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2J-Submodules and 2J- ideals in Commutative Rings

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ABSTRACT

Let A be a commutative ring. In this paper, we introduce and study the concept of $2J$-submodule. A submodule U of a A-module Tis called a $2J$-submodule) if for all $a \in A$ and $t \in T$, whenever $at \in U$ with $a^2 \notin (J(A)T:T)$, then $t \in U$. In this work examine the characteristics of the 2J-submodule as a generalization of the J-submodule. This paper provides various characterizations and properties of $2J$- submodule.

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1. Introduction

In this article, T stands for a left A-module and A for a commutative ring with unity 1≠0. The symbols $U \leq T(U < T)$ stand for U is a submodule of $T(U$is proper submodule). The notations $(U :_A T) = \{a \in A: aT \subseteq U\}$, $(U :_T I) = \{x \in T: Ix \subseteq U\}$, where I is an ideal of $A(I \leq A)$.

Many kinds of ideals have been established over time to enable us to properly comprehend the structures of rings generally. Prime, primary, and maximal

ideals are a few examples of this class; they are important concepts within the commutative algebraic theory. In the last thirty years, a great lot of generalizations and related ideal types have been examined, including see [1],[5],[6],[7]

The notion of r-ideals in commutative rings was first presented by Mohamadian [9]. An ideal I of a ring A is called an r-ideal if $a, b \in A$ with $ab \in I$ and Ann $(a) = 0$, then $b \in I$. Later, this approach was extended to J-ideals and J-submodules by Khashan and Bani-Ata [8]. An ideal I of a ring A is called a J-ideal if $a, b \in A$ such that $ab \in I$ and $a \notin \mathrm{J}(A)$, then $b \in I$. By $J(A)$ the Jacobian radical of A. [8] , they presented the following: $U < T$ is said to be a J -submodule if whenever $a \in A$ and $t \in T$ with $at \in U$ and $a \notin (J(A)T : T)$, then $a \in U$. In [4], we presented 2J-ideal, where an ideal I is called $2J$-ideal if whenever $a, b \in A$ such that $ab \in I$ *and* $a \notin J(A)$, then $b^2 \in I$ was extended of J-ideals.

These ideas motivate us to develop new kinds of submodules, like as 2J-submodule. A submodule $U < T$ is called 2J-submodule,

if for all $a \in A$ and $t \in T$, whenever $at \in U$ with $a^2 \notin (J(A)T: T)$, then $t \in U$.

Here, we examine the characteristics of the 2j-submodule, which is comparable to the J-submodule. Also, we study the properties of $2J$- submodule similar to $2J$ ideals.

2. 2J-Submodules over commutative rings

Definition 2.1: A submodule U of an A-module T is said to be 2J-submodule if $a \in A$ and $t \in T$, with $at \in U$ and $a^2 \notin (J(A)T{:}T)$, then $t \in U$.

Remark 2.2 If U is a J-submodule of an A-module T, then U is 2J-submodule.

Proof: Let $a \in A$ and $t \in T$, with $at \in U$ such that $a^2 \notin (J(A)T{:}T)$. As U is a J-submodule we

have $t \in U$ thus U is a 2J-submodule .■

The converse of Remark 2.2 is not true in general.

Example 2.3 : $<\bar{0}>$ is a 2J-submodule of $\mathbb{Z}_4$ as $\mathbb{Z}$-module. But it is not J-submodule, since $2.\bar{2} \in <\bar{0}>$ and neither $2 \notin (J(Z)T{:}T) =< 4 >$ nor $\bar{2} \notin <\bar{0}>$.

Proposition 2.4: Let T be an A-module, U a submodule of T, and I_1 an ideal of A. Then:

(1) $(U{:}_A T)$ is a 2J-ideal of A if U is a 2J-submodule of T and $(J(A)T : T) = J(A)$.

(2) $(U :_T I_1)$ is a 2J-submodule of T if U is a 2J-submodule of T.

Proof:(1) Let $ab \in (U{:}_A T)$ where $a,b \in A$ and $a \notin J(A)$, then we have $abT \subseteq U$ and so $abt \in U$ for all $t \in T$ .Since U is a 2J- submodule of T and $a^2 \notin (J(A)T{:}T)$,$bt \in U$ for all $t \in T$, thus $bT \subseteq U$ and so $b \in (U{:}_A T)$, so $b^2 \in (U{:}_A T)$ therefore$(U{:}_A T)$ is a 2J-ideal .

(2) Let $at \in (U{:}_T I_1)$ where $a \in A$ and $t \in T$ with $a^2 \notin (J(A)T{:}T)$, then we have $atI_1 \subseteq U$ and so $ati \in U$ for all $i \in I_1$,since U is a 2J-submodule of T and $a^2 \notin (J(A)T{:}T)$,$ti \in U$ for all $i \in I_1$,thus $tI_1 \subseteq U$ and so $t \in (U{:}_T I_1)$ therefore $(U{:}_T I_1)$ is a 2J -submodule . ■

If $(J(A)T{:}T) \nsubseteq J(A)$, then it is not necessary for component (1) of Proposition (2.4) to be valid.

As an illustration, the Z-module $T=Z_2$ then $2Z = (U{:}_A T) = (J(Z)T{:}T) \nsubseteq J(Z) = \{0\}$ now $U = \{0\}$ is clearly a 2J-submodule of Z_2 but $2Z = (U{:}_A T)$ ideal of Z is not a 2J.

A-module T is called a multiplication module if for every submodule N of T, there exists an ideal B of A such $that\ N = TB$. Equivalently, M is a
multiplication module if and only if $U = (U{:}_A T)T$ M, for each submodule U of T [3].

Proposition 2.5: (If $(U{:}_A T)$ is a 2J-ideal of A where U is a submodule of a multiplication module T, then U is a 2J-submodule of T.

Proof: Let $a \in A$ and $t \in T$ such that $at \in U$ and $a^2 \notin (J(A)T{:}T)$. Then $a(<t> : T) \subseteq (<at> : T) \subseteq (U{:}_A T)$. Since $J(A) \subseteq (J(A)T{:}\ T)$, $a^2 \notin J(A)$ and so $(<t>{:}\ T) \subseteq (U{:}_A T)$ as $(U{:}_A T)$ is a 2J-ideal of A. Given that T is a multiplication module at this point $<t>= (<t>{:}\ T)\ T \subseteq (U{:}_A T)\ T = U$. Thus $t \in U$ and U is a 2J-submodule of T. ■

Corollary 2.6: Assume that U is a proper submodule of T and that T is a finitely generated faithful multiplication A-module. Then the next conditions are equivalent:

(1)U is a 2J-submodule of T.

(2) $(U:_A T)$ is a 2J-ideal of A.

(3) $U = LT$ where L is a 2J-ideal of A.

Proof: (1) ⟺ (2) Propositions 2.4 and 2.5 come next, together with the information that

$(I_1T : T) = I_1$for any ideal I_1 of A.

(2)⟺ (3) We just choose $L = (U:_A T)$. ■

Proposition 2.7: Let T be a finitely generated faithful multiplication A-module If U is a 2J-submodule of T, then $U \subseteq J(T)$.

Proof: Notably, we observe $J(T) = J(A)T$. Assume that $U \nsubseteq J(T)$.

Clearly, $(U:_A T) \nsubseteq (J(A)T: T) = J(A)$. But $(U:_A T)$is a 2J-ideal by Proposition 2.4 which contradicts[4, Remark and example .2.2(3)] Thus, $U \subseteq J(T)$as required. ■

Recall that a proper submodule U of a module T over commutative ring A is said to be n-submodule, if $a \in R$ and $x \in T, ax \in U$ with $a \notin \sqrt{Ann_R(M)}$, then $x \in U$[10]

Proposition 2.8: Let T be an A-module satisfying $J(T) \subseteq J(A)T$. Then any n-submodule of T is a 2J-submodule.

Proof: Since U is n-submodule ,then U is J-submodule by [9, Proposition 3.7].Therefore It is 2J-submodule by Remark 2.2. ■

Corollary 2.9: The n-submodules of a finitely generated faithful multiplication module are 2J-submodules.

Let Let A be a ring and T be a A-module. The idealization ring $A(T)$ of T in A is defined as the set $\{(a,b): a \in A, b \in T\}$ with the usual componentwise addition and multiplication are defined as $(a,b)(s,n) = (as, bn + sb)$.

Let T be an A-module and A a ring. The set $\{(a,b): a \in A, b \in T\}$ is the idealization ring $A(T)$ of T in A. Its component wise addition and multiplication are defined as $(a,b)(s,n) = (as, bn + sb)$. It is easily verifiable that $A(T)$ is an identity commutative ring $(1_A, 0_T)$. If K ideal and U is a submodule of T, then $K(U) = \{(a,b): a \in K, b \in U\}$ is an ideal of $A(T)$ if and only if $kT \subseteq U$ [11]

Proposition 2.10: Let T be an A-module, U a submodule of T, and I_1 a 2J-ideal of A. Then:

(1) $I_1(T)$ is a 2J-ideal of $A(T)$.

(2) If $(J(A)T : T) = J(A)$ and U is a 2J-submodule of T with $I_1 T \subseteq U$, then $I_1(U)$ is a 2J-ideal of $A(T)$.

Proof: (1) Let (A_1, t_1),$(A_2, t_2) \in A(T)$ such that $(A_1, t_1)(A_2\ t_2) \in I_1(T)$ with $(A_1, t_1) \notin J(A(T))$. Then we have $a_1a_2 \in I_1$ and $a_1 \notin J(A)$. Since $I_1$is a 2J-ideal of A, we conclude that $a_2^2 \in I_1$ and so $(A_2, t_2)^2 \in I_1(T)$. Consequently, $I_1(T)$ is a 2J-ideal of $A(T)$.

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(2)Let (A_1,t_1),(A_2,t_2)∈ $A(T)$ such that $(A_1,t_1)(A_2,t_2) \in I_1(U)$ and $(A_1,t_1) \notin J(A(T)) = J(A)(T)$. Then we have $a_1a_2 \in I_1$ $and\ a_1 \notin J(A)$. As I_1 is 2J-ideal of A, then we $a_2^2 \in I_1$ and so $a_2t_1 \in I_1T \subseteq U$. Since $a_1t_2 + a_2t_1 \in U$, $a_1t_2 \in U$. But $a_1^2 \notin (J(A)T : T)$ and so $t_2 \in U$ as U is a 2J-submodule of T. Therefore, $(A_2, t_2)^2 \in I_1(U)$ and $I_1(U)$ is a 2J-ideal of $A(T)$. ■

Corollary 2.11: Consider a multiplication A-module, denoted by T, that is both finitely generated and faithful. If I_1I is a 2J-ideal of A and U is a 2J-submodule of T such that $I_1T \subseteq U$, then $I_1(U)$ is a 2J-submodule of $A(T)$.

Proposition 2.12: Let I_1 be an ideal of a ring A, , and let U be a proper submodule of A -module T. It follows that $I_1$is a 2J-ideal of A if I_1 (U) is a 2J-ideal of $A(T)$.

Proof: Assuming that $I_1(U)$ represents a 2 J-ideal of $A(T)$,we demonstrate that I_1 is a 2 J-ideal of A. Assume that $ab \in I_1$, and that $a \notin J(A)$. Next up, we got $(a, 0_T)\ (b, 0_T) = (ab, 0_T) \in I_1(U)$ with $(a, 0_T) \notin J(A)(T) = J(A(T))$. Since $I_1(U)$ is a 2J-ideal of $A(T)$, we conclude that $(b, 0_T)^2 \in I_1(U)$ and so $b^2 \in I_1$. ■

Theorem 2.13: Let U be a proper submodule of an A-module T. Then, the following statements are equivalent:

(i)T has 2J-submodule, named U

(ii) $U = (U:_T a)$, for every $a^2 \notin (J(A)T:T)$.

(iii) For every submodule $K\ of\ T$ and ideal $I_1$of A , $I_1K \subseteq U$ with $I_1^2 \nsubseteq (J(A)T:T)$ implies $K \subseteq U$.

Proof: (i) ⟹ (ii) Give U the form of a 2J-submodule of T . $\forall a \in A$, the inclusion $U \subseteq (U:_T a)$ always holds. Let $a^2 \notin (J\ (A)T:T)$ and $t \in (U:_T a\ )$. Then we have $at \in U$. Meanwhile U is an 2J-submodule, Finally, we determine that $t \in U$ and thus $U = (U:_T a)$.

(ii) ⇒ (iii) Presume that $I_1K \subseteq U$ where$I_1^2 \nsubseteq (J(A)T:T)$, for ideal I_1 of A and submodule K of T. Since $I_1^2 \nsubseteq (J(A)T:T)$, there exists $a \in I_1$ such that $a^2 \notin (J(A)T:T)$ Then we have $aK \subseteq U$, and so $K \subseteq (U:_T a)=U$ by (ii).

(iii) ⇒ (i) Let $at \in U$ with $a^2 \notin (J(A)T:T)$ for $a \in A$ and $t \in T$. That is adequate to take

$I_1 = Aa$ and $K = Ax$ to show that the result. ■

Proposition 2.14: If $U$is a 2J-submodule of an A-module T, then $(U:_A T) \subseteq (J(A)T:T)$.

Proof: Presume that U is an 2J-submodule; and $(U:_A T) \nsubseteq (J(A)T:T)$. Then there exists $a \in (U:_A T)$ such that $a \notin (J(A)T:T)$so $a^2 \in (U:_A T)$such that $a^2 \notin (J(A)T:T)$ Thus $a^2T \subseteq U$ and since U is an 2J-submodule, In the end, we ascertain that $U = T$, a contradiction. Hence $(U:_A T) \subseteq (J(A)T:T)$. ■

Lemma 2.15: Let T be a torsion-free A-module, then the zero submodule of T is a 2J-submodule **Proof:** Given $a \in A\ and\ t \in T$, let $at = 0$, with $a^2 \notin (J(A)T:T)$. Torsion-free T implies that t=0. Zero submodule is a 2J-submodule. ■

Lemma 2.16: Zero submodule is the only 2J-submodule of T if T is a torsion-free multiplication A-module

Proof: submodule is a 2J-submodule. . Therefore, by Proposition 2.14 $(U:_A T) \subseteq (J(A)T:T) = 0$, Consequently, then $(U:_A T)=0$. U=0 meanwhile T is a multiplication. Thus, the zero submodule is the only 2J-submodule according to Lemma 2.15. ■

Proposition 2.17: Let T be an A-module and that I_1 an ideal of A with $I_1 \nsubseteq (U:_A T)$. If U is a 2J-submodule of T, then $(U:_T I_1)$ is a 2J-submodule of T.

Proof: Let $at \in (U:_T I_1)$ with $a^2 \notin (J(A)T:T)$, for $a \in A$ and $t \in T$. So $aI_1 t \subseteq U$ and as U is an 2J-submodule, $I_1 t \subseteq U$. Hence $t \in (U :_T I_1)$. ■

Proposition 2.18: Let U be a proper submodule of an A-module T, then U is an 2J-submodule

if and only if for every t ∈ T, $(U:_A t) = A$ or $(U:_A t) \subseteq (J(A)T:T)$.

Proof: Assume that U is an 2J-submodule. If $(U:_A t) \nsubseteq (J(A)T:T)$, then there exists $a \in (U:_A t)$- $(J(A)T:T)$so $a^2 \in (U:_A t) - (J(A)T:T)$. Let $at \in U$ where a^2 $(J(A)T:T)$ since U is an 2J -submodule, $t \in U$. Hence $(U:_A t) = A$ Conversely, let $at \in U$ where $a^2 \notin (J(A)T:T)$, for $a \in A$ and $t \in T$. So $a^2 \in (U:_A t)$- $(J(A)T:T)$. By assumption, we have $(U:_A t) = A$ and so $t \in U$. ■

Corollary 2.19: Let U be a proper submodule of an A-module T. Then U is an 2J-submodule of T if and only if for every $t \in T - U$, $(U:_A t) \subseteq (J(A)T:T)$.

Theorem 2.20: If U is a maximal 2J-submodule of an A-module T, then U is a prime submodule of T.

of T.

Proof: Assume that U is the a maximal 2J-submodule of T and that t∈T and $a \notin (U:_A T)$ where $at \in U$. $(U:_T a)$ is a 2J-submodule by Proposition 2.17. According to maximality of U,

$t \in (U:_T a) = U$. Therefore, U is prime submodule. ■

Theorem 2.21:Let T be finitely generated A-module. Then T has a prime submodule if it has a 2J-submodule.

Proof: Let $\Omega = \{ L : L$ is a 2J- submodule of $T; U \subseteq L \}$ and that U is a 2J-submodule. Zorn's Lemma states that Ω has a maximal element $K \in \Omega$. Therefore, K is a prime submodule of T by Theorem 2.21. ■

A proper submodule N of M is called primary) if $rx \in N$, for $r \in R$ and $x \in M$, implies that either $x \in N$ or $r^n \in (N:_R M)$ for $n \in N$ [2].

Theorem 2.22: Let U be a submodule of T such that$(U:_A T) \subseteq (J(A)T:T)$. Then the following statements are equivalent:

(i)U is 2J. submodule .

(ii) A primary submodule of T has the name U

Proof (i)⟹(ii) Let $at \in U$ with $a^n \notin (U:_A T)$ for
$a \in$ and $t \in T$. As U is a $2J -$ submodule, then $t \in U$ and hence U is a primary submodule

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(ii) $\Longrightarrow$ (i) Let $at \in U$ with $a^2 \notin (J(A)T: T)$ for $a \in A$ and $t \in T$. As $(U:_A T) = (J(A)T:T)$ we have $a^2 \notin (J(A)T: T)$, since U is a primary submodule, we get $t \in U$. Therefore U is a 2J-submodule.■

<u>Proposition 2.24:</u> If L is a primary submodule of an A-module T and U is a 2J-submodule such that $(L:_A T) \subseteq (J(A)T:T)$, then $U \cap L$ is a 2J-submodule of T .

Proof: Let $at \in U \cap L$ where $a^2 \notin (J(A)T:T)$ for $a \in A, t \in T$. Then $a^2 \notin (L:_A T)$. Since L is primary, $t \in L$. Also, since U is a 2J-submodule, $t \in U$. Thus $t \in U \cap L$. ■

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