Bounds on the Initial Coefficients for New Subclasses of m-Fold Symmetric Bi-univalent Functions

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\textbf{ARTICLE INFO}

\textbf{Article history:}
Received: 29/5/2024
Revised form: 21/6/2024
Accepted: 23/6/2024
Available online: 30/6/24

\textbf{Keywords:}
Analytic function, Coefficient bounds, m-Fold symmetric analytic function, Bi-univalent function, m-Fold symmetric analytic biunivalent function.

\textbf{ABSTRACT}

The aim from this study is to propose and explore two new classes \( \mathcal{BM}_m(\lambda, \delta; \alpha) \) and \( \mathcal{BM}_m(\lambda, \delta; \beta) \) of \( \mathcal{S}_m \) made up of bi-univalent functions that are \( m \)-fold symmetric and analytic which are defined within the unit disk \( \mathbb{U} \) that is open. For each functions that correspond to these subclasses we derive upper bounds for the coefficients \( |d_{m+1}| \) and \( |d_{2m+1}| \). Many of the newly discovered and well-known outcomes are demonstrated to be special cases of our findings.

\textbf{MSC.}

https://doi.org/10.29304/jqcsm.2024.16.21551

1. Introduction

Suppose that \( \mathcal{A} = \{ k : \mathbb{U} \rightarrow \mathbb{C} : k \text{ is analytic within } \mathbb{U}, \ k(0) = k'(0) - 1 = 0 \} \) is the class of the form’s functions

\[ k(s) = s + \sum_{n=2}^{\infty} d_n s^n \quad (1.1) \]

and let \( \mathcal{S} \) be a subclass of \( \mathcal{A} \) that include every \( k \) univalent functions in \( \mathbb{U} \). Every function \( k \) in \( \mathcal{S} \) has an inverse \( k^{-1} \), which makes sure that the Koebe one quarter theorem (see to [4]) satisfies

\[ k^{-1}(k(s)) = s, \quad (s \in \mathbb{U}) \]

and

\[ k(k^{-1}(r)) = r, \quad (|r| < r_0(k), r_0(k) \geq 1/4). \]
In fact, the analytic expansion from \( k^{-1} \) to \( \mathbb{U} \) is
\[
k^{-1}(r) = h(r) = r - d_2 r^2 + (2d_2^2 - d_3) r^3 - (5d_2^3 - 5d_2 d_3 + d_4) r^4 + \ldots.
\] (1.2)

Within \( \mathbb{U} \), if \((k^{-1} \text{ and } k)\) are both univalent, then the function \( k \in \mathcal{A} \) is considered to be biunivalent in \( \mathbb{U} \). The class of biunivalent functions in \( \mathbb{U} \) provided based on (1.1) is denoted by \( \Sigma \). See [18] for a abbreviated background and engaging instances from class \( \Sigma \) (also look to [5, 6, 15, 16, 17]).

In \( \mathcal{S} \), for any function \( k \), following
\[
f(s) = \sqrt[m]{k(s^m)} \quad (m \in \mathbb{N}, s \in \mathbb{U}),
\]
is a function that’s univalent and transfers the unit disc \( \mathbb{U} \) to an area with \( m \)-fold symmetric. If the function has the normalized form given below, it is considered \( m \)-fold symmetric (see to [8]).

\[
k(s) = s + \sum_{n=1}^{\infty} d_{mn+1} s^{mn+1} \quad (s \in \mathbb{U}, m \in \mathbb{N}).
\] (1.3)

The class of \( m \)-fold symmetric univalent functions in \( \mathbb{U} \) is denoted by \( \Sigma_m \), and they are normalized by the series expansion (1.3). Actually, the class \( \mathcal{S} \) functions are one-fold symmetric.

“In [20] Srivastava et al. defined \( m \)-fold symmetric biunivalent functions analogues to the concept of \( m \)-fold symmetric univalent functions. They gave some important results such as each function \( k \in \Sigma \) generates an \( m \)-fold symmetric biunivalent function for each \( m \in \mathbb{N} \). Furthermore, for the normalized form of \( k(s) \) given by (1.3), they obtained the series expansion for \( k^{-1} \) as follows:"

\[
h(r) = r - d_{m+1} r^{m+1} + [(m+1)d_{m+1}^2 - d_{2m+1}] r^{2m+1} - \frac{1}{2} (2 + 3m)(1 + m)d_{m+1}^3 - (2 + 3m)d_{m+1} d_{2m+1} + d_{3m+1} r^{3m+1} + \ldots,
\] (1.4)

wherever \( k^{-1} = h \). The class of biunivalent \( m \)-fold symmetric function in \( \mathbb{U} \) is indicated by \( \Sigma_m \). It’s clear that the formula (1.4) and the formula (1.2) of the class \( \Sigma \) coincide for \( m = 1 \). The following some illustrations of biunivalent functions that are \( m \)-fold symmetry:

\[
(-\log(1 - s^m))^\frac{1}{m}, \quad \left(\frac{s^m}{1 - s^m}\right)^\frac{1}{m} \quad \text{and} \quad \left(\frac{1}{2} \log\left(\frac{s^m + 1}{1 - s^m}\right)\right)^\frac{1}{m},
\]
together with their respective inverse functions

\[
\left(\frac{e^{r^m} - 1}{e^{r^m}}\right)^\frac{1}{m}, \quad \left(\frac{r^m}{1 + r^m}\right)^\frac{1}{m} \quad \text{and} \quad \left(\frac{e^{2r^m} - 1}{e^{2r^m} + 1}\right)^\frac{1}{m},
\]
respectively. Estimates for different classes of analytic bi-univalent functions that are \( m \)-fold symmetric have recently been inverted by many penmen (see to [3,13,21,7,1,19]).

In this work, two new subclasses of function class \( \Sigma_m \) will be formed, and find estimates of initial coefficients for the functions in these new subclasses, \(|b_{m+1}|\) and \(|d_{2m+1}|\), will be obtained. Links to previously published results are established, and several related classes are also found.

We need the following lemma to obtain our major conclusions.

“Lemma (1.1) [4]: If \( h \in \mathcal{P} \), then
where the Carathéodary class \( \mathcal{P} \) is the family of all functions \( h \), analytic in \( \mathbb{U} \), for which
\[
Re\{h(s)\} > 0, \quad (s \in \mathbb{U})
\]
and
\[
h(s) = 1 + c_1 s + c_2 s^2 + \cdots \quad (s \in \mathbb{U}),
\]
the extremal function being given by
\[
h(s) = \frac{1 + s}{1 - s}, \quad (s \in \mathbb{U}).
\]

### 2. Bounds on the coefficients for the function subclass \( \mathbb{B}\mathcal{M}_{\Sigma_m}(\lambda, \delta; \alpha) \)

**Definition 2.1.** A function \( k(s) \in \Sigma_m \), specified based on (1.3), is said being in the subclass \( \mathbb{B}\mathcal{M}_{\Sigma_m}(\delta, \lambda; \alpha) \) for \( \delta, \lambda \geq 0 \) and \( 0 < \alpha \leq 1 \), if the following conditions are satisfied:

\[
\left| \arg\left( \frac{(k(s)}{s} \right)^{\delta} \frac{k'(s)}{k(s)} + \lambda \left[ 1 + \frac{sk''(s)}{k'(s)} + \delta \left( \frac{sk'(s)}{k(s)} - 1 \right) - \frac{sk'(s)}{k(s)} \right] \right| < \frac{\alpha \pi}{2} \quad (s \in \mathbb{U}) \tag{2.1}
\]
and
\[
\left| \arg\left( \frac{(h(r)}{r} \right)^{\delta} \frac{h'(r)}{h(r)} + \lambda \left[ 1 + \frac{rh''(r)}{h'(r)} + \delta \left( \frac{rh'(r)}{h(r)} - 1 \right) - \frac{rh'(r)}{h(r)} \right] \right| < \frac{\alpha \pi}{2} \quad (r \in \mathbb{U}), \tag{2.2}
\]
whereby (1.4) gives the function \( h(r) \).

**Theorem 2.1.** Let \( k \in \mathbb{B}\mathcal{M}_{\Sigma_m}(\lambda, \delta; \alpha) \), \( s, r \in \mathbb{U}, \lambda \geq 0, \delta \geq 0, 0 < \alpha \leq 1 \) and \( m \in \mathbb{N} \), be specified based on (1.3). Then
\[
|d_{m+1}| \leq \frac{2 \alpha}{\sqrt{\alpha(2 \lambda m^3 + \delta^2 + 3 \delta m + 2 m^2 + 2 \lambda \delta m^2) + (1 - \alpha)(\delta + m)^2(1 + \lambda m)^2}} \tag{2.3}
\]
and
\[
|d_{2m+1}| \leq \frac{2 \alpha}{\delta + 2m}(2m \lambda + 1) + \frac{2 \alpha^2 (m + 1)}{(\delta + m)^2 (1 + \lambda m)^2}. \tag{2.4}
\]

**Proof.** From (2.1 and 2.2), we get
\[
\left( \frac{k(s)}{s} \right)^{\delta} \frac{k'(s)}{k(s)} + \lambda \left[ 1 + \frac{sk''(s)}{k'(s)} + \delta \left( \frac{sk'(s)}{k(s)} - 1 \right) - \frac{sk'(s)}{k(s)} \right] = [p(s)]^a \tag{2.5}
\]
and
\[
\left( \frac{h(r)}{r} \right)^{\delta} \frac{h'(r)}{h(r)} + \lambda \left[ 1 + \frac{rh''(r)}{h'(r)} + \delta \left( \frac{rh'(r)}{h(r)} - 1 \right) - \frac{rh'(r)}{h(r)} \right] = [q(r)]^a, \tag{2.6}
\]
the functions \( p(s) \) and \( q(r) \), which belong to class \( \mathcal{P} \), possess the depictions of the series shown below:
\[
p(s) = 1 + p_m s^m + p_{2m} s^{2m} + p_{3m} s^{3m} + \cdots \tag{2.7}
\]
and
\[ q(r) = 1 + q_m r^m + q_{2m} r^{2m} + q_{3m} r^{3m} + \ldots. \quad (2.8) \]

In (2.5 and 2.6), the coefficients are equated, therefore we find that
\[ (\delta m + 1)(\alpha + m) d_{m+1} = \alpha p_m, \quad (2.9) \]
\[ (\delta + 2m)(1 + 2\lambda m)d_{2m+1} + \left( \delta m - \lambda m^3 - m + \frac{(\delta^2 - \delta)}{2} - 2\lambda m^2 - \delta \lambda m \right) d_{m+1}^2 = \alpha p_{2m} + \frac{\alpha(\alpha - 1)}{2} p_m^2, \quad (2.10) \]
and
\[ - (\delta + m)(1 + \lambda m)d_{m+1} = \alpha q_m, \quad (2.11) \]

Additionally, a quick calculation based on (2.10), (2.12) and (2.14) reveals that
\[ (2\delta m + \frac{(\delta^2 - \delta)}{2} + 2m^2 + m + 3\lambda m^2 + 2\lambda m^2 + \delta + 2\delta m^2 + \lambda \delta m) d_{m+1}^2 - (\delta + 2m)(1 + 2\lambda m)d_{2m+1} \]
\[ = \alpha q_{2m} + \frac{\alpha(\alpha - 1)}{2} q_m^2. \quad (2.12) \]

Using (2.9 and 2.11), we have
\[ p_m = -q_m \quad (2.13) \]
and
\[ 2(1 + \lambda m)^2(m + \delta)d_{m+1}^3 = \alpha^2 (p_m^2 + q_m^2). \quad (2.14) \]

Additionally, a quick calculation based on (2.10), (2.12) and (2.14) reveals that
\[ \left( 2\lambda m^3 + \delta^2 + 3\delta m + 2m^2 + 2m^2 \lambda \delta \right) d_{m+1}^2 = \alpha (p_{2m} + q_{2m}) + \frac{\alpha(\alpha - 1)}{2} (p_m^2 + q_m^2) \]
\[ = \alpha (p_{2m} + q_{2m}) + (\alpha - 1) \frac{(m + \delta)^2(1 + \lambda m)^2}{\alpha} d_{m+1}^2. \]

Therefore, we have
\[ d_{m+1}^2 = \frac{\alpha^2 (p_{2m} + q_{2m})}{\alpha (2\lambda m^3 + \delta^2 + 3\delta m + 2m^2 + 2m^2 \lambda \delta) + (1 - \alpha)(m + \delta)^2(1 + \lambda m)^2}. \]

Lemma (1.1) applied to the coefficient \( p_{2m} \) and \( q_{2m} \) yields
\[ |d_{m+1}| \leq \frac{2\alpha}{\sqrt{\alpha (2\lambda m^3 + \delta^2 + 3\delta m + 2m^2 + 2m^2 \lambda \delta) + (1 - \alpha)(m + \delta)^2(\lambda m + 1)^2}}. \]

As stated in (2.3), this provides the required estimate for \(|d_{m+1}|\).

Next, by deducting (2.12) from (2.10), we can establish the bound on \(|d_{2m+1}|\)
\[ 2((1 + 2\lambda m)(\delta + 2m))d_{2m+1} - (4\lambda m^2 + 2m + \delta m + 2\delta \lambda m + 4\lambda m^3 + 2m^2 + 2\lambda \delta m^2 + \delta) d_{m+1}^2 \]
\[ = \alpha (p_{2m} - q_{2m}) + \frac{\alpha(\alpha - 1)}{2} (p_m^2 - q_m^2). \quad (2.15) \]

By a simple computation, and using (2.13) to (2.15), we get
Consequently, we see that by reapplying Lemma (1.1) to the coefficients $p_m, p_{2m}, q_m$ and $q_{2m}$, we get

$$|d_{2m+1}| \leq \frac{2\alpha}{(\delta + 2m)(2\lambda m + 1)} + \frac{2\alpha^2(1 + m)}{((\delta + m)(1 + \lambda m))^2}.$$

This concludes the theorem (2.1) proof.

3. Bounds on the coefficients for the function subclass $\mathcal{BM}_\Sigma_m(\lambda, \delta; \beta)$

**Definition 3.1.** A function $k(s) \in \Sigma_m$, specified based on (1.4), is said being in the subclass $\mathcal{BM}_\Sigma_m(\lambda, \delta; \beta)$ for $\delta, \lambda \geq 0$ and $0 \leq \beta < 1$, if the following condition are satisfied:

$$\operatorname{Re}\left(\left(\frac{k(s)}{s}\right)^\delta \frac{sk'(s)}{k(s)} + \lambda \left[1 + \frac{sk''(s)}{k'(s)} + \delta \left(\frac{sk'(s)}{k(s)} - 1\right) - \frac{sk'(s)}{k(s)}\right]\right) > \beta \quad (s \in \mathbb{U}) \quad (3.1)$$

and

$$\operatorname{Re}\left(\left(\frac{h(r)}{r}\right)^\delta \frac{rh'(r)}{h(r)} + \lambda \left[1 + \frac{rh''(r)}{h'(r)} + \delta \left(\frac{rh'(r)}{h(r)} - 1\right) - \frac{rh'(r)}{h(r)}\right]\right) > \beta \quad (r \in \mathbb{U}), \quad (3.2)$$

whereby (1.4) gives the function $h(r)$.

**Theorem 3.1.** Let $k \in \mathcal{BM}_\Sigma_m(\lambda, \delta; \beta), (s, r \in \mathbb{U}, \lambda \geq 0, \delta \geq 0, 0 \leq \beta < 1$ and $m \in \mathbb{N}$) be specified based on (1.4). Then

$$|d_{m+1}| \leq 2 \frac{1 - \beta}{\sqrt{2\lambda m^3 + \delta^2 + 3\delta m + 2m^2 + 2m^2 \lambda \delta}} \tag{3.3}$$

and

$$|d_{2m+1}| \leq \frac{2(1 - \beta)}{(\delta + 2m)(2\lambda m + 1)} + \frac{2(m + 1)(1 - \beta)^2}{((\delta + m)(\lambda m + 1))^2}. \tag{3.4}$$

**Proof.** Initially, the argument inequalities in equations (3.1) and (3.2) can be expressed in the following ways.

$$\left(\frac{k(s)}{s}\right)^\delta \frac{sk'(s)}{k(s)} + \lambda \left[1 + \frac{sk''(s)}{k'(s)} + \delta \left(\frac{sk'(s)}{k(s)} - 1\right) - \frac{sk'(s)}{k(s)}\right] = \beta + (1 - \beta)p(s) \tag{3.5}$$

and

$$\left(\frac{h(r)}{r}\right)^\delta \frac{rh'(r)}{h(r)} + \lambda \left[1 + \frac{rh''(r)}{h'(r)} + \delta \left(\frac{rh'(r)}{h(r)} - 1\right) - \frac{rh'(r)}{h(r)}\right] = \beta + (1 - \beta)q(r), \tag{3.6}$$

where the forms of the functions $p(s)$ and $q(r)$, which belong to class $\mathcal{P}$, are (2.7 and 2.8), respectively. Now, by equating the coefficient in (3.5) and (3.6), just as in the Theorem (2.1) proof, we have

$$(1 + \lambda m)(\delta + m)d_{m+1} = (1 - \beta)p_m. \tag{3.7}$$
Using (3.7) and (3.9), we obtain
\[ p_m = -q_m \quad (3.11) \]
and
\[ 2(1 + \lambda m)(\delta + m)^2d_{m+1}^2 = (1 - \beta)^2(p_m^2 + q_m^2). \quad (3.12) \]

Also, from (3.8) and (3.10), we obtain
\[ (2\lambda m^3 + \delta^2 + 3\delta m + 2m^2 + 2m^2\lambda\delta)d_{m+1}^2 = (1 - \beta)(p_{2m} + q_{2m}). \]

Thus, clearly, we have
\[ |d_{m+1}|^2 \leq \frac{(1 - \beta)(p_{2m} + |q_{2m}|)}{2\lambda m^3 + \delta^2 + 3\delta m + 2m^2 + 2m^2\lambda\delta} \leq \frac{4(1 - \beta)}{2\lambda m^3 + \delta^2 + 3\delta m + 2m^2 + 2m^2\lambda\delta}. \]

As stated in (3.3), this provides the required estimate for $|d_{m+1}|$.

Next, by deducting (3.10) from (3.8), we can establish the bound on $|d_{2m+1}|$
\[ 2(1 + 2\lambda m)(\delta + 2m)d_{2m+1} - ((1 + m)(2\lambda m + 1)(2m + \delta))d_{m+1}^2 = (1 - \beta)(p_{2m} - q_{2m}). \]

Alternatively, equivalently
\[ d_{2m+1} = \frac{(1 - \beta)(p_{2m} - q_{2m})}{2(1 + 2\lambda m)(\delta + 2m)} + \frac{(1 + m)}{2}q_{m+1}^2. \quad (3.13) \]

When we replace the value of $d_{m+1}^2$ in (3.12), we obtain
\[ d_{2m+1} = \frac{(p_{2m} - q_{2m})(1 - \beta)}{2(\delta + 2m)(1 + 2\lambda m)} + \frac{(1 - \beta)^2(m + 1)(p_m^2 + q_m^2)}{4((\delta + m)(1 + \lambda m))^2}. \quad (3.14) \]

Consequently, we see that by reapplying Lemma (1.1) to the coefficient $p_m, p_{2m}, q_m$ and $q_{2m}$, we get
\[ |d_{2m+1}| \leq \frac{2(1 - \beta)}{(2\lambda m + 1)(\delta + 2m)} + \frac{2(1 - \beta)^2(m + 1)}{((1 + \lambda m)(\delta + m))^2}. \quad (3.15) \]

This concludes the theorem (3.1) proof.
4. Consequences and Corollaries

The subclasses $\mathcal{BM}_{\Sigma_m}(\lambda, \delta; \alpha)$ and $\mathcal{BM}_{\Sigma_m}(\lambda, \delta; \beta)$ reduce to the subclasses $\mathcal{BM}_{\Sigma_m}(\delta; \alpha)$ and $\mathcal{BM}_{\Sigma_m}(\delta; \beta)$, respectively, if $\lambda = 0$ is entered in Definitions (2.1) and (3.1). As a result, Theorems (2.1 and 3.1) reduce to Corollaries (4.1 and 4.2), respectively.

The subclasses $\mathcal{BM}_{\Sigma_m}(\delta; \alpha)$ and $\mathcal{BM}_{\Sigma_m}(\delta; \beta)$ have the following definitions:

**Definition 4.1.** A function $k(s) \in \Sigma_m$ specified based on (1.3) is said to belong to the subclass $\mathcal{BM}_x(\delta; \alpha)$ for $\delta \geq 0$ and $0 < \alpha \leq 1$, if each of the following requirements is met:

$$\left| \arg \left( \frac{k(s)}{s} \right)^\delta \right| \frac{s k'(s)}{k(s)} \left| s \in \mathbb{U} \right.$$

and

$$\left| \arg \left( \frac{h(r)}{r} \right)^\delta \right| \frac{r h'(r)}{h(r)} \left| r \in \mathbb{U}, \right.$$  

whereby (1.4) gives the function $h(r)$.

**Definition 4.2.** A function $k(s) \in \Sigma_m$ specified based on (1.3) is said to belong to the subclass $\mathcal{BM}_x(\delta; \beta)$ for $\delta \geq 0$ and $0 \leq \beta < 1$, if each of the following requirements is met:

$$Re \left( \frac{k(s)}{s} \right)^\delta \frac{s k'(s)}{k(s)} > \beta$$  

and

$$Re \left( \frac{h(r)}{r} \right)^\delta \frac{r h'(r)}{h(r)} > \beta$$  

whereby (1.4) gives the function $h(r)$.

**Corollary 4.1.** Assume that $k(s) \in \Sigma_m$, as shown based on (1.3), belongs to the class $\mathcal{BM}_x(\delta; \alpha)$. Then

$$|d_{m+1}| \leq \frac{2\alpha}{\sqrt{\alpha(\delta^2 + 3\delta m + 2m^2) + (1 - \alpha)(\delta + m)^2}}$$  

and

$$|d_{2m+1}| \leq \frac{2\alpha}{(\delta + 2m)} + \frac{2\alpha^2(1 + m)}{(\delta + m)^2}.$$

**Corollary 4.2.** Assume that $k(s) \in \Sigma_m$, as shown based on (1.3), belongs to the class $\mathcal{BM}_x(\delta; \alpha)$. Then

$$|d_{m+1}| \leq 2 \sqrt{\frac{1 - \beta}{\delta^2 + 3\delta m + 2m^2}}$$  

and

$$|d_{2m+1}| \leq \frac{2(1 - \beta)}{(\delta + 2m)} + \frac{2(m + 1)(1 - \beta)^2}{(\delta + m)^2}.$$
The subclasses $\mathcal{B}_m(\lambda, \delta; \alpha)$ and $\mathcal{B}_m^{(m)}(\lambda, \delta; \beta)$ reduce to the subclasses $\mathcal{B}_E(\lambda, \delta; \alpha)$ and $\mathcal{B}_E(\lambda, \delta; \beta)$, respectively, for one-fold symmetric analytic biunivalent functions. As a result, Theorems (2.1 and 3.1) reduce to Corollaries (4.3 and 4.4), respectively.

The subclasses $\mathcal{B}_E(\lambda, \delta; \alpha)$ and $\mathcal{B}_E(\lambda, \delta; \beta)$ have the following definitions:

**Definition 4.3.** A function $k(s) \in \Sigma$ specified based on (1.1) is said belong to the subclass $\mathcal{B}_E(\lambda, \delta; \alpha)$ for $\delta, \lambda \geq 0$ and $0 < \alpha \leq 1$, if each of the following requirements is met:

$$\left| \arg \left( \frac{(k(s))}{s} \right) \frac{k'(s)}{k(s)} + \lambda \left( 1 + \frac{k''(s)}{k'(s)} + \delta \left( \frac{k'(s)}{k(s)} - 1 \right) - \frac{k'(s)}{k(s)} \right) \right| < \frac{\alpha \pi}{2} \quad (s \in \mathbb{U})$$

and

$$\left| \arg \left( \frac{h(r)}{r} \right) \frac{h'(r)}{h(r)} + \lambda \left( 1 + \frac{h''(r)}{h'(r)} + \delta \left( \frac{h'(r)}{h(r)} - 1 \right) - \frac{h'(r)}{h(r)} \right) \right| < \frac{\alpha \pi}{2} \quad (r \in \mathbb{U}),$$

whereby (1.2) gives the function $h(r)$.

**Definition 4.4.** A function $k(s) \in \Sigma$ specified based on (1.1) is said belong to the subclass $\mathcal{B}_E(\lambda, \delta; \beta)$ for $\delta, \lambda \geq 0$ and $0 \leq \beta < 1$, if each of the following requirements is met:

$$\text{Re} \left\{ \frac{(k(s))}{s} \frac{k'(s)}{k(s)} + \lambda \left( 1 + \frac{k''(s)}{k'(s)} + \delta \left( \frac{k'(s)}{k(s)} - 1 \right) - \frac{k'(s)}{k(s)} \right) \right\} > \beta \quad (s \in \mathbb{U})$$

and

$$\text{Re} \left\{ \frac{h(r)}{r} \frac{h'(r)}{h(r)} + \lambda \left( 1 + \frac{h''(r)}{h'(r)} + \delta \left( \frac{h'(r)}{h(r)} - 1 \right) - \frac{h'(r)}{h(r)} \right) \right\} > \beta \quad (r \in \mathbb{U}),$$

whereby (1.2) gives the function $h(r)$.

**Corollary 4.3.** Assume that $k(s)$ as shown based on (1.1), belongs to the class $\mathcal{B}_E(\lambda, \delta; \alpha)$. Then

$$|d_2| \leq \frac{2\alpha}{\sqrt{\alpha(2\lambda + \delta^2 + 3\delta + 2 + 2\lambda\delta) + (1 - \alpha)(\delta + 1)^2(1 + \lambda)^2}}$$

and

$$|d_3| \leq \frac{2\alpha}{(\delta + 2)(2\lambda + 1)} + \frac{4\alpha^2}{(\delta + 1)^2(1 + \lambda)^2}.$$

**Corollary 4.4.** Assume that $k(s)$ as shown based on (1.1), belongs to the class $\mathcal{B}_E(\lambda, \delta; \beta)$. Then

$$|d_2| \leq 2 \left\{ \frac{1 - \beta}{\sqrt{(2\lambda + \delta^2 + 3\delta + 2 + 2\lambda\delta)}} \right\}$$

and

$$|d_3| \leq \frac{2(1 - \beta)}{(2 + \delta)(2\lambda + 1)} + \frac{4(1 - \beta)^2}{(1 + \delta)^2(1 + \lambda)^2}.$$

**Remark 4.1.** For m-fold symmetric biunivalent functions:

1) If we make $\lambda = 0$ and $\delta = 0$ in Th. (2.1) and Th. (3.1), we acquire the matching outcomes provided by Altinkaya and Yalcin [2].
2) If we make $\lambda = 1$ and $\delta = 0$ in Th. (2.1) and Th. (3.1), we acquire the matching outcomes provided by Kumar et al. [9].

3) If we make $\lambda = 0$ and $\delta = 1$ in Th. (2.1) and Th. (3.1), we acquire the matching outcomes provided by Srivastava et al. [20].

4) If we make $\delta = 0$ in Th. (2.1) and Th. (3.1), we acquire the matching outcomes provided by Sivasubramanian and Sivakumar [14].

**Remark 4.2.** For one-fold symmetric biunivalent functions:

1) If we make $\delta = 0$ in Th. (2.1) and Th. (3.1), we acquire the matching outcomes provided by Li and Wang [10].

2) If we make $\lambda = 0$ in Th. (2.1) and Th. (3.1), we acquire the matching outcomes provided by Prima and Keerthi [12].

3) If we make $\delta = 0, \lambda = 0$ in Th. (2.1) and Th. (3.1), we acquire the matching outcomes provided by Murugusundaramoorthy et al. [11].

4) If we make $\delta = 0, \lambda = 1$ in Th. (2.1) and Th. (3.1), we acquire the matching outcomes provided by Kumar et al. [9].

5) If we make $\delta = 1, \lambda = 0$ in Th. (2.1) and Th. (3.1), we acquire the matching outcomes provided by Srivastava et al. [18].

**References**


