\Rightarrow$) Suppose that $P$ is $O$-module of $G$, and $LD \subseteq P$ for $L, D$ are submodules of $G$. Since $G$ is multiplication, then $L = SG, D = JG$ for some ideal $S, J$ of $O$. That $S(JG) \subseteq P$. Since $P$ is $O$-module of $G$, then by proposition 2.3 either $JG \subseteq P \cap (G)$ or $SG \subseteq P \cap (J(G))$. It follows either $D \subseteq P \cap (J(G))$ or $L \subseteq P \cap (J(G))$.

($\Leftarrow$) Suppose $SD \subseteq P$ for $D$ is a submodule of $G$, and $S$ is ideal of $O$. Since $G$ is a multiplication, then $D = QG$ for some ideal $Q$ of $O$. That $S(QG) \subseteq P$, take $C = SG$, so $CD \subseteq P$. By hypothesis, we have either $D \subseteq P \cap (G)$ or $C \subseteq P \cap (J(G))$. Thus either $D \subseteq P \cap (J(G))$ or $SG \subseteq P \cap (J(G))$. Hence by proposition 2.3 $P$ is $O$-module of $G$.

We gave the corollaries a direct application of Proposition 3.1.

**Corollary 3.2**

Let $G$ be multiplication $O$-module,. Then $P$ is MP submodule of $G$ if and only if where $x_1, x_2 \subseteq P$ for $x_1, x_2 \in G$, implies that either $x_1 \subseteq P \cap (G)$ or $x_2 \subseteq P \cap (J(G))$.

**Corollary 3.3**

Let $G$ be a multiplication $O$-module, and $x \in G$, implies that either $x \subseteq P \cap (G)$ or $x \subseteq P \cap (J(G))$.

**Corollary 3.4**

Let $G$ be a multiplication $O$-module,. Then $P$ is MP submodule of $G$ if and only if whenever $xD \subseteq P$ for $D$ is submodules of $G$ and $x \in G$, implies that either $x \subseteq P \cap (G)$ or $x \subseteq P \cap (J(G))$.

**Remark3.5**

The residuals of MP submodule of an $O$-module $G$ need n't to be MP ideal of $O$.

The following example shows that:

Consider the $O = Z, G = Z_4$, the submodule $P = \langle 2 \rangle$ of $Z_4$ is a MP submodule by part (1) of remarks and examples 2.2. But $[P:O,G] = \langle 2 \rangle$ is not MP ideal of $O$, since $2.2 \in \langle 2 \rangle$ for $2.2 \in O$, but $2 \notin \langle 2 \rangle \cap (J(O)) = \langle 2 \rangle \cap (O) = \langle 0 \rangle$ and $2 \notin [(\langle 2 \rangle \cap (O)):O] = [(0):O] = \langle 0 \rangle$. 

Proposition 3.6

Let $G$ be a projective multiplication $O$-module. Then the proper submodule $P$ is MP submodule of $G$ if and only if $[P_{:O}G]$ is MP ideal of $O$.

Proof

($\Rightarrow$) Let $S \subseteq [P_{:O}G]$ for $S$ and $Q$ are ideals of $O$, implies that $SQG \subseteq P$. Since $G$ is multiplication, then $SQG = LK$ by taking $L = SG$, $K = QG$ are submodules of $G$, hence $LK \subseteq P$. But $P$ is MP submodule of multiplication $O$-module $G$, then by proposition 3.1 either $L \subseteq P \cap J(G)$ or $K \subseteq P \cap J(G)$. Since $G$ is multiplication, then $P = [P_{:O}G]$ is projective then by Proposition 1.3 $J(G) = J(O)$. Thus either $SG \subseteq [P_{:O}G]G \cap J(O)G$ or $QG \subseteq [P_{:O}G]G \cap J(O)G$. It follows that either $S \subseteq [P_{:O}G] \cap J(G)$ or $Q \subseteq [P_{:O}G] \cap J(G) = [P_{:O}G] \cap J(O)$. By Proposition 2.3 $[P_{:O}G]$ is MP ideal of $O$.

($\Leftarrow$) Let $LK \subseteq P$ where $L$ and $K$ are submodules of $G$. Since $G$ is a multiplication, then $L = SG$ and $K = QG$ for some ideals $S$ and $Q$ of $O$, that is $SQG \subseteq P$, implies that $SQ \subseteq [P_{:O}G]$, but $[P_{:O}G]$ is MP ideal of $O$, then by proposition 2.3 either $Q \subseteq [P_{:O}G] \cap J(G)$ or $S \subseteq [P_{:O}G] \cap J(O) = [P_{:O}G] \cap J(O)$. Hence either $QG \subseteq [P_{:O}G]G \cap J(O)G$ or $SG \subseteq [P_{:O}G]G \cap J(O)G$. Since $G$ is projective then by Proposition 1.3 $J(G) = J(O)$. Either $QG \subseteq P \cap J(G)$ or $SG \subseteq P \cap J(G)$. That is either $K \subseteq P \cap J(G)$ or $L \subseteq P \cap J(G)$. By proposition 3.1 $P$ is MP submodule of $G$.

Proposition 3.7

A proper submodule $P$ of faithful multiplication $O$-module $G$ is MP submodule of $G$ if and only if $[P_{:O}G]$ is MP ideal of $O$.

Proof

($\Rightarrow$) Suppose that $P$ is MP submodule of $G$, and let $rS \in [P_{:O}G]$ for $r \in O$, and $S$ is an ideal of $O$, implies that $r(SG) \subseteq P$. But $P$ is MP submodule of $G$, then by Corollary 2.4 either $SG \subseteq P \cap J(G)$ or $rG \subseteq P \cap J(G)$. Since $G$ is multiplication, then $P = [P_{:O}G]$ is multiplication, and $G$ is faithful multiplication, then by Proposition 1.4 $J(G) = J(O)$. Thus either $SG \subseteq [P_{:O}G]G \cap J(O)G$ or $rG \subseteq [P_{:O}G]G \cap J(O)G$. It follows that either $S \subseteq [P_{:O}G] \cap J(O)$ or $r \in [P_{:O}G] \cap J(O) = [P_{:O}G] \cap J(O)$. Hence by Corollary 2.4 $[P_{:O}G]$ is MP ideal of $O$.

($\Leftarrow$) Let $mD \subseteq P$ for $m \in G$ and $D$ is submodule of $G$. Since $G$ is multiplication, then $m = Om = SG$ and $D = JG$ for some ideals $S, J$ of $O$, that is $SJG \subseteq P$, implies that $SJ \subseteq [P_{:O}G]$, but $[P_{:O}G]$ is MP ideal of $O$, then by proposition 2.3 either $J \subseteq [P_{:O}G] \cap J(G)$ or $S \subseteq [P_{:O}G] \cap J(O) = [P_{:O}G] \cap J(O)$. Hence either $JG \subseteq [P_{:O}G]G \cap J(O)G$ or $SG \subseteq [P_{:O}G]G \cap J(O)G$. Hence $G$ is faithful multiplication, then by Proposition 1.4 either $JG \subseteq P \cap J(G)$ or $SG \subseteq P \cap J(G)$. That is either $D \subseteq P \cap J(G)$ or $m \subseteq P \cap J(G)$. Thus by Corollary 3.4 $P$ is MP submodule of $G$.

Proposition 3.8

Let $G$ be content multiplication $O$-module, a proper submodule $P$ of $G$ is MP submodule of $G$ if and only if $[P_{:O}G]$ is MP ideal of $O$.

Proof

($\Rightarrow$) Suppose that $P$ is MP submodule of $G$, and let $Qa \subseteq [P_{:O}G]$ for $Q$ is ideal of $O$ and $a \in O$, so $Q(aG) \subseteq P$. But $P$ is MP submodule of $G$, then by proposition 2.3 either $aG \subseteq P \cap J(G)$ or $QG \subseteq P \cap J(G)$. Since $G$ is multiplication, then $P = [P_{:O}G]$ is multiplication, and $G$ is content $O$-module then by Proposition 1.5 $J(G) = J(O)$. Thus either $aG \subseteq [P_{:O}G]G \cap J(O)G$ or $QG \subseteq [P_{:O}G]G \cap J(O)G$. It follows that either $a \subseteq [P_{:O}G] \cap J(O)$ or $Q \subseteq [P_{:O}G] \cap J(O) = [P_{:O}G] \cap J(O)$. Hence by Corollary 2.4 $[P_{:O}G]$ is MP ideal of $O$.

($\Leftarrow$) Let $mD \subseteq P$ for $L$ is submodule of $G$ and $m \in G$. Since $G$ is multiplication, then $L = SG$ and $m = Om = JG$ for some ideals $S, J$ of $O$, that is $SJG \subseteq P$, implies that $SJ \subseteq [P_{:O}G]$, but $[P_{:O}G]$ is MP ideal of $O$, then by proposition 2.3 either $J \subseteq [P_{:O}G] \cap J(G)$ or $S \subseteq [P_{:O}G] \cap J(O) = [P_{:O}G] \cap J(O)$. Hence either $JG \subseteq [P_{:O}G]G \cap J(O)G$ or $SG \subseteq [P_{:O}G]G \cap J(O)G$. Hence $G$ is content module then by Proposition 1.5 either $JG \subseteq P \cap J(G)$ or $SG \subseteq P \cap J(G)$. That is either $m \subseteq P \cap J(G)$ or $L \subseteq P \cap J(G)$. Thus by Corollary 3.3 $P$ is MP submodule of $G$.

Proposition 3.9

Let $G$ be multiplication module over good ring $O$, and $P$ be a proper submodule of $G$. Then $P$ is MP submodule of $G$ if and only if $[P_{:O}G]$ is MP ideal of $O$.

Proof

($\Rightarrow$) Suppose $P$ is MP submodule of $G$, and let $rs \in [P_{:O}G]$ for $r, s \in O$, implies that $r(sG) \subseteq P$. But $P$ is MP submodule of $G$, by Corollary 2.4 either $sG \subseteq P \cap J(G)$ or $rG \subseteq P \cap J(G)$. Since $G$ is multiplication, then $P = [P_{:O}G]$ is multiplication, and $O$ is a good ring then $J(G) = J(O)$. Thus either $sG \subseteq [P_{:O}G]G \cap J(O)G$ or $rG \subseteq [P_{:O}G]G \cap J(O)G$. It follows that either $s \subseteq [P_{:O}G] \cap J(O)$ or $r \in [P_{:O}G] \cap J(O) = [P_{:O}G] \cap J(O)$. Hence $[P_{:O}G]$ is MP ideal of $O$.
Let $x_1x_2 \subseteq P$ for $x_1, x_2 \in G$. Since $G$ is a multiplication, then $x_1 = O_{x_1} = SG, x_2 = O_{x_2} = JG$ for some ideals $S, J$ of $O$, follows that is $S[J] \subseteq P$, implies that $S[J] \not\subseteq [P_{i_0} G]$, but $[P_{i_0} G]$ is MP ideal of $O$, then by proposition 2.3 either $J \subseteq [P_{i_0} G] \cap J(O)$ or $S \subseteq [P_{i_0} G] \cap J(O) \cap O = [P_{i_0} G] \cap J(O)$. Hence either $JG \subseteq [P_{i_0} G]G \cap J(O)G$ or $SG \subseteq [P_{i_0} G]G \cap J(O)G$. Hence $O$ is a good ring either $JG \subseteq P \cap J(G)$ or $SG \subseteq P \cap J(G)$. That is either $x_2 \subseteq P \cap J(G)$ or $x_1 \subseteq P \cap J(G)$. By Corollary 3.2 $P$ is MP submodule of $G$.

**Proposition 3.10**

Let $G$ be a multiplication finitely generated $O$-module, with $JG \not\subseteq G$, for all maximal ideal $J$ of $O$, and $P$ be a proper submodule of $G$. Then $P$ is MP submodule of $G$ if and only if $[P_{i_0} G]$ is MP ideal of $O$.

**Proof**

The same proof of Proposition 3.9, and using of Proposition 1.6 we can obtain the result.

**Proposition 3.11**

Let $G$ be a projective finitely generated multiplication $O$-module, and $B$ is an ideal of $O$ with $ann_O(G) \subseteq U$. Then $U$ is MP ideal of $O$ if and only if $UG$ is MP submodule of $G$.

**Proof**

($\Rightarrow$) Let $DK \subseteq UG$, for $D, K$ are submodules of $G$. Since $G$ is a multiplication, then $D = SG, K = JG$ for some ideals $S, J$ of $O$, that is $S[J] \not\subseteq UG$. Hence either $JG \subseteq [P_{i_0} G] \cap J(O)G$ or $SG \subseteq [P_{i_0} G] \cap J(O)G$. Since $G$ is a multiplication then by Proposition 3.9, it follows that either $K \subseteq UG \cap J(G)$ or $D \subseteq UG \cap J(G)$. By proposition 3.1 $UG$ is MP submodule of $G$.

($\Leftarrow$) Let $S[J] \subseteq U$, for $S$ and $J$ are ideals in $O$, implies that $S[JG] \subseteq UG$. But $UG$ is MP submodule of $G$, then by proposition 2.3 either $(JG) \subseteq UG \cap J(G)$ or $S \subseteq [UG \cap J(G)]_O$, that is either $JG \subseteq UG \cap J(G)$ or $SG \subseteq UG \cap J(G)$. But $G$ is a projective then Thus either $JG \subseteq UG \cap J(O)G$ or $SG \subseteq UG \cap J(O)G$, it follows that either $J \subseteq U \cap J(O)$ or $S \subseteq U \cap J(O) \cap O$. By proposition 2.3 $U$ is MP ideal of $O$.

**Proposition 3.12**

Let $G$ be multiplication finitely generated module over a good ring $O$, and $U$ is an ideal of $O$ with $ann_O(G) \subseteq U$. Then $U$ is MP ideal of $O$ if and only if $UG$ is MP submodule of $G$.

**Proof**

($\Rightarrow$) Let $x_1x_2 \subseteq UG$, for $x_1, x_2 \in G$. Since $G$ is a multiplication, then $x_1 = O_{x_1} = SG, x_2 = O_{x_2} = JG$ for some ideals $S, J$ of $O$, that is $S[J] \not\subseteq UG$. But $G$ is multiplication finitely generated $O$-module then by Proposition 1.7 $S[J] \subseteq U + ann_O(G)$, since $ann_G(G) \subseteq U$, implies that $U + ann_O(G) = U$ implies that $S[J] \subseteq U$. But $U$ is MP ideal of $O$ then by Proposition 2.3 either $JG \subseteq UG \cap J(O)G$ or $SG \subseteq UG \cap J(O)G$. Since $G$ is a good ring then $JG \subseteq J(O)G$. Hence either $JG \subseteq UG \cap J(G)$ or $SG \subseteq UG \cap J(G)$. That is either $x_2 \subseteq UG \cap J(G)$ or $x_1 \subseteq UG \cap J(G)$. Therefore by Corollary 3.2 $UG$ is MP submodule of $G$.

($\Leftarrow$) Let $rS \subseteq U$, for $r \in O$, and $S$ is an ideal of $O$, implies that $r(SG) \subseteq UG$. Since $UG$ is MP submodule of $G$, then by Corollary 2.4 either $(SG) \subseteq UG \cap J(O)G$ or $r \subseteq [UG \cap J(G)]_O$, that is either $SG \subseteq UG \cap J(G)$ or $rG \subseteq UG \cap J(G)$. But $O$ is a good ring then $J(O)G = J(G)$. Hence either $JG \subseteq UG \cap J(G)$ or $SG \subseteq UG \cap J(G)$. That is either $S \subseteq B \cap J(O)$ or $r \in B \cap J(O) \subseteq [B \cap J(O)]_O$. Hence by Corollary 2.4 $B$ is MP ideal of $O$.

**Proposition 3.13**

Let $G$ be a multiplication finitely generated $O$-module with $JG \not\subseteq G$ for all maximal ideal $J$ of $O$, and $U$ is an ideal of $O$ with $ann_O(G) \subseteq U$. Then $U$ is MP ideal of $O$ if and only if $UG$ is MP submodule of $G$.

**Proof**

($\Rightarrow$) Let $aD \subseteq UG$, for $a \in O$, and $D$ is submodule of $G$. Since $G$ is multiplication, then $D = SG$ for some ideal $S$ of $O$, that is $SG \subseteq UG$. Since $G$ is a multiplication finitely generated $O$-module then by Proposition 1.7 $aS \subseteq U + ann_O(G)$, since $ann_O(G) \subseteq U$, implies that $U + ann_O(G) = U$ and hence $aS \subseteq U$. By hypothesis $U$ is MP ideal of $O$ by Corollary 2.4 either $S \subseteq U \cap J(O)$ or $a \in [U \cap J(O)]_O = U \cap J(O)$. Thus either $SG \subseteq UG \cap J(O)G$ or $aG \subseteq UG \cap J(O)G$. Since $G$ is a multiplication finitely generated module with $JG \not\subseteq G$ for all maximal ideal $J$ of $O$ then by Proposition 1.6 $J(G) = J(O)G$. Hence either $SG \subseteq UG \cap J(G)$ or $aG \subseteq UG \cap J(G)$. That is either $D \subseteq UG \cap J(G)$ or $a \in [UG \cap J(G)]_O$. Therefore by Corollary 2.4 $UG$ is MP submodule of $G$. 


Let $a \in U$, for $a \in O$, implies that $o(aG) \subseteq UG$. Since $UG$ is MP submodule of $G$, then by Corollary 2.4 either $aG \subseteq UG \cap J(G)$ or $a \in [UG \cap J(G);o]G$. That is either $aG \subseteq UG \cap J(G)$ or $aG \subseteq UG \cap J(G)$. By Proposition 1.6 $J(G) = J(O)G$. Hence either $aG \subseteq UG \cap J(O)G$ or $rG \subseteq UG \cap J(O)G$, it follows either $a \in U \cap J(O)$ or $r \in U \cap J(O) \subseteq [U \cap J(O);o]0$. Therefore $U$ is MP ideal of $O$.

**Proposition 3.14**

Let $G$ be multiplication finitely generated content module and $U$ is ideal of $O$ with $ann_O(G) \subseteq U$. Then $U$ is MP ideal of $O$ if and only if $UG$ is MP submodule of $G$.

**Proof**

$(\Rightarrow)$ Let $x_1, x_2 \subseteq UG$, for $x_1, x_2 \in G$. Since $G$ is a multiplication, then $x_1 = Ox_1 = SG, x_2 = Ox_2 = JG$ for some ideals $S, J$ of $O$, that is $SG \subseteq UG$. But $G$ is multiplication finitely generated $O$-module by Proposition 1.7 $SJ \subseteq U + ann_O(G)$, since $ann_O(G) \subseteq U$, implies that $U + ann_O(G) = U$ and hence $SJ \subseteq U$. But $U$ is MP ideal of $O$ by Proposition 2.3 either $J \subseteq U \cap J(O)$ or $S \subseteq [U \cap J(O);o]0 = U \cap J(O)$. Thus either $JG \subseteq UG \cap J(O)G$ or $SG \subseteq UG \cap J(O)G$. Since $G$ be a content module then by Proposition 1.5 $J(O)G = J(G)$. Hence either $JG \subseteq UG \cap J(G)G$ or $SG \subseteq UG \cap J(G)G$. That is either $x_1 \subseteq UG \cap J(G)G$ or $x_2 \subseteq UG \cap J(G)G$. Therefore by Corollary 3.2 $UG$ is MP submodule of $G$.

$(\Leftarrow)$ Let $s \subseteq U$, for $a \in O$, and $S$ is an ideal of $O$, implies that $a(SG) \subseteq UG$. Since $UG$ is MP submodule of $G$, then by Corollary 2.4 either $(SG) \subseteq UG \cap J(G)G$ or $a \in [UG \cap J(G);o]G$. That is either $SG \subseteq UG \cap J(G)$ or $aG \subseteq UG \cap J(G)$. But $G$ be a content module then by Proposition 1.5. Thus either $SG \subseteq UG \cap J(O)G$ or $aG \subseteq UG \cap J(O)G$, it follows that either $S \subseteq U \cap J(O)$ or $a \subseteq U \cap J(O) \subseteq [U \cap J(O);o]0$. Hence by Corollary 2.4 $U$ is MP ideal of $O$.

**Proposition 3.15**

Let $G$ be a faithful finitely generated multiplication $O$-module. Then $U$ is MP ideal of $O$ if and only if $UG$ is MP submodule of $G$.

**Proof**

$(\Rightarrow)$ Let $x_1, x_2 \subseteq UG$, for $x_1, x_2 \in G$. Since $G$ is a multiplication, then $x_1 = Ox_1 = SG, x_2 = Ox_2 = JG$ for some ideals $S, J$ of $O$, that is $SG \subseteq UG$. But $G$ is finitely generated multiplication $O$-module then by Proposition 1.7 $SJ \subseteq U + ann_O(G)$, since $G$ is faithful then $ann_O(G) = (0)$, implies that $SJ \subseteq U$. But $U$ is MP ideal of $O$ by Proposition 2.3 either $J \subseteq U \cap J(O)$ or $S \subseteq [U \cap J(O);o]0 = U \cap J(O)$. Thus either $JG \subseteq UG \cap J(O)G$ or $SG \subseteq UG \cap J(O)G$. But $G$ is faithful multiplication, by Proposition 1.4 $J(G) = J(O)G$. Hence either $JG \subseteq UG \cap J(G)$ or $SG \subseteq UG \cap J(G)$. That is either $x_1 \subseteq UG \cap J(G)G$ or $x_2 \subseteq UG \cap J(G)G$. Therefore by Corollary 3.2 $UG$ is MP submodule of $G$.

$(\Leftarrow)$ Let $ra \subseteq U$, for $r, a \in O$, implies that $r(aG) \subseteq UG$. But $UG$ is MP submodule of $G$, then by Corollary 2.4 either $aG \subseteq UG \cap J(G)$ or $r \in [UG \cap J(G);o]G$. That is either $aG \subseteq UG \cap J(G)$ or $rG \subseteq UG \cap J(G)$. Hence $G$ is faithful multiplication $O$-module by Proposition 1.4 either $aG \subseteq UG \cap J(O)G$ or $rG \subseteq UG \cap J(O)G$, it follows that either $a \subseteq U \cap J(O)$ or $r \subseteq U \cap J(O) \subseteq [U \cap J(O);o]0$. Therefore $U$ is MP ideal of $O$.

**Proposition 3.16**

Let $G$ be finitely generated projective multiplication $O$-module and $P$ be a proper submodule of $G$ then the statements that follow are equivalent:

1. $P$ is MP submodule of $G$.
2. $P_{\cdot o}G$ is MP ideal of $O$.
3. $P = UG$ for some MP ideal $U$ of $O$ with $ann_O(G) \subseteq U$.

**Proof**

It follows by Propositions [ Proposition 3.6 and Proposition 3.11 ].

**Proposition 3.17**

Let $G$ be content finitely generated multiplication $O$-module, and $P$ be a proper submodule of $G$ then the statement that follow are equivalent:

1. $P$ is MP submodule of $V$.
2. $[P_{\cdot o}V]$ is MP ideal of $O$.
3. $P = UG$ for some MP ideal $U$ of $O$ with $ann_O(G) \subseteq U$.

**Proof**

It follows by Propositions [ Proposition 3.8 and Proposition 3.14 ].

**Proposition 3.18**
Let $G$ be faithful multiplication finitely generated $O$-module, and $P$ be a proper submodule of $G$, then the statements that follow are equivalent:

1. $P$ is MP submodule of $G$.
2. $[P_{O}G]$ is MP ideal of $O$.

**Proof**

It follows by Propositions [Proposition 3.7 and Proposition 3.15]

**Proposition 3.19**

Let $G$ be finitely generated multiplication module over good ring $O$, and $P$ be a proper submodule of $G$. Then the statements that follow are equivalent:

1. $P$ is MP submodule of $G$.
2. $[P_{O}G]$ is MP ideal of $O$.
3. $P = UG$ for some MP ideal $U$ of $O$ with $ann_{O}(G) \subseteq U$.

**Proof**

It follows by Propositions [Proposition 3.9 and Proposition 3.12].

**Proposition 3.20**

Let $G$ be finitely generated multiplication module with $SG \neq G$ for all maximal ideal $S$ of $O$, and $P$ be a proper submodule of $G$. Then the statements that follow are equivalent:

1. $P$ is MP submodule of $G$.
2. $[P_{O}G]$ is MP ideal of $O$.
3. $P = UG$ for some MP ideal $U$ of $O$ with $ann_{O}(G) \subseteq U$.

**Proof**

It follows by Propositions [Proposition 3.10 and Proposition 3.13].

**References**