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Maclaurin Coefficients Estimates for New classes of m -Fold Symmetric Bi-Univalent Functions

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ABSTRACT

The purpose of this study is to establish new subclasses within the function class Σ_m, which consists of analytic as well as m-fold symmetric bi-univalent functions expressed within the open unit disk $\mathcal{Q}$. Additionally, for functions belonging to each of the newly established subclasses, this paper establishes estimates with regards to the Taylor-Maclaurin coefficients given by $|a_{m+1}|$ as well as $|a_{2m+1}|$. Moreover, we take into consideration of specific as well as existing special cases for our respective findings.

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1. Introduction

Let $\mathcal{A}$ represent the class of functions f analytic within the open unit disk $\mathcal{Q} = \{\mathfrak{z} \in \mathbb{C} : |\mathfrak{z}| < 1\}$, normalized by conditions given by $f(0) = f'(0) = -1$ expressed given by:

$$f(\mathfrak{z}) = \mathfrak{z} + \sum_{k=2}^{\infty} a_k \mathfrak{z}^k . \tag{1.1}$$

Let S denote the subclass of $\mathcal{A}$ comprising functions given in equation (1.1) that are also univalent in $\mathcal{Q}$. As per the Koebe one-quarter theorem (refer to [9]), the image of $\mathcal{Q}$ under any function f belonging to S includes a disk with a radius of $\frac{1}{4}$. Consequently, every function $f \in S$ possesses an inverse, denoted as f^{-1}, which satisfies the relationship

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$$f^{-1}(f(z)) = z,\ (z \in Q)\ and\ f(f^{-1}(\omega)) = \omega.\left(|\omega| < r_0(f), r_0(f) \geq \frac{1}{4}\right).$$

This property ensures that each function in S is not only univalent but also bijectively maps Q onto its image, allowing for the reversal of this mapping within the unit disk, where

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots. \quad (1.2)$$

A function f that belongs to $\mathcal{A}$ is considered bi-univalent in Q provided that both f as well as its inverse f^{-1} are univalent within Q. The class of these bi-univalent functions in Q, represented as Σ and described by form (1.1), includes various noteworthy examples and historical context, which can be explored in detail in [1, 4, 11, 12, 13, 14, 17, 18, 20, 21, 22, 23, 25, 27].

For every function f within the class S, the function $h(z) = \sqrt[m]{f(z^m)}$, where $z \in Q, m \in \mathbb{N}$ is univalent as well as projects the unit disk Q onto a region that exhibits m-fold symmetry. Moreover, a function is described as m-fold symmetric (refer to [15]) provided that it adheres to a specific normalized form given by:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \quad (z \in Q, m \in \mathbb{N}). \quad (1.3)$$

We refer to S_m as the class consisting m-fold symmetric univalent functions within Q, characterized by the series expansion given in equation (1.3). Indeed, the functions belonging to the class S exhibit one-fold symmetry.

Srivastava et al. [26] extended the concept of m-fold symmetric univalent functions to include m-fold symmetric bi-univalent functions. They presented significant findings, noting that every function $f \in \Sigma$ creates an m-fold symmetric bi-univalent function for every $m \in \mathbb{N}$. Additionally, the authors specified that for the normalized form of f, depicted in (1.3), the series expansion for the inverse function f^{-1} is provided given by:

$$g(\omega) = \omega - a_{m+1}\omega^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]\omega^{2m+1}$$
$$-\left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right]\omega^{3m+1} + \cdots, \quad (1.4)$$

in which $f^{-1} = g$. We refer to Σ_m as the class containing m-fold symmetric bi-univalent functions within Q. Moreover, it is straightforward to observe that when $m = 1$, equation (1.4) aligns with equation (1.2) from the Σ class. Moreover, examples of m-fold symmetric bi-univalent functions are provided below:

$$\left(\frac{z^m}{1 - z^m}\right)^{\frac{1}{m}}, \quad \left[\frac{1}{2}\log\left(\frac{1 + z^m}{1 - z^m}\right)\right]^{\frac{1}{m}} \text{ as well as } [-\log(1 - z^m)]^{\frac{1}{m}}$$

having the inverse functions given below:

$$\left(\frac{\omega^m}{1 + \omega^m}\right)^{\frac{1}{m}}, \quad \left(\frac{e^{2\omega^m} - 1}{e^{2\omega^m} + 1}\right)^{\frac{1}{m}} \text{ as well as } \left(\frac{e^{\omega^m} - 1}{e^{\omega^m}}\right)^{\frac{1}{m}},$$

accordingly.

In recent years, several researchers have explored bounds for different subclasses of m-fold bi-univalent functions, as noted in various studies ([2, 3, 5, 6, 8, 10, 16, 19, 24, 26]). The purpose of this study is to present new subclasses $\mathcal{E}\gamma_{\Sigma_m}(\alpha, \lambda, \wp, \delta, \rho)$ as well as $\mathcal{E}\gamma^*_{\Sigma_m}(\beta, \lambda, \wp, \delta, \rho)$ of Σ_m. They also determine estimates for the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions within each of these newly introduced subclasses.

To establish our primary findings, we need to apply the lemma stated below.

Lemma 1.1 [3]. Provided that $h \in \mathcal{P}$. Thus, $|c_k| \leq 2$ for every $k \in \mathbb{N}$, in which $\mathcal{P}$ denotes the family of all $\mathrm{Re}(h(z)) > 0, (z \in Q)$, in which

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots, (z \in Q).$$

2. Coefficient Estimates pertaining the Function Class $\mathcal{E}\gamma_{\Sigma_m}(\alpha, \lambda, \wp, \delta, \rho)$

Definition 2.1. A function $f \in \Sigma_m$ and expressed as in equation (1.3) is considered part of the class $\mathcal{E}\gamma_{\Sigma_m}(\alpha, \lambda, \wp, \delta, \rho)$ if it meets specific conditions. This classification applies when $(m \in \mathbb{N}, \wp \in \mathbb{C}\backslash\{0\}, \lambda \geq 0, 0 < \alpha \leq 1, 0 \leq \delta \leq 1, 0 \leq \rho \leq 1, (z, w) \in \mathcal{Q})$. The function f must fulfill these criteria to be categorized within this subclass.

$$\left|\arg\left[1 + \frac{1}{\wp}\left[\frac{z[f'(z) + \rho z f''(z)]^{\lambda}}{(1-\delta)z + \rho z f'(z) + \delta(1-\rho)f(z)} - 1\right]\right]\right| < \frac{\alpha\pi}{2}, (z \in \mathcal{Q}) \tag{2.1}$$

as well as

$$\left|\arg\left[1 + \frac{1}{\wp}\left[\frac{w[g'(w) + \rho w g''(w)]^{\lambda}}{(1-\delta)w + \rho w g'(w) + \delta(1-\rho)g(w)} - 1\right]\right]\right| < \frac{\alpha\pi}{2}, (w \in \mathcal{Q}), \tag{2.2}$$

in which the function $g = f^{-1}$ is expressed as in equation (1.4).

Specifically, for one-fold symmetric bi-univalent functions, we express the class $\mathcal{E}\gamma_{\Sigma_1}(\alpha, \lambda, \wp, \delta, \rho) = \mathcal{E}\gamma_{\Sigma_m}(\alpha, \lambda, \wp, \delta, \rho)$, indicating that the parameters and characteristics for one-fold symmetry are directly aligned with those defined for m-fold symmetry within the same subclass.

Remark 2.1. By specifying the parameters $\alpha, \lambda, \wp, \delta$and ρ, it is possible to define various new and previously known subclasses of analytic bi-univalent functions that have been explored in earlier studies:

1- In the case of $m = 1$, a new the class of bi-univalent function is introduced given by
$$\mathcal{E}\gamma_{\Sigma_m}(\alpha, \lambda, \wp, \delta, \rho) = \gamma_{\Sigma}(\alpha, \lambda, \wp, \delta, \rho).$$
2- In the case of $\delta = 0$, a new class emerges, encompassing m-fold symmetric bi-starlike functions given by
$$\mathcal{E}\gamma_{\Sigma_m}(\alpha, \lambda, \wp, \delta, \rho) = \gamma_{\Sigma}^{*}(\alpha, \lambda, \wp, \delta, \rho).$$
3- In the case of $\delta = 0$ and $\wp = 1$, a new class is established that includes m-fold symmetric convex bi-univalent functions given by
$$\mathcal{E}\gamma_{\Sigma_m}(\alpha, \lambda, \wp, \delta, \rho) = \mathcal{C}_{\Sigma}^{*}(\alpha, \lambda, \wp, \delta, \rho).$$
4- In the case of $\lambda = \delta = 1$, and $\rho = 0$, a new class containing m-fold symmetric bi-starlike functions, as expressed by Kumar et al. [16], is recognized.
$$\mathcal{E}\gamma_{\Sigma_m}(\alpha, \lambda, \wp, \delta, \rho) = \mathcal{S}_{\Sigma_m}(\alpha, \wp)$$
5- In the case of $\lambda = \delta = \wp = 1$, and $\rho = 0$, we identify a class consisting of bi-univalent functions as described by S. Altinkaya and S. Yalcin [2]
$$\mathcal{E}\gamma_{\Sigma_m}(\alpha, \lambda, \wp, \delta, \rho) = \mathcal{S}_{\Sigma_m}^{\alpha}$$
6- In the case of $\lambda = 1, m = 1, \rho = 0, \delta = 1$ and $\wp = 1$, we recognize a class of bi-univalent functions established by Brannan and Taha [7].
$$\mathcal{E}\gamma_{\Sigma_m}(\alpha, \lambda, \wp, \delta, \rho) = \mathcal{S}_{\Sigma}^{*}(\alpha).$$
7- In the case of $\lambda = \rho = \delta = \wp = 1$, and $m = 1$, a new class that includes convex bi-univalent functions as introduced by Brannan and Taha [7] emerged
$$\mathcal{E}\gamma_{\Sigma_m}(\alpha, \lambda, \wp, \delta, \rho) = \mathcal{S}_{\Sigma_1}(\alpha).$$

Theorem 2.1. Let $f \in \mathcal{E}\gamma_{\Sigma_m}(\alpha, \lambda, \wp, \delta, \rho)$ $(m \in \mathbb{N}, \wp \in \mathbb{C}\backslash\{0\}, \lambda \geq 0, 0 < \alpha \leq 1, 0 \leq \delta \leq 1, 0 \leq \rho \leq 1, (z, w) \in \mathcal{Q})$, be given by (1.3). Then

$$|a_{m+1}| \leq \frac{2\alpha|\wp|}{\sqrt{\alpha\wp\left[\begin{array}{c}(\rho m+1)^2[\lambda(\lambda-1)(1+m)^2 - 2\delta[\lambda(1+m)-\delta](2\rho m+1)(1+m)[\lambda(1+2m)-\delta]] \\ -(\delta-1)[\lambda(1+m)-\delta]^2(\rho m+1)^2\end{array}\right]}}, \tag{2.3}$$

and

$$|a_{2m+1}| \leq \frac{2\wp^2\alpha^2(m+1)}{(\lambda(1+m)-\delta)^2(1+\rho m)^2} + \frac{2\alpha|\wp|}{[\lambda(1+m)-\delta](2\rho m+1)}. \tag{2.4}$$

Proof. Conditions (2.1) as well as (2.2) indicates that

$$1+\frac{1}{\wp}\left[\frac{z[f'(z)+\rho z f''(z)]^\lambda}{(1-\delta)z+\rho z f'(z)+\delta(1-\rho)f(z)}-1\right]=[p(z)]^\alpha \tag{2.5}$$

and

$$1+\frac{1}{\wp}\left[\frac{w[g'(w)+\rho w g''(w)]^\lambda}{(1-\delta)w+\rho w g'(w)+\delta(1-\rho)g(w)}-1\right]=[q(w)]^\alpha, \tag{2.6}$$

in which $g = f^{-1}$ while $p, q \in \mathcal{P}$ possess the series representations given below:

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots \tag{2.7}$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots . \tag{2.8}$$

Comparing the respective coefficients from equations (2.5) as well as (2.6) results in:

$$\frac{[\lambda(1+m)-\delta](\rho m+1)}{\wp}a_{m+1} = \alpha p_m, \tag{2.9}$$

$$\frac{\left\{[(2\rho m+1)(\lambda(1+2m)-\delta)]a_{2m+1}+\frac{(2\rho m+1)^2}{2}[\lambda(\lambda-1)(1+m)^2-2\delta[\lambda(1+m)-\delta]]a_{m+1}^2\right\}}{\wp}$$

$$= \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2}p_m^2, \tag{2.10}$$

and

$$-\frac{[\lambda(1+m)-\delta](\rho m+1)}{\wp}a_{m+1} = \alpha q_m \tag{2.11}$$

$$\frac{\left\{(2\rho m+1)(1+m)[\lambda(1+2m)-\delta]+\frac{(2\rho m+1)^2}{2}[\lambda(\lambda-1)(1+m)^2-2\delta[\lambda(1+m)-\delta]]\right\}a_{m+1}^2 - (2\rho m+1)[\lambda(1+2m)-\delta]a_{2m+1}}{\wp}$$

$$= \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2}q_m^2. \tag{2.12}$$

From use of (2.9) and (2.11), we get

$$p_m = -q_m \tag{2.13}$$

and

$$\frac{2[\lambda(1+m)-\delta]^2(\rho m+1)^2}{\wp^2}a_{m+1}^2 = \alpha^2(p_m^2+q_m^2). \tag{2.14}$$

Also, from (2.10), (2.12) and (2.14), we have we get the next relation

$$a_{m+1}^2=\frac{\wp^2\alpha^2(p_{2m}+q_{2m})}{\wp\alpha[(\rho m+1)^2[\lambda(\lambda-1)(1+m)^2-2\delta(\lambda(1+m)-\delta)]+(2\rho m+1)(1+m)[\lambda(1+m)-\delta]] \\ -(\alpha-1)[\lambda(1+m)-\delta]^2(\rho m+1)^2}. \quad (2.15)$$

By taking the absolute value of equation (2.15) and utilizing Lemma 1.1 to assess the coefficients p_{2m} as well as q_{2m}, we derive the following results:

$$|a_{m+1}|\le\frac{2\alpha|\wp|}{\sqrt{\alpha\wp\left[\begin{array}{c}(\rho m+1)^2[\lambda(\lambda-1)(1+m)^2-2\delta[\lambda(1+m)-\delta](2\rho m+1)(1+m)[\lambda(1+m)-\delta]]\\ -(\alpha-1)[\lambda(1+m)-\delta]^2(\rho m+1)^2\end{array}\right.}}.$$

This process yields the desired estimate for $|a_{m+1}|$ as proposed in equation (2.3). To determine the $|a_{2m+1}|$ bound, we subtract equation (2.12) from equation (2.10), resulting in:

$$\frac{(2\rho m+1)[\lambda(1+2m)-\delta]}{\wp}[2a_{2m+1}-(m+1)a_{m+1}^2]$$

$$=\alpha(p_{2m}-q_{2m})+\frac{\alpha(\alpha-1)}{2}(p_m^2-q_m^2). \quad (2.16)$$

It follows from (2.13), (2.14) and (2.16) that

$$a_{2m+1}=\frac{\wp^2\alpha^2(p_m^2+q_m^2)(m+1)}{4[\lambda(1+2m)-\delta]^2(1+\rho m)^2}+\frac{\wp\alpha(p_{2m}-q_{2m})}{2[\lambda(1+2m)-\delta](2\rho m+1)}. \quad (2.17)$$

By taking the absolute value of equation (2.17) and implementing Lemma 1.1 once more to the coefficients p_m, p_{2m}, q_m as well as q_{2m}, we obtain the necessary bounds for these coefficients.

$$|a_{2m+1}|\le\frac{2\wp^2\alpha^2(m+1)}{[\lambda(1+2m)-\delta]^2(2\rho m+1)}+\frac{2\alpha|\wp|}{[\lambda(1+2m)-\delta](1+\rho m)^2},$$

completing the proof for Theorem 2.1.

Remark 2.2. By selecting the following condition in Theorem 2.1 with $\lambda=1$ and $\delta=1$. Therefore, we arrive at results that align with those provided by Kumar et al. in [16].

For $m=1$, concerning one-fold symmetric bi-univalent functions, Theorem 2.1 simplifies to the following corollary:

Corollary 2.1. Let $f\in\mathcal{E}\gamma_{\Sigma_m}(\alpha,\lambda,\wp,\delta,\rho)(\wp\in\mathbb{C}\backslash\{0\},0<\alpha\le 1,\ \lambda\ge 0,0\le\delta\le 1,0\le\rho\le 1)$ be given by (1.1). Then

$$|a_2|\le\frac{2\alpha|\wp|}{\sqrt{\alpha\wp[(\rho+1)^2[4\lambda(\lambda-1)-4\delta[2\lambda-\delta](2\rho+1)[3\lambda-\delta]]-(\alpha-1)[2\lambda-\delta]^2(\rho+1)^2]}}$$

and

$$|a_3|\le\frac{4\wp^2\alpha^2}{[2\lambda-\delta]^2(1+\rho)^2}+\frac{2\alpha|\wp|}{[3\lambda-\delta](2\rho+1)}.$$

Remark 2.3. In Corollary 2.1, by setting $\lambda=1$ and $\delta=1$, we achieve results that correspond to those established by Kumar et al. in [16, Theorem 2.1].

3. Coefficient Estimates pertaining the Function Class $\mathcal{E}\gamma^{*}_{\Sigma_m}(\beta,\lambda,\wp,\delta,\rho)$

Definition 3.1. A function $f \in \Sigma_m$ and expressed as in (1.3) is classified under $\mathcal{E}\gamma^{*}_{\Sigma_m}(\beta,\lambda,\wp,\delta,\rho)$ if it meets specific criteria. This classification is applicable when ($m \in \mathbb{N}, \wp \in \mathbb{C}\backslash\{0\}, \lambda \geq 0, 0 < \beta \leq 1, 0 \leq \delta \leq 1, 0 \leq \rho \leq 1, (z, w) \in \mathcal{Q}$). The function must fulfill these conditions to be considered part of this subclass.

$$\operatorname{Re}\left[1+\frac{1}{\wp}\left[\frac{z[f'(z)+\rho z f''(z)]^{\lambda}}{(1-\delta)z+\rho z f'(z)+\delta(1-\rho)f(z)}-1\right]\right] > \beta, (z \in \mathcal{Q}) \tag{3.1}$$

and

$$\operatorname{Re}\left[1+\frac{1}{\wp}\left[\frac{w[g'(w)+\rho w g''(w)]^{\lambda}}{(1-\delta)w+\rho w g'(w)+\delta(1-\rho)g(w)}-1\right]\right] > \beta, (w \in \mathcal{Q}), \tag{3.2}$$

in which the function $g = f^{-1}$ is provided by equation (1.4).

Specifically, for $m = 1$ which pertains to one-fold symmetric bi-univalent functions, the class is denoted as $\mathcal{E}\gamma^{*}_{\Sigma_1}(\beta,\lambda,\wp,\delta,\rho) = \mathcal{E}\gamma^{*}_{\Sigma}(\beta,\lambda,\wp,\delta,\rho)$.

Remark 3.1. By specifying the parameters $\beta,\lambda,\wp,\delta$ and ρ it is possible to define various new and previously recognized subclasses of analytic bi-univalent functions that have been examined in earlier research.

1. In the case of $m = 1$, a new the class of bi-univalent function is introduced given by:
$$\mathcal{E}\gamma_{\Sigma_m}(\beta,\lambda,\wp,\delta,\rho) = \mathcal{E}\gamma_{\Sigma}(\beta,\lambda,\wp,\delta,\rho).$$
2. In the case of $\delta = 0$, a new class emerges, encompassing m-fold symmetric bi-starlike functions given by
$$\mathcal{E}\gamma_{\Sigma_m}(\beta,\lambda,\wp,\delta,\rho) = \mathcal{E}\gamma^{*}_{\Sigma}(\beta,\lambda,\wp,\rho).$$
3. In the case of $\delta = 0$ and $\wp = 1$, a new class is established that includes m-fold symmetric convex bi-univalent functions given by
$$\mathcal{E}\gamma_{\Sigma_m}(\beta,\lambda,\wp,\delta,\rho) = \mathcal{E}\mathcal{C}^{*}_{\Sigma}(\beta,\lambda,\rho).$$
4. In the case of $\lambda = \delta = 1$, and $\gamma = 0$, a new class containing m-fold symmetric bi-starlike functions, as expressed by Kumar et al. [16], is recognized.
$$\mathcal{E}\gamma_{\Sigma_m}(\beta,\lambda,\wp,\delta,\rho) = \mathcal{S}_{\Sigma_m}(\beta,\wp).$$
5. In the case of $\lambda = \delta = \wp = 1$, and $\rho = 0$, we identify a class consisting of bi-univalent functions as described by S. Altinkaya and S. Yalcin [2]
$$\mathcal{E}\gamma_{\Sigma_m}(\beta,\lambda,\wp,\delta,\rho) = \mathcal{S}^{\beta}_{\Sigma_m}.$$
6. In the case of $\lambda = \delta = \wp = m = 1$, and $\rho = 0$, we recognize a class of bi-univalent functions established by Brannan and Taha [7].
$$\mathcal{E}\gamma_{\Sigma_m}(\beta,\lambda,\wp,\delta,\rho) = \mathcal{S}^{*}_{\Sigma}(\beta).$$
7. In the case of $\lambda = \rho = \wp = \delta = m = 1$, a new class that includes convex bi-univalent functions as introduced by Brannan and Taha [7] emerged
$$\mathcal{E}\gamma_{\Sigma_m}(\beta,\lambda,\wp,\delta,\rho) = \mathcal{S}_{\Sigma_1}(\beta).$$

Theorem 3.1. Let $f \in \mathcal{E}\gamma^{*}_{\Sigma_m}(\beta,\lambda,\wp,\delta,\rho)(m \in \mathbb{N}, \wp \in \mathbb{C}\backslash\{0\}, \lambda \geq 0, 0 < \beta \leq 1, 0 \leq \delta \leq 1, 0 \leq \rho \leq 1, (z, w) \in \mathcal{Q})$,be given by (1.3). Then

$$|a_{m+1}| \leq \sqrt{\frac{4\wp(1-\beta)}{(\rho m+1)^2[\lambda(\lambda-1)(1+m)^2 - 2\delta[\lambda(1+m)-\delta] + (2\rho m+1)(1+m)[\lambda(1+2m)-\delta]}}. \tag{3.3}$$

and

$$|a_{2m+1}| \le \frac{2(1-\beta)^2\wp^2(m+1)}{(\rho m+1)^2[\lambda(1+2m)-\delta]^2} + \frac{2(1-\beta)|\wp|}{(2\rho m+1)[\lambda(1+2m)-\delta]}. \tag{3.4}$$

Proof. Using relations (3.1) and (3.2) that there exist $p, q \in \mathcal{P}$ such that

$$1+\frac{1}{\wp}\left[\frac{z[f'(z)+\rho z f''(z)]^\lambda}{(1-\delta)z+\rho z f'(z)+\delta(1-\rho)f(z)}-1\right] = \beta+(1-\beta)p(z) \tag{3.5}$$

and

$$1+\frac{1}{\wp}\left[\frac{w[g'(w)+\rho w g''(w)]^\lambda}{(1-\delta)w+\rho w g'(w)+\delta(1-\rho)g(w)}-1\right] = \beta+(1-\beta)q(w), \tag{3.6}$$

in which $p(z)$ as well as $q(w)$ possess the forms given in equation (2.7) and (2.8), accordingly. Matching the coefficients from equations (3.5) and (3.6) results in:

$$\frac{[\lambda(1+m)-\delta](\rho m+1)}{\wp} a_{m+1} = (1-\beta)p_m, \tag{3.7}$$

$$\frac{(2\rho m+1)[\lambda(1+2m)-\delta]a_{2m+1} + \left[\frac{\lambda(\lambda-1)}{2}(1+m)^2-\delta[\lambda(1+m)-\delta]\right]a_{m+1}^2}{\wp} = (1-\beta)p_{2m}, \tag{3.8}$$

$$-\frac{[\lambda(1+m)-\delta](\rho m+1)}{\wp} = (1-\beta)q_m \tag{3.9}$$

and

$$\frac{(2\rho m+1)(1+m)[\lambda(1+2m)-\delta] + (2\rho m+1)^2\left[\frac{\lambda(\lambda-1)}{2}(1+m)^2-\delta[\lambda(1+m)-\delta]\right]a_{m+1}^2 - (2\rho m+1)[\lambda(1+2m)-\delta]a_{2m+1}}{\wp} = (1-\beta)q_{2m}. \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_m = -q_m \tag{3.11}$$

and

$$\frac{2[\lambda(1+m)-\delta]^2(\rho m+1)^2}{\wp^2} a_{m+1}^2 = (1-\beta)^2(p_m^2+q_m^2). \tag{3.12}$$

Upon adding equations (3.8) and (3.10) yields:

$$(2\rho m+1)(1+m)[\lambda(1+2m)-\delta] + 2(2\rho m+1)^2\left[\frac{\lambda(\lambda-1)}{2}(1+m)^2-\delta[\lambda(1+m)-\delta]\right]a_{m+1}^2 = (1-\beta)(p_{2m}+q_{2m}). \tag{3.13}$$

Therefore, we have

$$a_{m+1}^2 = \frac{\wp(1-\beta)(p_{2m}+q_{2m})}{(\rho m+1)^2[\lambda(\lambda-1)(1+m)^2 - 2\delta[\lambda(1+m)-\delta] + (2\rho m+1)(1+m)[\lambda(1+2m)-\delta]]}.$$

Applying Lemma 1.1 for the coefficients of p_{2m} as well as q_{2m} yields

$$|a_{m+1}| \leq \sqrt{\frac{4\wp(1-\beta)}{(\rho m+1)^2[\lambda(\lambda-1)(1+m)^2 - 2\delta[\lambda(1+m)-\delta] + (2\rho m+1)(1+m)[\lambda(1+2m)-\delta]]}}.$$

This yields the desired estimate for $|a_{m+1}|$ given in equation (3.3).

This calculation provides the desired estimate for $|a_{m+1}|$ as specified in equation (3.3).

By subtracting (3.10) from (3.8), we obtain the bound on $|a_{2m+1}|$,

$$\frac{(2\rho m+1)[\lambda(1+2m)-\delta]}{\wp}\{2a_{2m+1}-(m+1)a_{m+1}^2\} = (1-\beta)(p_{2m}-q_{2m}),$$

or equivalently

$$a_{2m+1} = \frac{(m+1)}{2}a_{m+1}^2 + \frac{(1-\beta)\wp(p_{2m}-q_{2m})}{2(2\gamma m+1)[\lambda(1+2m)-\delta]}.$$

From (3.12), we substituting the value of a_{m+1}^2 and get

$$a_{2m+1} = \frac{(1-\beta)^2\wp^2(p_m^2+q_m^2)(m+1)}{4(\rho m+1)^2[\lambda(1+2m)-\delta]^2} + \frac{(1-\beta)\wp(p_{2m}-q_{2m})}{2(2\rho m+1)[\lambda(1+2m)-\delta]}.$$

By applying Lemma 1.1 again to the coefficients p_m, p_{2m}, q_m as well as q_{2m}, we obtain the necessary results for these coefficients.

$$|a_{2m+1}| \leq \frac{2(1-\beta)^2\wp^2(m+1)}{(\rho m+1)^2[\lambda(1+2m)-\delta]^2} + \frac{2(1-\beta)|\wp|}{(2\rho m+1)[\lambda(1+2m)-\delta]},$$

As a result, we complete the proof pertaining to Theorem 3.1, yielding the desired estimate for $|a_{2m+1}|$ in (3.4).

Remark 3.2. In Theorem 3.1, by selecting $\lambda = 1$ and $\delta = 1$, the results align with those reported by Kumar et al. in [16].

For $m = 1$, pertaining to one-fold symmetric bi-univalent functions, Theorem 3.1 simplifies to the corollary given below:

Corollary 3.1. Let $f \in \mathcal{E}\gamma^*_{\Sigma_m}(\beta, \lambda, \wp, \delta, \rho)(\wp \in \mathbb{C}\backslash\{0\}, 0 \leq \beta < 1, \lambda \geq 0, 0 \leq \delta \leq 1, 0 \leq \rho \leq 1)$ be given by (1.1). Then

$$|a_2| \leq \sqrt{\frac{4\wp(1-\beta)}{2(\rho+1)^2[2\lambda(\lambda-1) - \delta[2\lambda-\delta] + 2(2\rho+1)[3\lambda-\delta]]}}.$$

and

$$|a_3| \leq \frac{4(1-\beta)^2\wp^2}{(\rho+1)^2[3\lambda-\delta]^2} + \frac{2(1-\beta)|\wp|}{(2\rho+1)[3\lambda-\delta]}.$$

Remark 3.3. In Theorem 3.1, setting $\lambda = 1$ and $\delta = 1$ yields result consistent with those presented by Kumar et al. in [16].

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