2J-Submodules and 2J-ideals in Commutative Rings

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\section*{ABSTRACT}
Let $A$ be a commutative ring. In this paper, we introduce and study the concept of $2J$-submodule. A submodule $U$ of an $A$-module $T$ is called a $2J$-submodule if for all $a \in A$ and $t \in T$, whenever $at \in U$ with $a^2 \notin (J(A):T)$, then $t \in U$. In this work examine the characteristics of the $2J$-submodule as a generalization of the $J$-submodule. This paper provides various characterizations and properties of $2J$-submodule.

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1. Introduction

In this article, $T$ stands for a left $A$-module and $A$ for a commutative ring with unity $1 \neq 0$. The symbols $U \leq T (U < T)$ stand for $U$ is a submodule of $T$ ($U$ is proper submodule). The notations $(U :_AT) = \{a \in A: aT \subseteq U\}, (U :_T I) = \{x \in T: Ix \subseteq U\}$, where $I$ is an ideal of $A (I \leq A)$.

Many kinds of ideals have been established over time to enable us to properly comprehend the structures of rings generally. Prime, primary, and maximal ideals are a few examples of this class; they are important concepts within the commutative algebraic theory. In the last thirty years, a great lot of generalizations and related ideal types have been examined, including see [1],[5],[6],[7]

The notion of $r$-ideals in commutative rings was first presented by Mohamadian [9]. An ideal $I$ of a ring $A$ is called an $r$-ideal if $a, b \in A$ with $ab \in I$ and $\text{Ann} (a) = 0$, then $b \in I$. Later, this approach was extended to $J$-ideals and $J$-submodules by Khashan and Bani-Ata [8]. An ideal $I$ of a ring $A$ is called a $J$-ideal if $a, b \in A$ such that $ab \in I$ and $a \notin J(A)$, then $b \in I$. By $J(A)$ the Jacobian radical of $A$. [8], they presented the following: $U < T$ is said to be a $J$-submodule if whenever $a \in A$ and $t \in T$ with $at \in U$ and $a \notin (J(A)T : T)$, then $a \notin U$. In [4], we presented $2J$-ideal, where an ideal $I$ is called $2J$-ideal if whenever $a, b \in A$ such that $ab \in I$ and $a \notin J(A)$, then $b^2 \in I$ was extended of $J$-ideals.

These ideas motivate us to develop new kinds of submodules, like as $2J$-submodule. A submodule $U < T$ is called $2J$-submodule,

if for all $a \in A$ and $t \in T$, whenever $at \in U$ with $a^2 \notin (J(A)T : T)$, then $t \in U$.

Here, we examine the characteristics of the $2J$-submodule, which is comparable to the $J$-submodule. Also, we study the properties of $2J$-submodule similar to $2J$ ideals.
2. 2J-Submodules over commutative rings

**Definition 2.1:** A submodule $U$ of an $A$-module $T$ is said to be 2J-submodule if $a \in A$ and $t \in T$, with $at \in U$ and $a^2 \notin (J(A)T{:}T)$, then $t \in U$.

**Remark 2.2** If $U$ is a J-submodule of an $A$-module $T$, then $U$ is 2J-submodule.

**Proof:** Let $a \in A$ and $t \in T$, with $at \in U$ such that $a^2 \notin (J(A)T{:}T)$. As $U$ is a J-submodule we have $t \in U$ thus $U$ is a 2J-submodule.

The converse of Remark 2.2 is not true in general.

**Example 2.3** $\langle 0 \rangle$ is a 2J-submodule of $\mathbb{Z}_a$ as $\mathbb{Z}$-module. But it is not J-submodule, since $2 \notin \langle 0 \rangle$ and neither $2 \notin (J(Z)T{:}T')$ nor $\bar{2} \notin < \bar{0} >$.

**Proposition 2.4:** Let $T$ be an $A$-module, $U$ a submodule of $T$, and $I_1$ an ideal of $A$. Then:

1. $(U{:}A^2T)$ is a 2J-ideal of $A$ if $U$ is a 2J-submodule of $T$ and $(J(A)T{:}T) = J(A)$.

2. $(U :_{T} I_1)$ is a 2J-submodule of $T$ if $U$ is a 2J-submodule of $T$.

**Proof:**

1. Let $ab \in (U{:}A^2T)$ where $a, b \in A$ and $a \notin J(A)$, then we have $abT \subseteq U$ and so $abt \in U$ for all $t \in T$. Since $U$ is a 2J-submodule of $T$ and $a^2 \notin (J(A)T{:}T)$, $bt \in U$ for all $t \in T$, thus $bT \subseteq U$ and so $b \in (U{:}A^2T)$, so $b^2 \in (U{:}A^2T)$ therefore $(U{:}A^2T)$ is a 2J-ideal.

2. Let $at \in (U :_{T} I_1)$ where $a \in A$ and $t \in T$ with $a^2 \notin (J(A)T{:}T)$, then we have $ait \subseteq U$ and so $ati \in U$ for all $i \in I_1$, since $U$ is a 2J-submodule of $T$ and $a^2 \notin (J(A)T{:}T)$, $ti \in U$ for all $i \in I_1$, thus $tI_1 \subseteq U$ and so $t \in (U :_{T} I_1)$ therefore $(U :_{T} I_1)$ is a 2J-submodule.

If $(J(A)T{:}T) \notin J(A)$, then it is not necessary for component (1) of Proposition (2.4) to be valid.

As an illustration, the $\mathbb{Z}$-module $T = \mathbb{Z}_2$ then $2\mathbb{Z} = (U{:}A^2T) = (J(Z)T{:}T) \notin J(Z) = \{0\}$ now $U = \{0\}$ is clearly a 2J-submodule of $\mathbb{Z}_2$ but $2\mathbb{Z} = (U{:}A^2T)$ ideal of $\mathbb{Z}$ is not a 2J.

A-module $T$ is called a multiplication module if for every submodule $N$ of $T$, there exists an ideal $B$ of $A$ such that $N = TB$. Equivalently, $M$ is a multiplication module if and only if $U = (U{:}A^2T)^M$, for each submodule $U$ of $T$ [3].

**Proposition 2.5:** If $(U{:}A^2T)$ is a 2J-ideal of $A$ where $U$ is a submodule of a multiplication module $T$, then $U$ is a 2J-submodule of $T$.

**Proof:** Let $a \in A$ and $t \in T$ such that $at \in U$ and $a^2 \notin (J(A)T{:}T)$. Then $a(\langle t \rangle : T) \subseteq (at : T) \subseteq (U{:}A^2T)$. Since $(J(A) \subseteq (J(A)T{:}T)$, $a^2 \notin J(A)$ and so $(\langle t \rangle : T) \subseteq (U{:}A^2T)$ is a 2J-ideal of $A$. Given that $T$ is a multiplication module at this point $\langle t \rangle = (\langle t \rangle : T) \subseteq (U{:}A^2T) T = U$. Thus $t \in U$ and $U$ is a 2J-submodule of $T$.

**Corollary 2.6:** Assume that $U$ is a proper submodule of $T$ and that $T$ is a finitely generated
faithful multiplication $A$-module. Then the next conditions are equivalent:

1) $U$ is a $2J$-submodule of $T$.

2) $(U;_A T)$ is a $2J$-ideal of $A$.

3) $U = LT$ where $L$ is a $2J$-ideal of $A$.

**Proof:** (1) $\iff$ (2) Propositions 2.4 and 2.5 come next, together with the information that

$(I_1 T : T) = I_1$ for any ideal $I_1$ of $A$.

(2) $\iff$ (3) We just choose $L = (U;_A T)$.

**Proposition 2.7:** Let $T$ be a finitely generated faithful multiplication $A$-module If $U$ is a $2J$-submodule of $T$, then $U \subseteq J(T)$.

**Proof:** Notably, we observe $J(T) = J(A)T$. Assume that $U \not\subseteq J(T)$.

Clearly, $(U;_A T) \not\subseteq (J(A)T : T) = J(A)$. But $(U;_A T)$ is a $2J$-ideal by Proposition 2.4 which contradicts [4, Remark and example 2.2(3)] Thus, $U \subseteq J(T)$ as required.

Recall that a proper submodule $U$ of a module $T$ over commutative ring $A$ is said to be $n$-submodule, if $a \in R$ and $x \in T$, $ax \in U$ with $a \notin \sqrt{\text{Ann}_R(M)}$, then $x \in U$.

**Proposition 2.8:** Let $T$ be an $A$-module satisfying $J(T) \subseteq J(A)T$. Then any $n$-submodule of $T$ is a $2J$-submodule.

**Proof:** Since $U$ is $n$-submodule, then $U$ is $J$-submodule by [9, Proposition 3.7]. Therefore it is $2J$-submodule by Remark 2.2.

**Corollary 2.9:** The $n$-submodules of a finitely generated faithful multiplication module are $2J$-submodules.

Let $A$ be a ring and $T$ be an $A$-module. The idealization ring $A(T)$ of $T$ in $A$ is defined as the set $\{(a, b) : a \in A, b \in T\}$ with the usual componentwise addition and multiplication defined as $(a, b)(s, n) = (as, bn + sb)$.

Let $T$ be an $A$-module and $A$ a ring. The set $\{(a, b) : a \in A, b \in T\}$ is the idealization ring $A(T)$ of $T$ in $A$. Its component wise addition and multiplication are defined as $(a, b)(s, n) = (as, bn + sb)$. It is easily verifiable that $A(T)$ is an identity commutative ring $(1_A, 0_T)$. If $K$ ideal and $U$ is a submodule of $T$, then $K(U) = \{(a, b) : a \in K, b \in U\}$ is an ideal of $A(T)$ if and only if $kT \subseteq U$.

**Proposition 2.10:** Let $T$ be an $A$-module, $U$ a submodule of $T$, and $I_1$ a $2J$-ideal of $A$. Then:

1) $I_1(T)$ is a $2J$-ideal of $A(T)$.

2) If $(J(A)T : T) = J(A)$ and $U$ is a $2J$-submodule of $T$ with $I_1 T \subseteq U$, then $I_1(U)$ is a $2J$-ideal of $A(T)$.

**Proof:** (1) Let $(A_1, t_1), (A_2, t_2) \in A(T)$ such that $(A_1, t_1)(A_2, t_2) \in I_1(T)$ with $(A_1, t_1) \notin J(A(T))$. Then we have $a_1a_2 \in I_1$ and $a_1 \notin J(A)$. Since $I_1$ is a $2J$-ideal of $A$, we conclude that $a_1^2 \in I_1$ and so $(A_2, t_2)^2 \in I_1(T)$. Consequently, $I_1(T)$ is a $2J$-ideal of $A(T)$.
Let \((A_1, t_1), (A_2, t_2) \in A(T)\) such that \((A_1, t_1)(A_2, t_2) \in I_1(U)\) and \((A_1, t_1) \notin J(A(T)) = J(A)(T)\). Then we have \(a_1a_2 \in I_1\) and \(a_3 \notin J(A)\). As \(I_1\) is a 2J-ideal of \(A\), then \(a_1 \in I_1\) and so \(a_2t_1 \in I_1T \subseteq U\). Since \(a_1t_2 + a_2t_1 \in U, a_1t_2 \notin U\). But \(a_2^2 \notin (J(A)T : T)\) and so \(t_2 \in U\) as \(U\) is a 2J-submodule of \(T\). Therefore, \((A_2, t_2)^2 \in I_1(U)\) and \(I_1(U)\) is a 2J-ideal of \(A(T)\).

**Corollary 2.11:** Consider a multiplication \(A\)-module, denoted by \(T\), that is both finitely generated and faithful. If \(I_1I\) is a 2J-ideal of \(A\) and \(U\) is a 2J-submodule of \(T\) such that \(I_1T \subseteq U\), then \(I_1(U)\) is a 2J-submodule of \(A(T)\).

**Proposition 2.12:** Let \(I_1\) be an ideal of a ring \(A\), and let \(U\) be a proper submodule of \(A\)-module \(T\). It follows that \(I_1\) is a 2J-ideal of \(A\) if \(I_1(U)\) is a 2J-ideal of \(A(T)\).

**Proof:** Assuming that \(I_1(U)\) represents a 2J-ideal of \(A(T)\), we demonstrate that \(I_1\) is a 2J-ideal of \(A\). Assume that \(ab \in I_1\), and that \(a \notin J(A)\). Next up, we get \((a, 0_T)(b, 0_T) = (ab, 0_T) \in I_1(U)\) with \((a, 0_T) \notin J(A)(T) = J(A(T))\). Since \(I_1(U)\) is a 2J-ideal of \(A(T)\), we conclude that \((b, 0_T)^2 \in I_1(U)\) and so \(b^2 \in I_1\).

**Theorem 2.13:** Let \(U\) be a proper submodule of an \(A\)-module \(T\). Then, the following statements are equivalent:

(i) \(U\) has 2J-submodule, named \(U\)

(ii) \(U = (U :_TA)\), for every \(a^2 \notin (J(A)T : T)\).

(iii) For every submodule \(K\) of \(A\) and ideal \(I_1\) of \(A\), \(I_1\) is a 2J-ideal of \(A\) if \(I_1(U)\) is a 2J-ideal of \(A(T)\).

**Proof:** (i) \(\Rightarrow\) (ii) Give \(U\) the form of a 2J-submodule of \(T\). \(\forall \alpha \in A\), the inclusion \(U \subseteq (U :_TA)\) always holds. Let \(a^2 \notin (J(A)T : T)\) and \(t \in (U :_TA)\). Then we have \(at \in U\). Meanwhile \(U\) is an 2J-submodule, Finally, we determine that \(t \in U\) and thus \(U = (U :_TA)\).

(ii) \(\Rightarrow\) (iii) Presume that \(I_1K \subseteq U\) where \(I_1^2 \subsetneq (J(A)T : T)\), for ideal \(I_1\) of \(A\) and submodule \(K\) of \(T\). Since \(I_1^2 \subsetneq (J(A)T : T)\), there exists \(a \in I_1\) such that \(a^2 \notin (J(A)T : T)\) then we have \(aK \subseteq U\), and so \(K \subseteq (U :_TA) = U\) by (ii).

(iii) \(\Rightarrow\) (i) Let \(at \in U\) with \(a^2 \notin (J(A)T : T)\) for \(a \in A\) and \(t \in T\). That is adequate to take \(I_1 = Aa\) and \(K = Ax\) to show that the result.

**Proposition 2.14:** If \(U\) is a 2J-submodule of an \(A\)-module \(T\), then \((U :_AT) \subseteq (J(A)T : T)\).

**Proof:** Presume that \(U\) is a 2J-submodule; and \((U :_AT) \subseteq (J(A)T : T)\). Then there exists \(a \in (U :_AT)\) such that \(a \notin (J(A)T : T)\), so \(a^2 \notin (U :_AT)\) such that \(a^2 \notin (J(A)T : T)\). Thus \(a^2 \subseteq U\) and since \(U\) is an 2J-submodule, In the end, we ascertain that \(U = T\), a contradiction. Hence \((U :_AT) \subseteq (J(A)T : T)\).

**Lemma 2.15:** Let \(T\) be a torsion-free \(A\)-module, then the zero submodule of \(T\) is a 2J-submodule

**Proof:** Given \(a \in A\) and \(t \in T, at = 0\), with \(a^2 \notin (J(A)T : T)\). Torsion-free \(T\) implies that \(t=0\). Zero submodule is a 2J-submodule.
Lemma 2.16: Zero submodule is the only $2J$-submodule of $T$ if $T$ is a torsion-free multiplication $A$-module.

Proof: Submodule is a $2J$-submodule. Therefore, by Proposition 2.14 $(U_A T) \subseteq (J(A)T:T) = 0$. Consequentially, then $(U_A T) = 0$. U=0 meanwhile T is a multiplication. Thus, the zero submodule is the only $2J$-submodule according to Lemma 2.15.

Proposition 2.17: Let $T$ be an $A$-module and that $I_1$ an ideal of $A$ with $I_1 \not\subseteq (U_A T)$. If $U$ is a $2J$-submodule of $T$, then $(U :_T I_1)$ is a $2J$-submodule of $T$.

Proof: Let $a \in (U :_T I_1)$ with $a^2 \in (J(A)T:T)$, for $a \in A$ and $t \in T$. So $aI_1 t \subseteq U$ and as $U$ is a $2J$-submodule, $I_1 t \subseteq U$. Hence $t \in (U :_T I_1)$.

Proposition 2.18: Let $U$ be a proper submodule of an $A$-module $T$, then $U$ is an $2J$-submodule if and only if for every $t \in T$, $(U : A) = A$ or $(U_A t) \subseteq (J(A)T:T)$.

Proof: Assume that $U$ is an $2J$-submodule. If $(U : A) \not\subseteq (J(A)T:T)$, then there exists $a \in (U : A)$, $(J(A)T:T)$ so $a^2 \in (U_A t) - (J(A)T:T)$. Let $a \in U$ where $a^2 \in (J(A)T:T)$. Since $U$ is an $2J$-submodule, $t \in U$. Hence $(U_A t) = A$ Conversely, let $a \in U$ where $a^2 \not\in (J(A)T:T)$, for $a \in A$ and $t \in T$. So $a^2 \in (U_A t) - (J(A)T:T)$. By assumption, we have $(U_A t) = A$ and so $t \in U$.

Corollary 2.19: Let $U$ be a proper submodule of an $A$-module $T$. Then $U$ is an $2J$-submodule of $T$ if and only if for every $t \in T - U$, $(U : A t) \subseteq (J(A)T:T)$.

Theorem 2.20: If $U$ is a maximal $2J$-submodule of an $A$-module $T$, then $U$ is a prime submodule of $T$.

Proof: Assume that $U$ is the a maximal $2J$-submodule of $T$ and that $t \in T$ and $a \not\in (U : A t)$ where $at \in U$. $(U :_T a)$ is a $2J$-submodule by Proposition 2.17. According to maximality of $U$,

$t \subseteq (U :_T a) = U$. Therefore, $U$ is prime submodule.

Theorem 2.21: Let $T$ be finitely generated $A$-module. Then $T$ has a prime submodule if it has a $2J$-submodule.

Proof: Let $\Omega = \{ L : L$ is a $2J$-submodule of $T; U \subseteq L \}$ and that $U$ is a $2J$-submodule. Zorn’s Lemma states that $\Omega$ has a maximal element $K \in \Omega$. Therefore, $K$ is a prime submodule of $T$ by Theorem 2.21.

A proper submodule $N$ of $M$ is called primary) if $rx \in N$, for $r \in R$ and $x \in M$, implies that either $x \in N$ or $r^n \in (N :_R M)$ for $n \in N$ [2].

Theorem 2.22: Let $U$ be a submodule of $T$ such that $(U : A T) \subseteq (J(A)T:T)$. Then the following statements are equivalent:

(i) $U$ is a $2J$-submodule.

(ii) A primary submodule of $T$ has the name $U$.

Proof (i) $\Rightarrow$ (ii) Let $at \in U$ with $a^n \not\in (U : A T)$ for $a \in A$ and $t \in T$. As $U$ is a $2J$-submodule, then $t \in U$ and hence $U$ is a primary submodule.
(ii)  \(\Rightarrow\) (i) Let \(a t \in U\) with \(a^2 \notin (J(A)T: T)\) for \(a \in A\) and \(t \in T\). As \((U:A T) = (J(A)T: T)\) we have \(a^2 \notin (J(A)T: T)\), since \(U\) is a primary submodule, we get \(t \in U\). Therefore \(U\) is a \(2J\)-submodule.

**Proposition 2.24:** If \(L\) is a primary submodule of an \(A\)-module \(T\) and \(U\) is a \(2J\)-submodule such that \((L:A T) \subseteq (J(A)T: T)\), then \(U \cap L\) is a \(2J\)-submodule of \(T\).

**Proof:** Let \(a t \in U \cap L\) where \(a^2 \notin (J(A)T: T)\) for \(a \in A\), \(t \in T\). Then \(a^2 \notin (L:A T)\). Since \(L\) is primary, \(t \in L\). Also, since \(U\) is a \(2J\)-submodule, \(t \in U\). Thus \(t \in U \cap L\).

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779.