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Mine-Prime Submodules

Ali Sabah Sadip[a], Haibat Karim Mohammadali [b]

a- *Republic of Iraq, Ministry of Education, Directorate General of Education in Diyala, alisabahsadickali@gmail.com*

b- *Department of Mathematics / College of Computer Science and Mathematics / Tikrit University / Iraq, dr.mohammadali2013@gmail.com.*

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ABSTRACT

Let O be commutative rings with identity, and all modules are (left) unitary $O-$module. A proper submodule P of an $O-$module G is called prime submodule, if for any $rm\epsilon P$, for $r\epsilon O$, $m\epsilon G$, imples that either $m\epsilon P$ or $r\mathrm{G} \subseteq P$.As strong from of prime sub modules we introduce in that paper the concept of Mine-Prime submodules and gave same basic properties , example and characterizations of this concept. Moreover we study be haver of Mine-Prime submodules in class of of multiplication modules, furthermore we prove that by examples the residual of Mine-Prime submodules not to be Mine-Prime ideal of O so we gave under sertion conditions several characterizations of Mine-Prime submodules

MSC..

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Introduction

Famous concept to start with in this paper was prime submodule this concept was first introduce by Dauns [1]. Many research interesting generalized prime submodules such as (semiprime , quasi prime) submodules see [2, 3].

I recent time this concept was generalized by (nearly prime , nearly semiprime , nearly quasi prime) submodules by see [4, 5 ,6].

As strong from prime submodule the introduce the concept of (restrict nearly semiprime, restrict nearly prime) submodules see [7,8]

In this paper we introduce new strong from of prime submodule wich we called Mine-Prime submodule we study this concept extensively.

∗Corresponding author: Ali Sabah Sadiq

Email addresses: alisabahsadickali@gmail.com

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This concept consist three parts , part one deal with reeling well-know definition , propositions that we need in the sequel. Part two corned with introduce the definition of Mine-Prime submodules and gave several importation ,characterization , basic proposition and example with of this concept.

Finally part three devoted to gave many characterization of Mine-Prime submodule in some types of module such as (multiplication , projective, faithful, content, ...) modules

1. Basic Concepts and Prelminaees

This part deal with reeling well-know definition , propositions that we need in the sequel.

"Recall that a proper ideal S of a ring O is called a prime ideal, if whenever $xy \in S$, for $x, y \in O$ implies that either $x \in S$ or $y \in S$ [9]".

" A proper submodule P of an O-module G is called prime submodule if $ox \in P$ for $o \in O$, $x \in G$ implies that either $x \in P$ or $o \in [P:_O G]$ [1]".

"Recall that the residual of a submodule P of an O-module G denoted by $[P:_O G]$ is an ideal of O defined by $[P:_O G] = \{o \in O: oG \subseteq P\}$[10]".

" A submodule P of an O-module G is called maximal submodule if $P \subsetneq D \subseteq G$, then $D = G$ [11]".

" the Jacobson radical of O-module G denoted by $J(G)$ is the intersection of all maximal submodule of G [11]".

" A proper submodule P of an O-module G is called nearly prime submodule, if whenever $ox \epsilon P$ for $o \epsilon O$, $x \epsilon G$ implies that either $x \epsilon P + J(G)$ or $o \epsilon [P+J(G):_O G]$ [5]".

"Recall that a subset S of a ring O is multipilcatively closed subset of O if $1 \epsilon S$ and $xy \epsilon S$ for every $x, y \epsilon S$. And if P is a submodule of an O-module G and S is multipilcatively closed subset of O, then $P_S = \{m \in G: \exists t \in S$ such that $tm \in P\}$ is a submodule of G and $P \subseteq P_S$ [12]".

Proposition1.1 [9, Th. (5.1)]

"Let S be a proper ideal of a ring O. Then S is maximal ideal $if\ and\ only\ if\ S + \langle a \rangle = O\ for\ any\ a \notin S$ ".

"Recall that a submodule P of an O-module G is called small if $P + D = G$ implies that $D = G$ for any proper submodule D of G [13]".

Proposition 1.2[14, Coro. (9.1.5)(a)]

"If $\mho: G \to G'$ be O-epimorphism and $Ker\mho$ is a small submodule of G, then $\mho\big(J(G)\big) = J(G')$ and $\mho^{-1}\big(J(G')\big) = J(G)$".

"We say a non-zero O-module G is called hollow if every proper submodule of G is small [13]".

"Recall that a submodule $[P:_G S] = \{x \in G: xS \subseteq P\}$, where S is an ideal of O and P is a submodule of G such that $P \subseteq [P:_G S]$ and $[P:_G O] = P$, $[S:O] = S$." [15, p.16]

"Recall that G is an multipilcation O-module G is, if every submodule P of G is of the form $P = SG$ for some ideal S of O, G is a multiplication O-module if $P = [\ P:_O G]\ G$ [16]".

"Recall that for any submodule P and D of a multiplication O-module G with $P = SG$ and $D = JG$ for some ideals S and J of O. The product $PD = SG.JG = SJG$, that is $PD = SD$. In particular $PG = SGG = SG = P$. Also for any $x \in G$ we have $P = Sx$ and $x = Ox$ as a submodule of G [17]".

"Recall that O-module G is projective if every O-epimorphism f from O-module G' into O-module G" and for any O-homomorphism g from O-module G into O- module G" there exists an O-homomorphism h from O-module G into O-module G' such that $f \circ h = g$ [14]".

Proposition1.3 [11, Pr. (17.10)]

"If G is a projective O-module, then $J(O)G = J(G)$".

"Recall that G is an faithful O-module G if $ann_O(G) = (0)$, where $ann_O(G) = \{r \in O: rG = (0)\}$, and $[0:_O G] = ann_O(G)$ [14]".

Proposition 1.4 [18, Re. p14]

"If G is multiplication faithful O-module, then $J(O)G = J(G)$."

"Recall that an O-module G is called content module if $(\bigcap_{i\in I} S_i)G = \bigcap_{i\in I} S_i\, G$ for each family of ideals S_i in O [19]".

Proposition1.5[18, Pr. (1.11)]

"If G is content module, then $J(O)G = J(G)$."

"Recall that a ring O is called a good ring if $J(G) = J(O)G$ for any O-module G [14]."

"Recall that an O-module G is finitely generated if $G = Ov_1 + Ov_2 + \cdots + Ov_n$ where $v_1, v_2,\ldots, v_n \in G$ [14]".

Proposition 1.6 [19, Co. (5)]

"Let G be multipilcation finitely generated O-module with $SG \neq G$ for all maximal ideal S of O, then $J(G) = J(O)G$."

Proposition1.7 [20, Co. of Th. (9)]

"Let G be generated multipilcation finitely O-module and S, J are ideals of O. Then $SG \subseteq JG$ if and only if $S \subseteq J + ann_O(G)$."

2.Basic Properties Mine-Prime

In this part of this research we intraduce the definition of Mine-Prime submodule ,and we give some properties , characterizations of this concept.

Definition 2.1

A proper submdule P for an $O-$ module G is called Mine-Prime (for short MP) submdule, if for any $rm\epsilon P$, for $r\epsilon O$, $m\epsilon G$, implies that either $m\epsilon P \cap J(G)$ or $rG \subseteq P \cap J(G)$.

And we called an idael S of a ring O is MP idael of if S is MP O-submdule for an $O-$ module O.

Remarks, and examples 2.2

1. Let $O = Z$, $G = Z_4$, the submdule $P = \langle\bar{2}\rangle$ is MP submdule of Z_4. Thus for each $s\epsilon Z, m\epsilon Z_4$, if $sm\epsilon P$, impales that either $m\epsilon P \cap J(G) = \langle\bar{2}\rangle \cap \langle\bar{2}\rangle = \langle\bar{2}\rangle$ or $s\epsilon[P \cap J(Z_4):_Z Z_4] = [\langle\bar{2}\rangle:_Z Z_4] = 2Z$.

2. Every MP submdule for an $O-$ module V is Prime submodul for G, but the opposite is not true

Proof: it is clear that every MP submdule for an $O-$ module G is prime submdule

For the converse consider the example:

Let $G = Z_{12}$, $O = Z$, the submdule $P = \langle\bar{2}\rangle$ it is clearly that P is prime submodule . But P isnot MP submdule of Z_{12} , since $2.\bar{2}\epsilon P$, for $2\epsilon Z$, $\bar{2}\epsilon Z_{12}$ but $\bar{2} \notin P \cap J(G) = \langle\bar{6}\rangle$ and $2Z_{12} \nsubseteq \langle\bar{6}\rangle$

3- Every MP submdule for $O-$ module G is nearly prime submdule for G, but contrariwise isn't true.

Proof: it is clear that every MP submdule for an O −module G is nearly prime submdule

For the converse consider the example

Let $G = Z_{12}$, $O = Z$, and the submdule $P = \langle\bar{2}\rangle$ it's nearly prime submdule of Z_{12}. Thus for each $s\epsilon Z, x\epsilon Z_{12}$, if $sx\epsilon P$, impales that either $x\epsilon P + J(G) = \langle\bar{2}\rangle + \langle\bar{6}\rangle = \langle\bar{2}\rangle$ or $s\epsilon[P + J(G):_Z G] = [\langle\bar{2}\rangle:_Z G] = 2Z$. But P isnot MP submdule of G (by Remarks and examples 2.2 (2))

4. If P and D are proper submdules of an O-module G with $D \subsetneq P$, and P is MP submdule for G, then$D$ is not to is MP submdule for G. The follawing example explain that:

Let $G = Z_4$, $O = Z$, the submdule $P = \langle\bar{2}\rangle$ is MP submdule of G(by Remarks and examples 2.2 (1)) and $D = \langle\bar{0}\rangle$ is submdule of G such that $D \subsetneq P$, but D is not MP submdule of G , since $2.\bar{2}\epsilon D$, for $2\epsilon Z, \bar{2}\epsilon Z_4$ but $\bar{2} \notin D \cap J(G) = \langle\bar{0}\rangle$ and $2Z_4 \not\subseteq \langle\bar{0}\rangle$.

Proposition 2.3

Let G is an O − module, and P is submdule for G. Then P is MP submdule for M ,*if and* only if for any submdule $D$for G and any idael S of O with $SD\subseteq P$, implies that either $D \subseteq P \cap J(G)$ or $S\subseteq [P \cap J(G):_O G]$.

Proof

(⇒) Suppose $SD\subseteq P$, for D is submdule for G and S is an idael of O, with $D \not\subseteq P \cap J(G)$, then $\exists\, x\epsilon D$ and $x \notin P \cap J(G)$. Since $SD \subseteq P$ then for any $a\epsilon S$, $ax\epsilon P$. But P is MP submdule for G and $x \notin P \cap J(G)$ then $a\epsilon[P \cap J(G):_O G]$ hence $S\subseteq [P \cap J(G):_O G]$.

(⇐) Suppose $rx\epsilon P$, for $r\epsilon O, x\epsilon G$, then $(r)(x) \subseteq P$, so by hypothesis either $(x) \subseteq P \cap J(G)$ or $(r) \subseteq [P \cap J(G):_O G]$. That is either $x\epsilon P \cap J(G)$ or $r\epsilon[P \cap J(G):_O G]$. Hence P is MP submdule for G.

As direct application of Proposition 2.3 we gave the following corollaries.

Corollary 2.4

Let G is O − module ,and P is submdule for G. Then P is MP submdule for G *if and only* if for any submdule D for G and any $s\epsilon O$ with $sD \subseteq P$, implies that either $D \subseteq P \cap J(G)$ or $s\epsilon[P \cap J(G):_O G]$.

Corollary2.5

Let G is O − module ,and P is submdule for M. Then P is MP submdule for G if and only if for any $s\epsilon O$ with $sG \subseteq P$, implies that either $G \subseteq P \cap J(G)$ or $s\epsilon[P \cap J(G):_O G]$.

Corollary 2.6

Let G is O − module ,and P is submdule for G. Then P *is MP* submdule for G i*f and only* if for any ideal S of O , $m \in G$ with $Sm \subseteq P$, implies that either $m \in P \cap J(G)$ or $S \subseteq [P \cap J(G):_O G]$.

Proposition 2.7

Let P is a proper submdule for an O − module G, and $[P \cap J(G):_O G]$ is a prime idael of O. Then P is MP submdule for G *if and only if* $P(S) \subseteq P \cap J(G)$ for each multiplicatively closed subset S of O such that $S \cap [P \cap J(G):_O G] = \phi$.

Proof

(⇒) Suppose P is MP submdule for G, and let $x\epsilon P(S)$, then there exists $s\epsilon S$ such that $sx\epsilon P$. But P is MP submdule for G, so either $x\epsilon P \cap J(G)$ or $s\epsilon[P \cap J(G):_O G]$. But if $s\epsilon[P \cap J(G):_O G]$, imples that $s\epsilon S \cap [P \cap J(G):_O G] = \phi$, which is a contradiction. Thus $x\epsilon P \cap J(G)$ and hence $P(S) \subseteq P \cap J(G)$.

(⇐) Suppose $rx\epsilon P$, for $r\epsilon O$, $x\epsilon G$, such that $x \notin P\cap J(G)$ and $r \notin [P\cap J(G):_O G]$. But S is a multiplicatively closed subset for O, then $S = \{1, r, r^2, r^3, \dots\}$, and since $[P\cap J(G):_O G]$ is a prime idael of O, then $S \cap [P\cap J(G):_O G] = \phi$. But $x \notin P\cap J(G)$, imples that $x \notin P(S)$ and hence $rx \notin P$ which is a contradiction. Thus, either $x\epsilon P\cap J(G)$ or $r\epsilon[P\cap J(G):_O G]$, therefore P Is MP submdule for G.

Proposition 2.8

Let G is $O-$ module ,and P is submdule for G with $[P\cap J(G):_O G]$ is a maximal ideal for O. Then P is MP submdule for G.

Proof

let $sx\epsilon P$, for $s\epsilon O$, $x\epsilon G$, with $s \notin [P\cap J(G):_O G]$. Since $[P\cap J(G):_O G]$ is maximal ideal of O, by Proposition 1.1 $O = \langle s\rangle + [P\cap J(G):_O G]$, wherever $\langle s\rangle$ is ideal of O generated by s, we obtain $\exists\ a\epsilon O$ and $b\epsilon[P\cap J(G):_O G]$ such that $1 = as + b$, hence $x = asx + bx\epsilon P\cap J(G)$. Hence P is MP submdule for G.

Proposition 2.9

Let G is an $O-$module, and P is a proper submdule for G, with $[D:_O G] \nsubseteq [P\cap J(G):_O G]$, and $P\cap J(G)$ is a proper submdule of D for each submdule D for M such that $[P\cap J(G):_O G]$ is a prime ideal of O. Then P is MP submdule for G.

Proof

Suppose $rx\epsilon P$, for $r\epsilon O$, $x\epsilon G$, and $x \notin P\cap J(G)$. Then $P\cap J(G) \subsetneq P\cap J(G) + \langle x\rangle = D$ and so $[D:_O G] \nsubseteq [P\cap J(G):_O G]$, then there exists $a\epsilon[D:_O G]$ and $a \notin [P\cap J(G):_O G]$. That is $aG \subseteq D$ and $aG \nsubseteq P\cap J(G)$. Thus $aG \subseteq D$, imples that $raG \subseteq r(P\cap J(G) + \langle x\rangle) \subseteq P\cap J(G)$. It follows that $ra\epsilon[P\cap J(G):_O G]$. But $[P\cap J(G):_O G]$ is a prime idael of O, and $a \notin [P\cap J(G):_O G]$ then $r\epsilon[P\cap J(G):_O G]$. Hence P is MP submdule for G.

Proposition 2.10

Let G be an O-module, and P be a submodule of G with $J(G) \subseteq P$. Then P is an MP submodule of G if and only if $[P:_G S]$ is MP submodule of G, for every nonzero ideal , S of O.

Proof

(⇒) Suppose that P is MP submodule of G, and let $rm \in [P:_G S]$, for $r \in O, m \in G$, and S is an ideal of O, then $r(mS) \subseteq P$. But P is MP submodule of G, then by Corollary 2.4 either $(mS) \subseteq P\cap J(G)$ or $rG \subseteq P\cap J(G))$, but $J(G) \subseteq P$, implies that $P\cap J(G) \subseteq P$. Thus either $(mS) \subseteq P$ or $rG \subseteq P$, it follows that either $m \in [P:_G S]$ or $rG \subseteq P \subseteq [P:_G S]$. That is either $m \in [P:_G S]\cap J(G)$ or $rG \subseteq P \subseteq [P:_G S]\cap J(G)$. Hence $[P:_G S]$ is MP submodule of G.

(⇐) Suppose $[P:_G S]$ *is* MP submodule of G, for every nonzero ideal S of O, hence ,put $S = O$, we $[P:_G O] = P$ is MP submodule of G.

Proposition2.11

Let $\mho: G \to G'$be $O-$ *epimorphism* and $Ker\mho$ is small submodule of G, and P be MP submodule of G'. Then $\mho^{-1}(P)$ is a MP submodule of G.

Proof

Let $rx \in \mho^{-1}(P)$ for $r \in O, x \in G$ with $x \notin \mho^{-1}(P)\cap J(G)$, it follows $\mho(x) \notin P\cap \mho(J(G)) = P\cap J(G')$ by Proposition 1.2 . Now, since $rx \in \mho^{-1}(P)$, implies that $r\,\mho(x) \in P$. But P is a MP submodule of $G'$and $\mho(x) \notin P\cap J(G')$, it follows $r \in [P\cap J(G'):_O G']$, that is $r\,G' \subseteq P\cap J(G')$, hence $r\,\mho(G) = \mho(r\,G) \subseteq P\cap J(G')$. Implies that $rG \subseteq \mho^{-1}(P)\cap J(G)$ by Proposition 1.2. Thus $\mho^{-1}(P)$ is a MP submodule G.

Proposition 2.12

Let $\mho : G \longrightarrow G'$ be an O-epimorphism and $Ker\mho$ small submodule of G. If P is a MP submodule of G with $Ker\mho \subseteq P$. Then $\mho(P)$ is a MP *sub*module of G'.

Proof

$\mho(P)$ is proper submodule of G', if not $\mho(P) = G'$, that is for each $x \in G, \mho(x) \in G' = \mho(P)$, it $follows\ that$ $\exists b \in P$ s.t $\mho(b) = \mho(x)$, hence $\mho(b - x) = 0$, then $b - x \in Ker\mho \subseteq P$, hence $x \in P$, that is $G \subseteq P$, but $P \subseteq G$, it follows $P = G$ contradiction since P is a $proper\ submodule$ of G.
Let $sx' \in \mho(P)$, for $s \in O$, $x' \in G'$. Since $\mho$ is epimorphism there exist none zero $x \in G$ such that $\mho(x) = x'$, so $sx' = s\mho(x) = \mho(sx) \in \mho(P)$,then there exist none zero $b' \in P$ s.t $\mho(sx) = \mho(b')$, implies that $\mho(rx - b') = 0$, hence $rx - b' \in Ker\ \mho \subseteq P$, implies that $rx \in P$. but P is a MP submodule of G, then either $x \in P \cap J(G)$ or $sG \subseteq P \cap J(G)$, it follows that either $x' = \mho(x) \in \mho(P) \cap \mho(J(G))$ or$s\mho(G) \subseteq \mho(P) \cap \mho(J(G))$. Hence by Proposition 1.2 . we have either $x' \in \mho(P) \cap J(G')$ or $sG' \subseteq \mho(P) \cap J(G')$. That is $\mho(P)$ is a MP submodule of G'.
The following corollaries a direct application of Proposition 2.12 .

Corollary 2.13

Let G be hollow O-module and $\mho : G \longrightarrow G'$ be a O-epimorphism , and P is a MP submodule of G with $Ker\mho \subseteq P$. Then $\mho(P)$ is MP submodule of G'.

Corollary2.14

Let P be a submodule of $a\ hollow$ O-module G and C be a submodule of G with $C \subseteq P$. If P is MP submodule of G, then $\frac{P}{C}$ is MP submodule of $\frac{G}{C}$.

Proof

Follow from Corollary 2.13 by setting $\gamma: G \to \frac{G}{C}$ be an epimorphism with $Ker\ \gamma = C \subseteq P$.

3: Characterizations of Mine-Prime submodules in some types of module.

We start this part by following characterization of Mine-Prime submodule in class of multipilcation module.

Proposition 3.1

Let G be a multipilcation O-module, and P is proper submodule of G. Then P is MP submodule of G $if\ and\ only\ if$ whenever $LD \subseteq P$ for L, D are submodules of G, implies that either $D \subseteq P \cap J(G)$ or $L \subseteq P \cap \boldsymbol{J}(\boldsymbol{M})$

Proof

($\Rightarrow$) Suppose that P is MP submodule of G, and $LD \subseteq P$ for L, D are submodules of G. Since G is multipilcation , then $L = SG$, $D = JG$ for some ideal S, J of O. That $S(JG) \subseteq P$. Since P is MP submodule of G, then by proposition 2.3) either $JG \subseteq P \cap J(G)$ or $SG \subseteq P \cap J(G)$. It follows either $D \subseteq P \cap J(G)$ or $L \subseteq P \cap J(G)$.

($\Leftarrow$) Suppose $SD \subseteq P$ for D is a submodule of G, and S is ideal of O. Since G is a multiplication , then $D = QG$ for some ideal Q of O, that is $S(QG) \subseteq P$,take $C = SG$, so $CD \subseteq P$. By hypothesis, we have either $D \subseteq P \cap J(G)$ or $C \subseteq P \cap J(G)$.Thus either $D \subseteq P \cap J(G)$ or $SG \subseteq P \cap J(G)$. Hence by proposition 2.3 P is MP submodule of G.

We gave the corollaries a direct application of Proposition 3.1 .

Corollary 3.2

Let G be multiplication O-module,. Then P is MP submodule of G $if\ and$ only if where $x_1x_2 \subseteq P$ for $x_1, x_2 \in G$, implies that $either$ $x_1 \subseteq P \cap J(G)$ or $x_2 \subseteq P \cap J(G)$

Corollary 3.3

Let G be a multipilcation O-module, Then P is MP submodule of G $if\ and\ only\ if$ whenever $Dx \subseteq P$ for D is submodules of G and $x \in G$, implies that either $x \subseteq P \cap J(G)$ or $D \subseteq P \cap J(G)$

Corollary 3.4

Let G be a multipilcation O-module,, Then P is MP submodule of G if and only if whenever $xD \subseteq P$ for $x \in G$ and D is a submodules of G, implies that either $D \subseteq P \cap J(G)$ or $x \subseteq P \cap J(G)$.

Remark3.5

The residuals of MP submodule of an O-module G need n't to be MP $ideal$ of O.
Thefollowing example shows that:
Consider the$O = Z$ $G = Z_4$, the submodule $P = \langle \bar{2} \rangle$ of Z_4 is a MP submodule by part (1)of remarks and examples 2.2 . But $[P:_O G] = \langle 2 \rangle$ is not MP ideal of O, since $2.2 \in \langle 2 \rangle$ for $2, 2 \in O$, but $2 \notin \langle 2 \rangle \cap J(O) = \langle 2 \rangle \cap \langle 0 \rangle = \langle 0 \rangle$ and $2 \notin [\langle 2 \rangle \cap J(O):_O O] = [\langle 0 \rangle:_O O] = \langle 0 \rangle$.

Proposition 3.6

Let G is a projective multiplication O-module G the proper submodule P is MP submodule of G if *and only if* $[P:_O G]$ is MP ideal of O.

Proof

(⇒) Let $SJ \subseteq [P:_O G]$ for S and Q are ideals of O, implies that $SQG \subseteq P$. Since G is multiplication, then $SQG = LK$ by taking $L = SG, K = QG$ are submodules of G, hence $LK \subseteq P$. But P is MP submodule of multiplication O-module G, then by proposition 3.1 either $L \subseteq P \cap J(G)$ or $K \subseteq P \cap J(G)$. since G is multiplication, then $P = [P:_O G]G$, since G is projective then by Proposition 1.3 $J(G) = J(O)G$ Thus either $SG \subseteq [P:_O G]G \cap J(O)G$ or $QG \subseteq [P:_O G]G \cap J(O)G$, it follows that either $S \subseteq [P:_O G] \cap J(O)$ or $Q \subseteq [P:_O G] \cap J(O) = [[P:_O G] \cap J(G):_O O]$. By Proposition 2.3 $[P:_O G]$ is MP ideal of O.

(⇐) Let $LK \subseteq P$ where L and K are submodules of G. Since G is a multipilcation, then $L = SG$ and $K = QG$ for some, ideals S and Q of O, that is $SQG \subseteq P$, implies that $SQ \subseteq [P:_O G]$, but $[P:_O G]$ is MP ideal of O, then by proposition 2.3 either $Q \subseteq [P:_O G] \cap J(O)$ or $S \subseteq [[P:_O G] \cap J(O):_O O] = [P:_O G] \cap JO)$. Hence either $QG \subseteq [P:_O G]G \cap J(O)G$ or $SG \subseteq [P:_O G]G \cap J(O)G$. since G is projective then by Proposition 1.3 $J(G) = J(O)G$. either $QG \subseteq P \cap J(G)$ or $SG \subseteq P \cap J(G)$. That is either $K \subseteq P \cap J(G)$ or $L \subseteq P \cap J(G)$. By proposition 3.1 P is MP submodule of G.

Proposition3.7

A proper submodule P of faithful multiplication O-module G is MP submodule of G *if and only if* $[P:_O G]$ is MP ideal of O.

Proof

(⇒) Suppose that P is MP submodule of G, and Let $rS \in [P:_O G]$ for $r \in O$, and S is an ideal of O, implies that $r(SG) \subseteq P$. But P is MP submodule of G, then by Corollary 2.4 either $SG \subseteq P \cap J(G)$or $rG \subseteq P \cap J(G)$Since G is multiplication, then $P = [P:_O G]G$, and since G is faithful multiplication, then by Proposition 1.4 $J(G) = J(O)G$. Thus either $SG \subseteq [P:_O G]G \cap J(O)G$ or $rG \subseteq [P:_O G]G \cap J(O)G$, it follows that either $S \subseteq [P:_O G] \cap J(O)$ or $r \in [P:_O G] \cap J(O) = [[P:_O G] \cap J(O):_O O]$. Hence by Corollary 2.4 $[P:_O G]$ is MP ideal of O.

(⇐) Let $mD \subseteq P$ for $m \in G$ and D is submodule of G. Since G is multiplication, then $m = Om = SG$ and $D = JG$ for, some ideals S, J of O, that is $SJG \subseteq P$, implies that $SJ \subseteq [P:_O G]$, but $[P:_O G]$ is MP ideal of O, then by proposition 2.3 either $J \subseteq [P:_O G] \cap J(O)$ or $S \subseteq [[P:_O G] \cap J(O):_O O] = [P:_O G] \cap J(O)$. Hence either $JG \subseteq [P:_O G]G \cap J(O)G$ or $SG \subseteq [P:_O G]G \cap J(O)G$. Hence G is faithful multiplication, then by Proposition 1.4 either $JG \subseteq P \cap J(G)$ or $SG \subseteq P \cap J(G)$. That is either $D \subseteq P \cap J(G)$ or $m \subseteq P \cap J(G)$. Thus by Corollary 3.4 P is MP submodule of G.

Proposition 3.8

Let G a content multiplication O-module, a proper submodule P of G is MP submodule of G *if and only* if $[P:_O G]$ is MP ideal of O.

Proof

(⇒) Suppose that P is MP submodule of G, and Let $Q\,a \subseteq [P:_O G]$ for Q is ideal of O and $a \in O$, so $Q(aG) \subseteq P$. But P is MP submodule of G, then by proposition 2.3 either $aG \subseteq P \cap J(G)$ or $QG \subseteq P \cap J(G)$. Since G is multiplication, then P=$[P:_O G]G$, and G is content O-module then by Proposition 1.5 $J(G) = J(O)G$. Thus either $aG \subseteq [P:_O G]G \cap J(O)G$ or $QG \subseteq [P:_O G]G \cap J(O)G$, it follows that either $a \in [P:_O G] \cap J(O)$ or $Q \subseteq [P:_O G] \cap J(O) = [[P:_O G] \cap J(O):_O O]$. Hence by Corollary 2.4 $[P:_O G]$ is MP ideal of O.

(⇐) Let $Lm \subseteq P$ for L is submodule of G and $m \in G$. Since G is multiplication, then $L = SG$ and $m = Om = JG$ for some ideals S, J of O, that is $SJG \subseteq P$, implies that $SJ \subseteq [P:_O G]$, but $[P:_O G]$ is MP ideal of O, then by proposition (2.3) either $J \subseteq [P:_O G] \cap J(O)$ or $S \subseteq= [[P:_O G] \cap J(O):_O O] = [P:_O G] \cap J(O)$. Hence either $JG \subseteq [P:_O G]G \cap J(O)G$ or $SG \subseteq [P:_O G]G \cap J(O)G$. Hence G is content module by Proposition 1.5 either $JG \subseteq P \cap J(G)$ or $SG \subseteq P \cap J(G)$. That is either $m \subseteq P \cap J(G)$ or $L \subseteq P \cap J(G)$. Thus by Corollary 3.3 P is MP submodule of G.

Proposition 3.9

Let G be multipilcation module over good ring O, and P be a proper *submodule* of G. Then P is MP submodule of G *if and only* if $[P:_O G]$ is MP ideal of O.

Proof

(⇒) Suppose P is MP submodule of G, and Let $rs \in [P:_O G]$ for $r,s \in O$, implies that $r(sG) \subseteq P$. But P is MP submodule of G, by Corollary 2.4 either $sG \subseteq P \cap J(G)$or $rG \subseteq P \cap J(G)$Since G is multiplication, then $P = [P:_O G]G$, and O is a good ring then $J(G) = J(O)G$. Thus either $sG \subseteq [P:_O G]G \cap J(O)G$ or $rG \subseteq [P:_O G]G \cap J(O)G$, it follows that either $s \in [P:_O G] \cap J(O)$ or $r \in [P:_O G] \cap J(O) = [[P:_O G] \cap J(O):_O O]$. Hence $[P:_O G]$ is MP ideal of O.

(⇐) Let $x_1x_2 \subseteq P$ for $x_1, x_2 \in G$. Since G is a multiplication, then $x_1 = Ox_1 = SG$, $x_2 = Ox_2 = JG$ for some ideals S, J of O, follows that is $SJG \subseteq P$, implies that $SJ \subseteq [P:_O G]$, but $[P:_O G]$ is MP ideal of O, then by proposition 2.3 either $J \subseteq [P:_O G] \cap J(O)$ or $S \subseteq [[P:_O G] \cap J(O):_O O] = [P:_O G] \cap J(O)$. Hence either $JG \subseteq [P:_O G]G \cap J(O)G$ or $SG \subseteq [P:_O G]G \cap J(O)G$. Hence O is a good ring either $JG \subseteq P \cap J(G)$or $SG \subseteq P \cap J(G)$. That is either $x_2 \subseteq P \cap J(G)$ or $x_1 \subseteq P \cap J(G)$. By Corollary 3.2 P is MP submodule of G.

Proposition 3.10

Let G be a multipilcation finitely generated O-module, with $JG \neq G$,for all maximal ideal J of O, and P be a proper submodule of G. Then P is MP submodule of G if and only if $[P:_O G]$ is MP ideal of O.

Proof

The same proof of Proposition 3.9, and using of Proposition 1.6 we can obtain the result.

Proposition 3.11

Let G be a projective finitely generated multipilcation O-module, and B is an ideal of O with $ann_O(G) \subseteq U$. Then U is MP ideal of O if and only if UG is MP submodule of G.

Proof

(⇒) Let $DK \subseteq UG$, for D, K are submodules of G. Since G is a multiplication, then $D = SG$, $K = JG$ for some ideals S, J of O, that is $SJG \subseteq UG$. But G is a finitely generated multipilcation O-module then by Proposition 1.7 $SJ \subseteq U + ann_O(G)$, but $ann_O(G) \subseteq U$, implies that $U + ann_O(G) = U$, thus $SJ \subseteq U$. By assumption U is MP ideal of O then by proposition 2.3 either $J \subseteq U \cap J(O)$ or $S \subseteq [U \cap J(O):_O O] = U \cap J(O)$, it follows that either $JG \subseteq UG \cap J(O)G$ or $SG \subseteq UG \cap J(O)G$. Since G is a projective then by Proposition 1.3 , it follows that either $K \subseteq UG \cap J(G)$ or $D \subseteq UG \cap J(G)$. By proposition 3.1 UG is MP submodule of G.

(⇐) Let $SJ \subseteq U$, for S and J are ideals in O, implies that $S(JG) \subseteq UG$. But UG is MP submodule of G, then by proposition 2.3 either $(JG) \subseteq UG \cap J(G)$ or $S \subseteq [UG \cap J(G):_O G]$. That is either $JG \subseteq UG \cap J(G)$ or $SG \subseteq UG \cap J(G)$. But G is a projective then Thus either $JG \subseteq UG \cap J(O)G$ or $SG \subseteq UG \cap J(O)G$, it follows that either $J \subseteq U \cap J(O)$ or $S \subseteq U \cap J(O) \subseteq [U \cap J(O):_O O]$. By proposition 2.3 U is MP ideal of O.

Proposition 3.12

Let G be multipilcation finitely generated module over a good ring O , and U is an ideal of O with $ann_O(G) \subseteq U$. Then U is MP ideal of O *if and only if* UG is MP submodule of G.

Proof

(⇒) Let $x_1x_2 \subseteq UG$, for $x_1, x_2 \in G$. Since G is a multiplication, then $x_1 = Ox_1 = SG$, $x_2 = Ox_2 = JG$ for some ideals S, J of O, that is $SJG \subseteq UG$. But G is multipilcation finitely generated O-module then by Proposition 1.7 $SJ \subseteq U + ann_O(G)$, since $ann_O(G) \subseteq U$, implies that $U + ann_O(G) = U$ implies that $SJ \subseteq U$. But U is MP ideal of O then by Proposition 2.3 either $J \subseteq U \cap J(O)$ or $S \subseteq [U \cap J(O):_O O] = U \cap J(O)$. Thus either $JG \subseteq UG \cap J(O)G$ or $SG \subseteq UG \cap J(O)G$. Since O is good ring then $J(O)G = J(G)$. Hence either $JG \subseteq UG \cap J(G)$ or $SG \subseteq UG \cap J(G)$. That is either $x_1 \subseteq UG \cap J(G)$ or $x_2 \subseteq UG \cap J(G)$. Therefore by Corollary 3.2 UG is MP submodule of G.

(⇐) Let $rS \subseteq U$, for $r \in O$, and S is an ideal of O, implies that $r(SG) \subseteq UG$. Since UG is MP submodule of G, then by Corollary 2.4 either $(SG) \subseteq UG \cap J(G)$ or $r \in [UG \cap J(G):_O G]$. That is either $SG \subseteq UG \cap J(G)$ or $rG \subseteq UG \cap J(G)$. But O is good ring then $J(O)G = J(G)$. Thus either $SG \subseteq BG \cap J(O)G$ or $rG \subseteq BG \cap J(O)G$, it follows that either $S \subseteq B \cap J(O)$ or $r \in B \cap J(O) \subseteq [B \cap J(O):_O O]$. Hence by Corollary 2.4 B is MP ideal of O.

Proposition 3.13

Let G be a multipilcation finitely generated O-module with $JG \neq G$ for all maximal ideal J of O, O, and U is an ideal of O with $ann_O(G) \subseteq U$. Then U is MP ideal of O *if and only if* UG is MP submodule of G.

Proof

(⇒) Let $oD \subseteq UG$, for $o \in O$, and D is submodule of G. Since G is multiplication, then $D = SG$ for some ideal S of O, that is $oSG \subseteq UG$. Since G is a multipilcation finitely generated O-module then by Proposition 1.7 $oS \subseteq U + ann_O(G)$, since $ann_O(G) \subseteq U$, implies that $U + ann_O(G) = U$ and hence $oS \subseteq U$. By hypothesis U is MP ideal of O by Corollary 2.4 either$S \subseteq U \cap J(O)$ or $o \in [U \cap J(O):_O O] = U \cap J(O)$. Thus either $SG \subseteq UG \cap J(O)G$ or $oG \subseteq UG \cap J(O)G$. Since G be a multipilcation finitely generated module with $JG \neq G$ for all maximal ideal J of O then by Proposition 1.6 $J(G) = J(O)G$. Hence either $SG \subseteq UG \cap J(G)$ or $oG \subseteq UG \cap J(G)$. That is either $D \subseteq UG \cap J(G)$ or $o \in [UG \cap J(G):_O G]$. Therefore by Corollary 2.4 UG is MP submodule of G.

(⇐) Let $oa \in U$, for $o, a \in O$, implies that $o(aG) \subseteq UG$. Since UG is MP submodule of G, then by Corollary 2.4 either $aG \subseteq UG \cap J(G)$ or $o \in [UG \cap J(G):_O G]$. That is either $aG \subseteq UG \cap J(G)$ or $oG \subseteq UG \cap J(G)$. By Proposition 1.6 $J(G) = J(O)G$.. Hence either $aG \subseteq UG \cap J(O)G$ or $rG \subseteq UG \cap J(O)G$, it follows either $a \in U \cap J(O)$ or $r \in U \cap J(O) \subseteq [U \cap J(O):_O O]$. Therefore U is MP ideal of O.

Proposition 3.14
Let G be multiplication finitely generated content module and U is ideal of O with $ann_O(G) \subseteq U$. Then U is MP ideal of O *if and only if* UG is MP submodule of G.
Proof
(⇒) Let $x_1x_2 \subseteq UG$, for $x_1, x_2 \in G$. Since G is a multiplication, then $x_1 = Ox_1 = SG$, $x_2 = Ox_2 = JG$ for some ideals S, J of O, that is $SJG \subseteq UG$. But G is multiplication finitely generated O-module by Proposition 1.7 $SJ \subseteq U + ann_O(G)$, since $ann_O(G) \subseteq U$, implies that $U + ann_O(G) = U$ and hence $SJ \subseteq U$. But U is MP ideal of O by Proposition 2.3 either $J \subseteq U \cap J(O)$ or $S \subseteq [U \cap J(O):_O O] = U \cap J(O)$. Thus either $JG \subseteq UG \cap J(O)G$ or $SG \subseteq UG \cap J(O)G$. Since G be a content module then by Proposition1.5 $J(O)G = J(G)$. Hence either $JG \subseteq UG \cap J(G)$ or $SG \subseteq UG \cap J(G)$. That is either $x_1 \subseteq UG \cap J(G)$ or $x_2 \subseteq UG \cap J(G)$. Therefore by Corollary 3.2 UG is MP submodule of G.

(⇐) Let $aS \subseteq U$, for $a \in O$, and S is an ideal of O, implies that $a(SG) \subseteq UG$. Since UG is MP submodule of G, then by Corollary 2.4 either $(SG) \subseteq UG \cap J(G)$ or $a \in [UG \cap J(G):_O G]$. That is either $SG \subseteq UG \cap J(G)$ or $aG \subseteq UG \cap J(G)$. But G be a content module then by Proposition 1.5. Thus either $SG \subseteq UG \cap J(O)G$ or $aG \subseteq UG \cap J(O)G$, it follows that either $S \subseteq U \cap J(O)$ or $a \in U \cap J(O) \subseteq [U \cap J(O):_O O]$. Hence by Corollary 2.4 U is MP ideal of O.

Proposition 3.15
Let G be a faithful finitely generated multipilcation O-module. Then U is MP ideal of Oif and only if UG is MP submodule of G.
Proof
(⇒) Let $x_1x_2 \subseteq UG$, for $x_1, x_2 \in G$. Since G is a multiplication, then $x_1 = Ox_1 = SG$, $x_2 = Ox_2 = JG$ for some ideals S, J of O, that is $SJG \subseteq UG$. But G is finitely generated multipilcation O-module then by proposition 1.7 $SJ \subseteq U + ann_O(G)$, since G is faithful then $ann_O(G) = (0)$, implies that $SJ \subseteq U$. But U is MP ideal of O by proposition 2.3 either $J \subseteq U \cap J(O)$ or $S \subseteq [U \cap J(O):_O O] = U \cap J(O)$. Thus either $JG \subseteq UG \cap J(O)G$ or $SG \subseteq UG \cap J(O)G$. But G is faithful multiplication, by Proposition 1.4 $J(G) = J(O)G$. Hence either $JG \subseteq UG \cap J(G)$ or $SG \subseteq UG \cap J(G)$. That is either $x_1 \subseteq UG \cap J(G)$ or $x_2 \subseteq UG \cap J(G)$. Therefore by Corollary 3.2 UG is MP submodule of G.

(⇐) Let $ra \in U$, for $r, a \in O$, implies that $r(aG) \subseteq UG$. But UG is MP submodule of G, then by Corollary 2.4 either $aG \subseteq UG \cap J(G)$ or $r \in [UG \cap J(G):_O G]$. That is either $aG \subseteq UG \cap J(G)$ or $rG \subseteq UG \cap J(G)$. Hence G faithful multiplication O-module by Proposition 1.4 either $aG \subseteq UG \cap J(O)G$ or $rG \subseteq UG \cap J(O)G$, it follows that either $a \in U \cap J(O)$ or $r \in U \cap J(O) \subseteq [U \cap J(O):_O O]$. Therefore U is MP ideal of O.

Proposition 3.16
Let G be finitely generated projective multipilcation O-module and P be a proper submodule of G then the statements that follow are equivalent:

1. P is *MP submodule of* G.
2. $[P:_O G]$ *is MP ideal of* O.
3. $P = UG$ for some MP ideal U of O with $ann_O(G) \subseteq U$.

Proof
It follows thar by Propositions [Proposition 3.6 and Proposition 3.11].

Proposition 3.17
Let G be content finitely generated multipilcation O-module, and P be a proper submodule of G then the statement that follow are equivalent :

1. P is MP submodule of V.
2. **[**P:_O V**]** is MP ideal of O.
3. $P = UG$ for some MP ideal U of O with $ann_O(G) \subseteq U$.

Proof
It follows by Propositions [Proposition 3.8 and Proposition 3.14]
Proposition 3.18

Let G be faithful multiplication finitely generated O-module, and P be a proper submodule of G, then the statement s that follow are equivalent :

1. P is MP submodule of G.

2. $[P:_O G]$ is MP ideal of O.

3. $P = UG$ for some MP ideal U of O.

Proof

It follows by Propositions [Proposition 3.7 and Proposition 3.15]

Proposition 3.19

Let G be finitely generated multipilcation module over good ring O ,and P be proper submodule of G**.** Then the statements that follow are equivalent:

1. P is MP submodule of G.

2. $[P:_O G]$ is MP ideal of O.

3. $P = UG$ for some MP ideal U of O with $ann_O(G) \subseteq U$.

Proof

It follows by Propositions [Proposition 3.9) and Proposition 3.12]

Proposition 3.20

Let G be finitely generated multipilcation module with $SG \neq G$ for all maximal ideal S of O, and P be a proper submodule of G**.** Then the statements that follow are equivalent:

1. P is MP submodule s of G.

2. $[P:_O G]$ is MP ideal of O.

3. $P = UG$ for some MP ideal U of O with $ann_O(G) \subseteq U$.

Proof

It follows by Propositions [Proposition 3.10 and Proposition 3.13].

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