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Results on New Subclasses for Bi-Univalent Functions Using Quasi-Subordination

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ABSTRACT

The present study introduces and examines two specific subclasses, $\mathrm{M}_{\Sigma}^{\mathrm{q}}(\beta, \rho, \lambda)$ and $\mathrm{W}_{\Sigma}^{\mathrm{q}}(\tau, \mu, \eta, \phi)$ of bi-univalent functions that are defined by quasi-subordination. We obtain approximations for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions within these subclasses.

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1. **Introduction**

Let A denote the class of all normalized analytic functions f in an open unit disk $U=\{z: z \in \mathbb{C}, |z|<1\}$ of the form:

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n, \qquad (z \in U). \tag{1.1}$$

A function f has an inverse f^{-1} is satisfying $f^{-1}(f(z))=z, (z \in U)$, and $f(f^{-1}(w))=w$, $\left(|w|<r_0(f), r_0(f) \geq \frac{1}{4}\right)$,

where

$$g(w)=f^{-1}(w)=w-a_2 w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots, (w \in U). \tag{1.2}$$

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If f as well as f^{-1} are both univalent functions in the open unit disk U, then f is said to be bi-univalent in U and the set of bi-univalent functions defined in U is denoted by $\sum$. Refer to reference [13].

Let f and g are analytic functions in A. Then f is said to be quasi-subordinate to g in U and written as follows:

$$f(z) \prec_q g(z), \qquad (z \in U),$$

if there exists $\theta(z)$ and $w(z)$ be two analytic functions in U, with $w(0) = 0$ such that $|\theta(z)| < 1, |w(z)| < 1$ and $f(z) = \theta(z)g\big(w(z)\big)$. If $\theta(z) = 1$, then $f(z) = g\big(w(z)\big)$, so that $f(z) \prec g(z)$ in U. If $w(z) = z$, then $f(z) = \theta(z)g(z)$, and it is said that f is majorized by g and written $f(z) \ll g(z)$ in U. (see [4], [15])

Ma and Manda [14] introduced a category of starlike and convex functions through the use of the subordination approach. They examined the classes $S^*(\phi)$ and $G^*(\phi)$ defined by this method.

$$S^*(\phi) = \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \phi(z), z \in U \right\},$$

and

$$G^*(\phi) = \left\{ f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), z \in U \right\}.$$

By $S^*_{\sum}(\phi)$ and $G^*_{\sum}(\phi)$, we denote to bi-starlike and bi-convex functions f is bi-starlike and bi-convex of Ma-Minda type respectively [14].

In the sequal, it assumed that ϕ of the form:

$$\phi(z) = 1 + E_1 z + E_2 z^2 + \cdots, \tag{1.3}$$

where $\phi(0) = 1$ and $\phi'(0) > 0$, also

$$\theta(z) = F_0 + \sum_{i=1}^{\infty} F_i z^i, \tag{1.4}$$

which are both analytical and bounded within the domain U. Nevertheless, the literature has only a limited number of studies that establish the overall coefficient bounds $|a_2|$ and $|a_3|$ for analytic bi-univalent functions. These studies are referenced in [1,2,3,5,6,10,11,14] and [12], while further references may be found in [5,6,7,8].

Lemma (1.1) [9]. Define the function $h(z) = 1 + h_1 z + h_2 z^2 + \cdots \in P$, where P represents the set of all functions h that are analytic in U and satisfy $\operatorname{Re}\{h(z)\} > 0$ for all $z \in U$. It follows that $|h_i| \leq 2$ for $i = 1,2,3,\cdots$.

2-Main Results

Definition (2.1). A function f, which is a member of the class $\sum$ and defined by equation (1.1), is considered to be in the class $M^q_{\sum}(\beta, \rho, \lambda)$ if it satisfies the following quasi-subordination conditions:

$$\frac{1}{\beta}\left[\left(\frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)}\right) + \left(\frac{\rho z^2 f''(z) + zf'(z)}{\rho z f'(z) + (1-\rho)f(z)}\right) - \left(\frac{zf'(z) + \lambda z^2 f''(z)}{f(z)}\right) - 1\right] \prec_q (\emptyset(z) - 1) \tag{2.1}$$

and

$$\frac{1}{\beta}\left[\left(\frac{wg'(w)}{g(w)}\right)\left(1 + \frac{wg''(w)}{g'(w)}\right) + \left(\frac{\rho w^2 g''(w) + wg'(w)}{\rho w g'(w) + (1-\rho)g(w)}\right) - \left(\frac{wg'(w) + \lambda w^2 g'(w)}{g(w)}\right) - 1\right] \prec_q (\emptyset(w) - 1), \tag{2.2}$$

where $(0 \leq \lambda \leq 1, 0 \leq \rho < 1)$, and $\beta \in \mathbb{C} \setminus \{0\}$, $z, w \in U$, and $g = f^{-1}$, and the functions g, $\emptyset$ are given by (1.2) and (**1.3**) respectively.

By substituting $\lambda = 0$ into Definition (2.1), we may deduce the following Remark: $M_{\Sigma}^{q}(\beta,\rho,0) = M_{\Sigma}^{q}(\beta,\rho,\lambda)$.

Remark (2.1). A function $f \in \sum$, as defined by (1.1), is believed to be in the class $M_{\Sigma}^{q}(\beta,\rho)$ if it fulfills the following conditions:

$$\frac{1}{\beta}\left[\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)+\left(\frac{\rho z^2 f''(z)+zf'(z)}{\rho zf'(z)+(1-\rho)f(z)}\right)-\left(\frac{zf'(z)}{f(z)}\right)-1\right] \prec_q (\emptyset(z)-1) \tag{2.3}$$

and

$$\frac{1}{\beta}\left[\left(\frac{wg'(w)}{g(w)}\right)\left(1+\frac{wg''(w)}{g'(w)}\right)+\left(\frac{\rho w^2 g''(w)+wg'(w)}{\rho wg'(w)+(1-\rho)g(w)}\right)-\left(\frac{wg'(w)}{g(w)}\right)-1\right] \prec_q (\emptyset(w)-1). \tag{2.4}$$

Theorem (2.1). Consider the function f defined by equation (1.1) to belong to the class $M_{\Sigma}^{q}(\beta,\rho,\lambda)$. Thus

$$|a_2| \le min\left\{\frac{|\beta F_0 E_1|}{|3+\rho-2\lambda|}, \sqrt{\frac{\beta F_0 E_2}{5+\rho(2-\rho)-6\lambda}}\right\} \tag{2.5}$$

and

$$|a_3| \le min\left\{\frac{\beta^2 F_0^2 E_1^2}{(3+\rho-2\lambda)^2}+\frac{\beta F_1 E_1}{8+4\rho-6\lambda}, \frac{\beta F_0 E_2}{5+\rho(2-\rho)-6\lambda}+\frac{\beta F_1 E_1}{8+4\rho-6\lambda}\right\} \tag{2.6}$$

Proof. Let $f \in \mathcal{L}_{\Sigma}^{q}(\beta,\lambda,\phi)$ and $g = f^{-1}$. Then, there are two analytic functions $u, v: U \to U$ with $v(0) = 0$ and $u(0) = 0, |u(z)| < 1, |v(w)| < 1$, satisfying

$$\frac{1}{\beta}\left[\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)+\left(\frac{\rho z^2 f''(z)+zf'(z)}{\rho zf'(z)+(1-\rho)f(z)}\right)-\left(\frac{zf'(z)+\lambda z^2 f''(z)}{f(z)}\right)-1\right] = \left(\theta(z)\left[\emptyset\left(u(z)\right)-1\right]\right) \tag{2.7}$$

and

$$\frac{1}{\beta}\left[\left(\frac{wg'(w)}{g(w)}\right)\left(1+\frac{wg''(w)}{g'(w)}\right)+\left(\frac{\rho w^2 g''(w)+wg'(w)}{\rho wg'(w)+(1-\rho)g(w)}\right)-\left(\frac{wg'(w)+\lambda w^2 g'(w)}{g(w)}\right)-1\right] = \left(\theta(w)\left[\emptyset\left(v(w)\right)-1\right]\right). \tag{2.8}$$

We define the function $p(z)$ and $q(w)$ by:

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + d_1 z + d_2 z^2 + \cdots, \tag{2.9}$$

and

$$q(w) = \frac{1+v(w)}{1-v(w)} = 1 + e_1 w + e_2 w^2 + \cdots. \tag{2.10}$$

Or equivalent,

$$u(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[d_1 z + \left(d_2 - \frac{d_1^2}{2}\right)z^2 + \cdots\right], \tag{2.11}$$

and

$$v(w) = \frac{q(w)-1}{q(w)+1} = \frac{1}{2}\left[e_1 w + \left(e_2 - \frac{e_1^2}{2}\right)w^2 + \cdots\right], \tag{2.12}$$

then $p(z)$ and $q(w)$ are analytic functions in U, with $p(0) = q(0) = 1$. Since, $v, u: U \to U$, have a positive real part in U, and $|d_i| \leq 2$ and $|e_i| \leq 2$, for $i = 1,2$. Using (2.11) and (2.12) in (2.7) and (2.8), respectively, we have

$$\frac{1}{\beta}\left[\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)+\left(\frac{\rho z^2 f''(z)+zf'(z)}{\rho zf'(z)+(1-\rho)f(z)}\right)-\left(\frac{zf'(z)+\lambda z^2 f''(z)}{f(z)}\right)-1\right]=\theta(z)\left(\phi\left[\frac{p(z)-1}{p(z)+1}\right]-1\right) \tag{2.13}$$

and

$$\frac{1}{\beta}\left[\left(\frac{wg'(w)}{g(w)}\right)\left(1+\frac{wg''(w)}{g'(w)}\right)+\left(\frac{\rho w^2 g''(w)+wg'(w)}{\rho wg'(w)+(1-\rho)g(w)}\right)-\left(\frac{wg'(w)+\lambda w^2 g'(w)}{g(w)}\right)-1\right]=\theta(w)\left(\phi\left[\frac{q(w)-1}{q(w)+1}\right]-1\right). \tag{2.14}$$

Given that $f \in \sum$ possesses the Maclurian series specified by equation (1.1), a calculation reveals that its inverse $g = f^{-1}$ can be expressed using the expansion described in equation (1.2). Therefore, we can conclude that

$$\frac{1}{\beta}\left[\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}\right)+\left(\frac{\rho z^2 f''(z)+zf'(z)}{\rho zf'(z)+(1-\rho)f(z)}\right)-\left(\frac{zf'(z)+\lambda z^2 f''(z)}{f(z)}\right)-1\right]=\frac{1}{\beta}[(3+\rho-2\lambda)a_2 z+[2(43+2\rho-3\lambda)a_3-(3+\rho(2+\rho)-2\lambda)a_2^2]z^2+\cdots] \tag{2.15}$$

and

$$\frac{1}{\beta}\left[\left(\frac{wg'(w)}{g(w)}\right)\left(1+\frac{wg''(w)}{g'(w)}\right)+\left(\frac{\rho w^2 g''(w)+wg'(w)}{\rho wg'(w)+(1-\rho)g(w)}\right)-\left(\frac{wg'(w)+\lambda w^2 g'(w)}{g(w)}\right)-1\right]=\frac{1}{\beta}[-(3+\rho-2\lambda)a_2 w+[(13+\rho(6-\rho)-10\lambda)a_2^2-2(4+2\rho-3\lambda)a_3]w^2+\cdots]. \tag{2.16}$$

By combining equations (2.11) and (2.12) with equations (1.3) and (1.4), it becomes clear that

$$\theta(z)\left(\phi\left[\frac{p(z)-1}{p(z)+1}\right]-1\right)=\frac{1}{2}F_0E_1d_1 z+\left(\frac{1}{2}F_1E_1d_1+\frac{1}{2}F_0E_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{1}{4}F_0E_2d_1^2\right)z^2+\cdots, \tag{2.17}$$

and

$$\theta(w)\left(\phi\left[\frac{q(w)-1}{q(w)+1}\right]-1\right)=\frac{1}{2}F_0E_1e_1 w+\left(\frac{1}{2}F_1E_1e_1+\frac{1}{2}F_0E_1\left(e_2-\frac{e_1^2}{2}\right)+\frac{1}{4}F_0E_2e_1^2\right)w^2+\cdots. \tag{2.18}$$

By utilising equations (2.17) and (2.15) and comparing the coefficients of z and z^2, we have

$$\frac{1}{\beta}(3+\rho-2\lambda)a_2=\frac{1}{2}F_0E_1d_1, \tag{2.19}$$

and

$$\frac{1}{\beta}[2(43+2\rho-3\lambda)a_3-(3+\rho(2+\rho)-2\lambda)a_2^2]=\frac{1}{2}F_1E_1d_1+\frac{1}{2}F_0E_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{1}{4}F_0E_2d_1^2. \tag{2.20}$$

By utilising equations (2.18) and (2.16) and comparing the coefficients of w and w^2, we obtain

$$-\frac{1}{\beta}(3+\rho-2\lambda)a_2=\frac{1}{2}F_0E_1e_1, \tag{2.21}$$

and

$$\frac{1}{\beta}[(13+\rho(6-\rho)-10\lambda)a_2^2-2(4+2\rho-3\lambda)a_3]=\frac{1}{2}F_1E_1e_1+\frac{1}{2}F_0E_1\left(e_2-\frac{e_1^2}{2}\right)+\frac{1}{4}F_0E_2e_1^2. \tag{2.22}$$

Making use of (2.19) and (2.21), we obtain

$$d_1=-e_1 \tag{2.23}$$

and

$$8(3-2\lambda+\delta)^2a_2^2=\beta^2F_0^2E_1^2(d_1^2+e_1^2). \tag{2.24}$$

Also, from (2.20), (2.22) and (2.24), we find that

$$a_2^2 = \frac{\beta[2F_0E_1(d_2+e_2)+F_0(E_2-E_1)(d_1^2+e_1^2)]}{8(5-6\lambda+\rho(2-\rho))}. \tag{2.25}$$

Applying $|d_i| \le 2$ and $|e_i| \le 2$ for the coefficients d_1, e_1, d_2 and e_2, we immediately have

$$|a_2| \le \frac{|\beta F_0 E_1|}{|3+\rho-2\lambda|}$$

and

$$|a_2| \le \sqrt{\frac{\beta F_0 E_2}{5-6\lambda+\rho(2-\rho)}}.$$

Furthermore, to determine the upper limit of $|a_3|$, we can subtract equations (2.20) and (2.22), resulting in

$$\frac{2}{\beta}(16+8\rho-12\lambda)(a_3-a_2) = 2F_1E_1d_1 + F_0E_1(d_2-e_2), \tag{2.26}$$

$$\frac{2}{\beta}(16+8\rho-12\lambda)a_3 = \frac{2}{\beta}(16+8\rho-12\lambda)a_2^2 + 2F_1E_1d_1 + F_0E_1(d_2-e_2),$$

therefore

$$a_3 = \frac{\beta[2F_1E_1d_1 + F_0E_1((d_2-e_2)]}{2(16-12\lambda+8\rho)} + a_2^2. \tag{2.27}$$

In light of (2.24), (2.25) and putting (2.27), we have

$$|a_3| \le \frac{\beta[F_1E_1]}{(8-6\lambda+4\rho)} + \frac{\beta[F_0E_2]}{(5-6\lambda+\rho(2-\rho))}$$

and

$$|a_3| \le \frac{\beta[F_1E_1]}{(8-6\lambda+4\rho)} + \frac{\beta^2F_0^2E_1^2}{(3-2\lambda\_+\rho)^2}.$$

This completes the proof Theorem (2.1).

By substituting $\lambda = 0$ into Theorem (2.1), we obtain the following corollary:

Corollary (2.1). Consider the function f defined by equation (1.1) to belong to the class $M_{\Sigma}^{q}(\beta,\rho,0)$. Subsequently

$$|a_2| \le \min\left\{\frac{|\beta F_0E_1|}{|3+\rho|}, \sqrt{\frac{\beta F_0E_2}{5+\rho(2-\rho)}}\right\}$$

and

$$|a_3| \le \min\left\{\frac{\beta^2F_0^2E_1^2}{(3+\rho)^2} + \frac{\beta F_1E_1}{8+4\rho}, \frac{\beta F_0E_2}{5+\rho(2-\rho)} + \frac{\beta F_1E_1}{8+4\rho}\right\}.$$

Definition (2.2). A function $f \in \Sigma$ is believed to be in the class $W_{\Sigma}^{q}(\tau,\mu,\eta,\phi)$ if it fulfills the quasi-subordination conditions described by equation (1.1).

$$\frac{1}{\tau}\left[(1-\mu+2\eta)\frac{f(z)}{z}+(\mu-2\eta)f'(z)+\eta zf''(z)-1\right]\prec_q(\phi(z)-1) \tag{2.28}$$

and

$$\frac{1}{\tau}\left[(1-\mu+2\eta)\frac{g(w)}{w}+(\mu-2\eta)g'(w)+\eta wg''(w)-1\right]\prec_q(\phi(w)-1), \tag{2.29}$$

where ($\tau\in\mathbb{C}\backslash\{0\},\mu\geq 0,\eta\geq 0,z,w\in U$) and $g=f^{-1}$ and the function g,ϕ are given by (1.2) and (1.3) respectively.

If we put $\mu=1+2\eta$ in Definition (2.2), we obtain the following Remark such that $W_{\Sigma}^{q}(\tau,\mu,\eta,\phi)=W_{\Sigma}^{q}(\tau,1+2\eta,\eta,\phi)$.

Remark (2.2). A function $f\in\sum$ is considered to be in the class $W_{\Sigma}^{q}(\tau,1+2\eta,\eta,\phi)$ if it fulfills the given quasi-subordination requirements defined by equation (1.1).

$$\frac{1}{\tau}[f'(z)+\eta zf''(z)-1]\prec_q(\phi(z)-1) \tag{2.30}$$

and

$$\frac{1}{\tau}[g'(w)+\eta wg''(w)-1]\prec_q(\phi(w)-1), \tag{2.31}$$

where ($\tau\in\mathbb{C}\backslash\{0\},,\eta\geq 0,z,w\in U$) and $g=f^{-1}$ and the function g,ϕ are given by (1.2) and (1.3) respectively.

Theorem (2.2). Let $f\in W_{\Sigma}^{q}(\tau,\mu,\eta,\phi)(\tau\in\mathbb{C}\backslash\{0\},,\eta\geq 0,z,w\in U)$ be given by (1.1). Then

$$|a_2|\leq min\left\{\frac{2\tau F_0E_1}{1+\mu},2\sqrt{\frac{|\tau F_0E_2|}{1+2\mu+2\eta}}\right\} \tag{2.32}$$

and

$$|a_3|\leq \min\left\{\frac{4F_0^2E_1^2\tau^2}{(1+\mu)^2}+\frac{\tau E_1(F_1+F_0)}{1+2\mu+2\eta},\frac{4\tau F_0E_2}{1+2\mu+2\eta}+\frac{\tau E_1(F_1+F_0)}{1+2\mu+2\eta}\right\}. \tag{2.33}$$

Proof. Let $f\in W_{\Sigma}^{q}(\tau,\mu,\eta,\phi)$ and $g=f^{-1}$. Then, there are two analytic functions $u,v:U\to U$ with $v(0)=0$ and $u(0)=0,|u(z)|<1,|v(w)|<1$, satisfying:

$$\frac{1}{\tau}\left[(1-\mu+2\eta)\frac{f(z)}{z}+(\mu-2\eta)f'(z)+\eta zf''(z)-1\right]=\theta(z)\big(\phi(u(z)-1)\big) \tag{2.34}$$

and

$$\frac{1}{\tau}\left[(1-\mu+2\eta)\frac{g(w)}{w}+(\mu-2\eta)g'(w)+\eta wg''(w)-1\right]=\theta(w)\big(\phi(v(w)-1)\big). \tag{2.35}$$

Since $f\in\sum$ has the Maclurian series defined by (1.1), a computation shows that its inverse $g=f^{-1}$ has the expansion by (1.2), we get

$$\frac{1}{\tau}\left[(1-\mu+2\eta)\frac{f(z)}{z}+(\mu-2\eta)f'(z)+\eta zf''(z)-1\right]=$$

$$\frac{1}{\tau}(1+\mu)a_2z+\frac{1}{\tau}(1+2\mu+2\eta)a_3z^2+\cdots, \tag{2.36}$$

and

$$\frac{1}{\tau}\left[(1-\mu+2\eta)\frac{g(w)}{w}+(\mu-2\eta)g'(w)+\eta w g''(w)-1\right]=$$

$$-\tfrac{1}{\tau}(1+\mu)a_2 w+\tfrac{1}{\tau}(1+2\mu+2\eta)(2a_2^2-a_3)w^2-\cdots. \quad (2.37)$$

By utilizing equations (2.17) and (2.36) and comparing the coefficients z and z^2, we obtain

$$\frac{1}{\tau}(1+\mu)a_2=\frac{F_0E_1}{2}d_1, \quad (2.38)$$

and

$$\frac{1}{\tau}(1+2\mu+2\eta)a_3=\frac{F_1E_1}{2}d_1+\frac{F_0E_1}{2}\left(d_2-\frac{d_1^2}{2}\right)+\frac{F_0E_2}{4}d_1^2. \quad (2.39)$$

Furthermore, by employing equations (2.18) and (2.37) and comparing the coefficients of w and w^2, we obtain

$$-\frac{1}{\tau}(1+\mu)a_2=\frac{F_0E_1}{2}e_1, \quad (2.40)$$

and

$$\tfrac{1}{\tau}(1+2\mu+2\eta)(2a_2^2-a_3)=\tfrac{F_1E_1}{2}e_1+\tfrac{F_0E_1}{2}\left(e_2-\tfrac{e_1^2}{2}\right)+\tfrac{F_0E_2}{4}e_1^2. \quad (2.41)$$

From (2.38) and (2.40), we obtain

$$d_1=-e_1 \quad (2.42)$$

and

$$\frac{2}{\tau^2}(1+\mu)^2a_2^2=F_0^2E_1^2(d_1^2+e_1^2). \quad (2.43)$$

Now, adding (2.39), (2.41), we obtain

$$\frac{2}{\tau}(1+2\mu+2\eta)a_2^2=2F_0E_1(d_2+e_2)+F_0(E_2-E_1)(d_1^2+e_1^2). \quad (2.44)$$

Applying $|d_i|\leq 2$ and $|e_i|\leq 2$ for the coefficients d_1, e_1, d_2 and e_2, we immediately have

$$|a_2|\leq\frac{2\tau F_0E_1}{1+\mu},$$

and

$$|a_2|\leq 2\sqrt{\frac{|\tau F_0E_2|}{1+2\mu+2\eta}}.$$

Furthermore, to derive the upper limit of $|a_3|$, we can remove equation (2.39) from equation (2.41), resulting in

$$a_3=a_2^2+\frac{2\tau F_1E_1d_1+\tau F_0E_1(d_2-e_2)}{4(1+2\mu+2\eta)}. \quad (2.45)$$

In light of (2.43), (2.44) and putting (2.45), we have

$$|a_3| \leq \frac{4F_0^2E_1^2\tau^2}{(1+\mu)^2} + \frac{\tau E_1(F_1+F_0)}{1+2\mu+2\eta}$$

and

$$|a_3| \leq \frac{4\tau F_0 E_2}{1+2\mu+2\eta} + \frac{\tau E_1(F_1+F_0)}{1+2\mu+2\eta}.$$

This completes the proof Theorem (2.1).

By putting $\mu = 1 + 2\eta$ in Theorem (1.1), we get the next Corollary:

Corollary (2.2). Let f defined by (1.1) belongs to the class $W_{\Sigma}^{q}(\tau, \eta, \phi)$. Then

$$|a_2| \leq min\left\{\frac{\tau F_0 E_1}{1+\eta}, 2\sqrt{\frac{|\tau F_0 E_2|}{3(1+2\eta)}}\right\}$$

and

$$|a_3| \leq min\left\{\frac{F_0^2E_1^2\tau^2}{(1+\eta)^2} + \frac{\tau E_1(F_1+F_0)}{3(1+2\eta)}, \frac{4\tau F_0 E_2}{3(1+2\eta)} + \frac{\tau E_1(F_1+F_0)}{3(1+2\eta)}\right\}.$$

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