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Bounds on the Initial Coefficients for New Subclasses of m-Fold Symmetric Bi-univalent Functions

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ABSTRACT

The aim from this study is to propose and explore two new classes $\mathfrak{BM}_{\Sigma_m}(\lambda,\delta;\alpha)$ and $\mathfrak{BM}_{\Sigma_m}(\lambda,\delta;\beta)$ of Σ_m made up of bi-univalent functions that are m-fold symmetric and analytic which are defined within the unit disk $\mathbb{U}$ that is open. For each functions that correspond to these subclasses we derive upper bounds for the coefficients $|d_{m+1}|$ and $|d_{2m+1}|$. Many of the newly discovered and well-known outcomes are demonstrated to be special cases of our findings.

MSC..

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1. Introduction

Suppose that $\mathcal{A} = \{k : \mathbb{U} \to C : k$ is anlytic within $\mathbb{U}$, $k(0) = k'(0) - 1 = 0\}$ is the class of the form's functions

$$k(s) = s + \sum_{n=2}^{\infty} d_n s^n \tag{1.1}$$

and let $\mathcal{S}$ be a subclass of $\mathcal{A}$ that include every k univalent functions in $\mathbb{U}$. Every function k in $\mathcal{S}$ has an inverse k^{-1}, which makes sure that the Koebe one quarter theorem (see to [4]) satisfies

$$k^{-1}(k(s)) = s, \qquad (s \in \mathbb{U})$$

and

$$k(k^{-1}(r)) = r, \qquad (|r| < r_0(k), r_0(k) \geq 1/4).$$

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In fact, the analytic expansion from k^{-1} to $\mathbb{U}$ is

$$k^{-1}(r) = h(r) = r - d_2 r^2 + (2d_2^2 - d_3)r^3 - (5d_2^3 - 5d_2 d_3 + d_4)r^4 + \cdots. \tag{1.2}$$

Within $\mathbb{U}$, if (k^{-1} and k) are both univalent, then the function $k \in \mathcal{A}$ is considered to be biunivalent in $\mathbb{U}$. The class of biunivalent functions in $\mathbb{U}$ provided based on (1.1) is denoted by Σ. See [18] for a abbreviated background and engaging instances from class Σ (also look to [5, 6, 15, 16, 17]).

In $\mathcal{S}$, for any function k, following

$$f(s) = \sqrt[m]{k(s^m)} \qquad (m \in \mathbb{N}, s \in \mathbb{U}),$$

is a function that's univalent and transfers the unit disc $\mathbb{U}$ to an area with m-fold symmetric. If the function has the normalized form given below, it is considered m-fold symmetric (see to [8]).

$$k(s) = s + \sum_{n=1}^{\infty} d_{mn+1} s^{mn+1} \qquad (s \in \mathbb{U}, m \in \mathbb{N}). \tag{1.3}$$

The class of m-fold symmetric univalent functions in $\mathbb{U}$ is denoted by Σ_m, and they are normalized by the series expansion (1.3). Actually, the class $\mathcal{S}$ functions are one-fold symmetric.

"In [20] Srivastava *et al.* defined m-fold symmetric biunivalent functions analogues to the concept of m-fold symmetric univalent functions. They gave some important results such as each function $k \in \Sigma$ generates an m-fold symmetric biunivalent function for each $m \in \mathbb{N}$. Furthermore, for the normalized form of $k(s)$ given by (1.3), they obtained the series expansion for k^{-1} as follows:"

$$h(r) = r - d_{m+1}r^{m+1} + [(m+1)d_{m+1}^2 - d_{2m+1}]r^{2m+1}$$
$$- \left[\frac{1}{2}(2+3m)(1+m)d_{m+1}^3 - (2+3m)d_{m+1}d_{2m+1} + d_{3m+1}\right]r^{3m+1} + \cdots, \tag{1.4}$$

wherever $k^{-1} = h$. The class of biunivalent m-fold symmetric function in $\mathbb{U}$ is indicated by Σ_m. It's clear that the formula (1.4) and the formula (1.2) of the class Σ coincide for $m = 1$. The following some illustrations of biunivalent functions that are m-fold symmetry:

$$(-log(1-s^m))^{\frac{1}{m}}, \qquad \left(\frac{s^m}{1-s^m}\right)^{\frac{1}{m}} \qquad \text{and} \qquad \left({}^{1}\!/_{2}\, log\left(\frac{s^m+1}{1-s^m}\right)\right)^{\frac{1}{m}},$$

together with their respective inverse functions

$$\left(\frac{e^{r^m}-1}{e^{r^m}}\right)^{\frac{1}{m}}, \qquad \left(\frac{r^m}{1+r^m}\right)^{\frac{1}{m}} \qquad \text{and} \qquad \left(\frac{e^{2r^m}-1}{e^{2r^m}+1}\right)^{\frac{1}{m}},$$

respectively. Estimates for different classes of analytic bi-univalent functions that are m-fold symmetric have recently been inverted by many penmen (see to [3,13,21,7,1,19]).

In this work, two new subclasses of function class Σ_m will be formed, and find estimates of initial coefficients for the functions in these new subclasses, $|b_{m+1}|$ and $|b_{2m+1}|$, will be obtained. Links to previously published results are established, and several related classes are also found.

We need the following lemma to obtain our major conclusions.

"Lemma (1.1) [4]: If $h \in \mathcal{P}$, then

$$|c_n| \leq 2 \text{ for each } n \in \mathbb{N},$$

where the Carathéodary class $\mathcal{P}$ is the family of all functions h, analytic in $\mathbb{U}$, for which

$$Re\{h(s)\} > 0, \qquad (s \in \mathbb{U})$$

and

$$h(s) = 1 + c_1 s + c_2 s^2 + \cdots \qquad (s \in \mathbb{U}),$$

the extremal function being given by

$$h(s) = \frac{1+s}{1-s}, \qquad (s \in \mathbb{U})."$$

2. Bounds on the coefficients for the function subclass $\mathfrak{B}\mathcal{M}_{\Sigma_m}(\lambda, \delta; \alpha)$

Definition 2.1. A function $k(s) \in \Sigma_m$, specified based on (1.3), is said being in the subclass $\mathfrak{B}\mathcal{M}_{\Sigma_m}(\delta, \lambda; \alpha)$ for $\delta, \lambda \geq 0$ and $0 < \alpha \leq 1$, if the following conditions are satisfied:

$$\left| arg\left(\left(\frac{k(s)}{s}\right)^{\delta} \frac{sk'(s)}{k(s)} + \lambda\left[1 + \frac{sk''(s)}{k'(s)} + \delta\left(\frac{sk'(s)}{k(s)} - 1\right) - \frac{sk'(s)}{k(s)}\right]\right)\right| < \frac{\alpha\pi}{2} \qquad (s \in \mathbb{U}) \tag{2.1}$$

and

$$\left| arg\left(\left(\frac{h(r)}{r}\right)^{\delta} \frac{rh'(r)}{h(r)} + \lambda\left[1 + \frac{rh''(r)}{h'(r)} + \delta\left(\frac{rh'(r)}{h(r)} - 1\right) - \frac{rh'(r)}{h(r)}\right]\right)\right| < \frac{\alpha\pi}{2} \qquad (r \in \mathbb{U}), \tag{2.2}$$

whereby (1.4) gives the function $h(r)$.

Theorem 2.1. Let $k \in \mathfrak{B}\mathcal{M}_{\Sigma_m}(\lambda, \delta; \alpha)$, $(s, r \in \mathbb{U}, \lambda \geq 0, \delta \geq 0, 0 < \alpha \leq 1 \; and \; m \in \mathbb{N})$, be specified based on (1.3). Then

$$|d_{m+1}| \leq \frac{2\alpha}{\sqrt{\alpha(2\lambda m^3 + \delta^2 + 3\delta m + 2m^2 + 2\lambda\delta m^2) + (1-\alpha)(\delta+m)^2(1+\lambda m)^2}} \tag{2.3}$$

and

$$|d_{2m+1}| \leq \frac{2\alpha}{(\delta + 2m)(2m\lambda + 1)} + \frac{2\alpha^2(m+1)}{(\delta+m)^2(1+\lambda m)^2}. \tag{2.4}$$

Proof. From (2.1 and 2.2), we get

$$\left(\frac{k(s)}{s}\right)^{\delta} \frac{sk'(s)}{k(s)} + \lambda\left[1 + \frac{sk''(s)}{k'(s)} + \delta\left(\frac{sk'(s)}{k(s)} - 1\right) - \frac{sk'(s)}{k(s)}\right] = [p(s)]^{\alpha} \tag{2.5}$$

and

$$\left(\frac{h(r)}{r}\right)^{\delta} \frac{rh'(r)}{h(r)} + \lambda\left[1 + \frac{rh''(r)}{h'(r)} + \delta\left(\frac{rh'(r)}{h(r)} - 1\right) - \frac{rh'(r)}{h(r)}\right] = [q(r)]^{\alpha}, \tag{2.6}$$

the functions $p(s)$ and $q(r)$, which belong to class $\mathcal{P}$, possess the depictions of the series shown below:

$$p(s) = 1 + p_m s^m + p_{2m} s^{2m} + p_{3m} s^{3m} + \cdots \tag{2.7}$$

and

$$q(r)=1+q_m r^m+q_{2m}r^{2m}+q_{3m}r^{3m}+\cdots. \tag{2.8}$$

In (2.5 and 2.6), the coefficients are equated, therefore we find that

$$(\lambda m+1)(\delta+m)d_{m+1}=\alpha p_m, \tag{2.9}$$

$$(\delta+2m)(1+2\lambda m)d_{2m+1}+\left(\delta m-\lambda m^3-m+\frac{(\delta^2-\delta)}{2}-2\lambda m^2-\delta\lambda m\right)d_{m+1}^2$$

$$=\alpha p_{2m}+\frac{\alpha(\alpha-1)}{2}p_m^2, \tag{2.10}$$

$$-(\delta+m)(1+\lambda m)d_{m+1}=\alpha q_m, \tag{2.11}$$

and

$$(2\delta m+\frac{(\delta^2-\delta)}{2}+2m^2+m+3\lambda m^3+2\lambda m^2+\delta+2\lambda\delta m^2+\lambda\delta m)d_{m+1}^2-(\delta+2m)(1+2\lambda m)d_{2m+1}$$

$$=\alpha q_{2m}+\frac{\alpha(\alpha-1)}{2}q_m^2. \tag{2.12}$$

Using (2.9 and 2.11), we have

$$p_m=-q_m \tag{2.13}$$

and

$$2(1+\lambda m)^2(m+\delta)d_{m+1}^2=\alpha^2(p_m^2+q_m^2). \tag{2.14}$$

Additionally, a quick calculation based on (2.10), (2.12) and (2.14) reveals that

$$\left(2\lambda m^3+\delta^2+3\delta m+2m^2+2m^2\lambda\delta\right)d_{m+1}^2=\alpha(p_{2m}+q_{2m})+\frac{\alpha(\alpha-1)}{2}(p_m^2+q_m^2)$$

$$=\alpha(p_{2m}+q_{2m})+(\alpha-1)\frac{(m+\delta)^2(1+\lambda m)^2}{\alpha}d_{m+1}^2.$$

Therefore, we have

$$d_{m+1}^2=\frac{\alpha^2(p_{2m}+q_{2m})}{\alpha(2\lambda m^3+\delta^2+3\delta m+2m^2+2m^2\lambda\delta)+(1-\alpha)(m+\delta)^2(1+\lambda m)^2}.$$

Lemma (1.1) applied to the coefficient p_{2m} and q_{2m} yields

$$|d_{m+1}|\leq\frac{2\alpha}{\sqrt{\alpha(2\lambda m^3+\delta^2+3\delta m+2m^2+2m^2\lambda\delta)+(1-\alpha)(m+\delta)^2(\lambda m+1)^2}}.$$

As stated in (2.3), this provides the required estimate for $|d_{m+1}|$.

Next, by deducting (2.12) from (2.10), we can establish the bound on $|d_{2m+1}|$

$$2\big((1+2\lambda m)(\delta+2m)\big)d_{2m+1}-(4\lambda m^2+2m+\delta m+2\delta\lambda m+4\lambda m^3+2m^2+2\lambda\delta m^2+\delta)d_{m+1}^2$$

$$=\alpha(p_{2m}-q_{2m})+\frac{\alpha(\alpha-1)}{2}(p_m^2-q_m^2). \tag{2.15}$$

By a simple computation, and using (2.13) to (2.15), we get

$$d_{2m+1} = \frac{\alpha(p_{2m} - q_{2m})}{2(\delta + 2m)(2\lambda m + 1)} + \frac{\alpha^2(p_m^2 + q_m^2)(1 + m)}{4\big((\delta + m)(1 + \lambda m)\big)^2}.$$

Consequently, we see that by reapplying Lemma (1.1) to the coefficients p_m, p_{2m}, q_m and q_{2m}, we get

$$|d_{2m+1}| \leq \frac{2\alpha}{(\delta + 2m)(2\lambda m + 1)} + \frac{2\alpha^2(1 + m)}{\big((\delta + m)(1 + \lambda m)\big)^2}.$$

This concludes the theorem (2.1) proof.

3. Bounds on the coefficients for the function subclass $\mathfrak{B}\mathcal{M}_{\Sigma_m}(\lambda, \delta; \beta)$

Definition 3.1. A function $k(s) \in \Sigma_m$, specified based on (1.4), is said being in the subclass $\mathfrak{B}\mathcal{M}_{\Sigma_m}(\lambda, \delta; \beta)$for $\delta, \lambda \geq 0$ and $0 \leq \beta < 1$, if the following condition are satisfied:

$$Re\left(\left(\frac{k(s)}{s}\right)^{\delta} \frac{sk'(s)}{k(s)} + \lambda\left[1 + \frac{sk''(s)}{k'(s)} + \delta\left(\frac{sk'(s)}{k(s)} - 1\right) - \frac{sk'(s)}{k(s)}\right]\right) > \beta \qquad (s \in \mathbb{U}) \tag{3.1}$$

and

$$Re\left(\left(\frac{h(r)}{r}\right)^{\delta} \frac{rh'(r)}{h(r)} + \lambda\left[1 + \frac{rh''(r)}{h'(r)} + \delta\left(\frac{rh'(r)}{h(r)} - 1\right) - \frac{rh'(r)}{h(r)}\right]\right) > \beta \qquad (r \in \mathbb{U}), \tag{3.2}$$

whereby (1.4)gives the function $h(r)$.

Theorem 3.1. Let $k \in \mathfrak{B}\mathcal{M}_{\Sigma_m}(\lambda, \delta; \beta)$, $(s, r \in \mathbb{U}, \lambda \geq 0, \delta \geq 0, 0 \leq \beta < 1 \; and \; m \in \mathbb{N})$ be specified based on (1.4). Then

$$|d_{m+1}| \leq 2\sqrt{\frac{1 - \beta}{2\lambda m^3 + \delta^2 + 3\delta m + 2m^2 + 2m^2\lambda\delta}} \tag{3.3}$$

and

$$|d_{2m+1}| \leq \frac{2(1 - \beta)}{(\delta + 2m)(2\lambda m + 1)} + \frac{2(m + 1)(1 - \beta)^2}{\big((\delta + m)(\lambda m + 1)\big)^2}. \tag{3.4}$$

Proof. Initially, the argument inequalities in equations (3.1) and (3.2) can be expressed in the following ways.

$$\left(\frac{k(s)}{s}\right)^{\delta} \frac{sk'(s)}{k(s)} + \lambda\left[1 + \frac{sk''(s)}{k'(s)} + \delta\left(\frac{sk'(s)}{k(s)} - 1\right) - \frac{sk'(s)}{k(s)}\right] = \beta + (1 - \beta)p(s) \tag{3.5}$$

and

$$\left(\frac{h(r)}{r}\right)^{\delta} \frac{rh'(r)}{h(r)} + \lambda\left[1 + \frac{rh''(r)}{h'(r)} + \delta\left(\frac{rh'(r)}{h(r)} - 1\right) - \frac{rh'(r)}{h(r)}\right] = \beta + (1 - \beta)q(r), \tag{3.6}$$

where the forms of the functions $p(s)$ and $q(r)$, which belong to class $\mathcal{P}$, are (2.7 and 2.8), respectively. Now, by equating the coefficient in (3.5) and (3.6), just as in the Theorem (2.1) proof, we have

$$(1 + \lambda m)(\delta + m)d_{m+1} = (1 - \beta)p_m, \tag{3.7}$$

$$(\delta+2m)(1+2\lambda m)d_{2m+1}+\left(\delta m-\lambda m^3-m+\frac{(\delta^2-\delta)}{2}-2\lambda m^2-\delta\lambda m\right)d_{m+1}^2$$

$$=(1-\beta)p_{2m}, \tag{3.8}$$

$$-(1+\lambda m)(\delta+m)d_{m+1}=(1-\beta)q_m, \tag{3.9}$$

and

$$\left(2\delta m+\frac{(\delta^2-\delta)}{2}+2m^2+m+3\lambda m^3+2\lambda m^2+\delta+2\lambda\delta m^2+\lambda\delta m\right)d_{m+1}^2-(\delta+2m)(1+2\lambda m)d_{2m+1}$$

$$=(1-\beta)q_{2m}. \tag{3.10}$$

Using (3.7) and (3.9), we obtain

$$p_m=-q_m \tag{3.11}$$

and

$$2\big((1+\lambda m)(\delta+m)\big)^2d_{m+1}^2=(1-\beta)^2(p_m^2+q_m^2). \tag{3.12}$$

Also, from (3.8) and (3.10), we obtain

$$\left(2\lambda m^3+\delta^2+3\delta m+2m^2+2m^2\lambda\delta\right)d_{m+1}^2=(1-\beta)(p_{2m}+q_{2m}).$$

Thus, clearly, we have

$$|d_{m+1}|^2\le\frac{(1-\beta)(|p_{2m}|+|q_{2m}|)}{2\lambda m^3+\delta^2+3\delta m+2m^2+2m^2\lambda\delta}\le\frac{4(1-\beta)}{2\lambda m^3+\delta^2+3\delta m+2m^2+2m^2\lambda\delta}.$$

As stated in (3.3), this provides the required estimate for $|d_{m+1}|$.

Next, by deducting (3.10) from (3.8), we can establish the bound on $|d_{2m+1}|$

$$2(1+2\lambda m)(\delta+2m)d_{2m+1}-\big((1+m)(2\lambda m+1)(2m+\delta)\big)d_{m+1}^2=(1-\beta)(p_{2m}-q_{2m}).$$

Alternatively, equivalently

$$d_{2m+1}=\frac{(1-\beta)(p_{2m}-q_{2m})}{2(1+2\lambda m)(\delta+2m)}+\frac{(1+m)}{2}d_{m+1}^2. \tag{3.13}$$

When we replace the value of d_{m+1}^2 in (3.12), we obtain

$$d_{2m+1}=\frac{(p_{2m}-q_{2m})(1-\beta)}{2(\delta+2m)(1+2\lambda m)}+\frac{(1-\beta)^2(m+1)(p_m^2+q_m^2)}{4\big((\delta+m)(1+\lambda m)\big)^2}. \tag{3.14}$$

Consequently, we see that by reapplying Lemma (1.1) to the coefficient p_m, p_{2m}, q_m and q_{2m}, we get

$$|d_{2m+1}|\le\frac{2(1-\beta)}{(2\lambda m+1)(\delta+2m)}+\frac{2(1-\beta)^2(m+1)}{\big((1+\lambda m)(\delta+m)\big)^2}. \tag{3.15}$$

This concludes the theorem (3.1) proof.

4. Consequences and Corollaries

The subclasses $\mathfrak{B}\mathcal{M}_{\Sigma_m}(\lambda,\delta;\alpha)$ and $\mathfrak{B}\mathcal{M}_{\Sigma_m}(\lambda,\delta;\beta)$ reduce to the subclasses $\mathfrak{B}\mathcal{M}_{\Sigma_m}(\delta;\alpha)$ and $\mathfrak{B}\mathcal{M}_{\Sigma_m}(\delta;\beta)$, respectively, if $\lambda = 0$ is entered in Definitions (2.1) and (3.1). As a result, Theorems (2.1 and 3.1) reduce to Corollaries (4.1 and 4.2), respectively.

The subclasses $\mathfrak{B}\mathcal{M}_{\Sigma_m}(\delta;\alpha)$ and $\mathfrak{B}\mathcal{M}_{\Sigma_m}(\delta;\beta)$ have the following definitions:

Definition 4.1. A function $k(s) \in \Sigma_m$ specified based on (1.3) is said belong to the subclass $\mathfrak{B}\mathcal{M}_{\Sigma}(\delta;\alpha)$ for $\delta \geq 0$ and $0 < \alpha \leq 1$, if each of the following requirements is met:

$$\left|arg\left(\left(\frac{k(s)}{s}\right)^{\delta}\frac{sk'(s)}{k(s)}\right)\right| < \frac{\alpha\pi}{2} \qquad (s \in \mathbb{U})$$

and

$$\left|arg\left(\left(\frac{h(r)}{r}\right)^{\delta}\frac{rh'(r)}{h(r)}\right)\right| < \frac{\alpha\pi}{2} \qquad (r \in \mathbb{U}),$$

whereby (1.4) gives the function $h(r)$.

Definition 4.2. A function $k(s) \in \Sigma_m$ specified based on (1.3) is said belong to the subclass $\mathfrak{B}\mathcal{M}_{\Sigma}(\delta;\beta)$ for $\delta \geq 0$ and $0 \leq \beta < 1$, if each of the following requirements is met:

$$Re\left(\left(\frac{k(s)}{s}\right)^{\delta}\frac{sk'(s)}{k(s)}\right) > \beta \qquad (s \in \mathbb{U})$$

and

$$Re\left(\left(\frac{h(r)}{r}\right)^{\delta}\frac{rh'(r)}{h(r)}\right) > \beta \qquad (r \in \mathbb{U}),$$

whereby (1.4) gives the function $h(r)$.

Corollary 4.1. Assume that $k(s) \in \Sigma_m$, as shown based on (1.3), belongs to the class $\mathfrak{B}\mathcal{M}_{\Sigma}(\delta;\alpha)$. Then

$$|d_{m+1}| \leq \frac{2\alpha}{\sqrt{\alpha(\delta^2+3\delta m+2m^2)+(1-\alpha)(\delta+m)^2}}$$

and

$$|d_{2m+1}| \leq \frac{2\alpha}{(\delta+2m)} + \frac{2\alpha^2(1+m)}{(\delta+m)^2}.$$

Corollary 4.2. Assume that $k(s) \in \Sigma_m$, as shown based on (1.3), belongs to the class $\mathfrak{B}\mathcal{M}_{\Sigma}(\delta;\alpha)$. Then

$$|d_{m+1}| \leq 2\sqrt{\frac{1-\beta}{\delta^2+3\delta m+2m^2}}$$

and

$$|d_{2m+1}| \leq \frac{2(1-\beta)}{(\delta+2m)} + \frac{2(m+1)(1-\beta)^2}{(\delta+m)^2}.$$

The subclasses $\mathfrak{B}\mathcal{M}_{\Sigma_m}(\lambda,\delta;\alpha)$ and $\mathfrak{B}\mathcal{M}_{\Sigma_m}(\lambda,\delta;\beta)$ reduce to the subclasses $\mathfrak{B}\mathcal{M}_{\Sigma}(\lambda,\delta;\alpha)$ and $\mathfrak{B}\mathcal{M}_{\Sigma}(\lambda,\delta;\beta)$, respectively, for one-fold symmetric analytic biunivalent functions. As a result, Theorems (2.1 and 3.1) reduce to Corollaries (4.3 and 4.4), respectively.

The subclasses $\mathfrak{B}\mathcal{M}_{\Sigma}(\lambda,\delta;\alpha)$ and $\mathfrak{B}\mathcal{M}_{\Sigma}(\lambda,\delta;\beta$ have the following definitions:

Definition 4.3. A function $k(s)\in\Sigma$ specified based on (1.1) is said belong to the subclass $\mathfrak{B}\mathcal{M}_{\Sigma}(\lambda,\delta;\alpha)$ for $\delta,\lambda\geq 0$ and $0<\alpha\leq 1$, if each of the following requirements is met:

$$\left|arg\left(\left(\frac{k(s)}{s}\right)^{\delta}\frac{sk'(s)}{k(s)}+\lambda\left[1+\frac{sk''(s)}{k'(s)}+\delta\left(\frac{sk'(s)}{k(s)}-1\right)-\frac{sk'(s)}{k(s)}\right]\right)\right|<\frac{\alpha\pi}{2}\qquad (s\in\mathbb{U})$$

and

$$\left|arg\left(\left(\frac{h(r)}{r}\right)^{\delta}\frac{rh'(r)}{h(r)}+\lambda\left[1+\frac{rh''(r)}{h'(r)}+\delta\left(\frac{rh'(r)}{h(r)}-1\right)-\frac{rh'(r)}{h(r)}\right]\right)\right|<\frac{\alpha\pi}{2}\qquad (r\in\mathbb{U}),$$

whereby (1.2) gives the function $h(r)$.

Definition 4.4. A function $k(s)\in\Sigma$ specified based on (1.1) is said belong to the subclass $\mathfrak{B}\mathcal{M}_{\Sigma}(\lambda,\delta;\beta)$ for $\delta,\lambda\geq 0$ and $0\leq\beta<1$, if each of the following requirements is met:

$$Re\left(\left(\frac{k(s)}{s}\right)^{\delta}\frac{sk'(s)}{k(s)}+\lambda\left[1+\frac{sk''(s)}{k'(s)}+\delta\left(\frac{sk'(s)}{k(s)}-1\right)-\frac{sk'(s)}{k(s)}\right]\right)>\beta\qquad (s\in\mathbb{U})$$

and

$$Re\left(\left(\frac{h(r)}{r}\right)^{\delta}\frac{rh'(r)}{h(r)}+\lambda\left[1+\frac{rh''(r)}{h'(r)}+\delta\left(\frac{rh'(r)}{h(r)}-1\right)-\frac{rh'(r)}{h(r)}\right]\right)>\beta\qquad (r\in\mathbb{U}),$$

whereby (1.2) gives the function $h(r)$.

Corollary 4.3. Assume that $k(s)$ as shown based on (1.1), belongs to the class $\mathfrak{B}\mathcal{M}_{\Sigma}(\lambda,\delta;\alpha)$. Then

$$|d_2|\leq\frac{2\alpha}{\sqrt{\alpha(2\lambda+\delta^2+3\delta+2+2\lambda\delta)+(1-\alpha)(\delta+1)^2(1+\lambda)^2}}$$

and

$$|d_3|\leq\frac{2\alpha}{(\delta+2)(2\lambda+1)}+\frac{4\alpha^2}{(\delta+1)^2(1+\lambda)^2}.$$

Corollary 4.4. Assume that $k(s)$ as shown based on (1.1)), belongs to the class $\mathfrak{B}\mathcal{M}_{\Sigma}(\lambda,\delta;\beta)$. Then

$$|d_2|\leq 2\sqrt{\frac{1-\beta}{\left(2\lambda+\delta^2+3\delta+2+2\lambda\delta\right)}}$$

and

$$|d_3|\leq\frac{2(1-\beta)}{(2+\delta)(2\lambda+1)}+\frac{4(1-\beta)^2}{(1+\delta)^2(1+\lambda)^2}.$$

Remark 4.1. For m-fold symmetric biunivalent functions:

1) If we make $\lambda=0$ *and* $\delta=0$ in Th. (2.1) and Th. (3.1), we acquire the matching outcomes provided by Altinkaya and Yalcin [2].

2) If we make $\lambda = 1$ *and* $\delta = 0$ in Th. (2.1) and Th. (3.1), we acquire the matching outcomes provided by Kumar et al. [9].
3) If we make $\lambda = 0$ *and* $\delta = 1$ in Th. (2.1) and Th. (3.1), we acquire the matching outcomes provided by Srivastava et al. [20].
4) If we make $\delta = 0$ in Th. (2.1) and Th. (3.1), we acquire the matching outcomes provided by Sivasubamanlan and Sivakumar [14].

Remark 4.2. For one-fold symmetric biunivalent functions:

1) If we make $\delta = 0$ in Th. (2.1) and Th. (3.1), we acquire the matching outcomes provided by Li and Wang [10].
2) If we make $\lambda = 0$ in Th. (2.1) and Th. (3.1), we acquire the matching outcomes provided by Prima and Keerthi [12].
3) If we make $\delta = 0, \lambda = 0$ in Th. (2.1) and Th. (3.1), we acquire the matching outcomes provided by Murugusundaramoorthy et al. [11].
4) If we make $\delta = 0, \lambda = 1$ in Th. (2.1) and Th. (3.1), we acquire the matching outcomes provided by Kumar et al. [9].
5) If we make $\delta = 1, \lambda = 0$ in Th. (2.1) and Th. (3.1), we acquire the matching outcomes provided by Srivastava et al. [18].

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