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Solvability of an Unbounded Operator Equation

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ABSTRACT

In this work, we will submit the form of the solution of the generalization kind of unbounded operator equation define on Hilbert space which is $X^*B^* + BX = C$, where X, is the bounded operator satisfy the above operator equations and showing this solution via application the semigroup theory, moreover we will discuss some properties of this solution and proved its unique. Also, it was proved that for a locally bounded operator $X(n)$ satisfying the equation.

MSC..

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1. Introduction

The solution of this equation is useful in differential equations and control theory see for example [1] and [2]. Rosenblum (1969) [3], provided the sufficient and necessary conditions for a bounded solution for the equation $AX - XB = C$. Phóng (1991) [4] tackled the case when A and B generate C_0- semigroup. S. Schweiker (2000) [5] studied the relation of the equation $AX - XB^2 = -\delta_0$ with other types, and the 2nd order ODEs the line of real numbers. Nguyen (2001) [6], established a criterion for the solvability of $AX - XB = C$ the results were then used to study the solvability of the non-homogeneous Cauchy problem. Caruso et. al. (2021) [7] discussed for the inverse problem $Af = g$ where A is a (possibly unbounded) linear operator on an infinite-dimensional Hilbert space and g is a datum in its range.

The main aim of the paragraph is to:

1. Present the solution form of a generalized unbounded operator equation defined on a Hilbert space, specifically $X^*B^* + BX = C$.
2. Show that the bounded operator X satisfies this operator equation using semigroup theory.
3. Discuss some properties of this solution.
4. Prove the uniqueness of the solution.

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5. Prove that for a locally bounded operator $X(n)$, the equation is satisfied.

In summary, the primary goal is to find and validate a solution to the specified operator equation, demonstrate its uniqueness, and explore its properties using semigroup theory.

First, we recall some basic definitions of this work. Let us start with the following definitions

Definition 1.1 [8] If X, Y are two normed spaces, the operator $T: X \to Y$ is bounded if $\|Tx\| \leq M\|x\|$ for all$x \in X$, where M is a positive real number. Otherwise $T: X \to Y$ is unbounded operator.

Definition 1.2 [8] A linear operator $T : D(T) \subset H \to H$ (where $D(T)$is the domain of T and H is a Hilbert space) is closed if the graph $G(T) = \{(x, y): x \in D(T), y = Tx\}$is closed in $H \times H$, where the norm on $H \times H$ is defined by, $\|(x, y)\| = (\|x\|^2 + \|y\|^2)$ and results from the inner product which is follow as $\langle(x_1, y_1), (x_2, y_2)\rangle = \langle x_1, x_2\rangle + \langle y_1, y_2\rangle$

Definition 1.3 [9] the family has one parameter summarize by $\{S(n)\}_{n\geq 0}$ of bounded linear operators from a Banach space X, into itself is said to be a semigroup has types of bounded linear operators on the Banach space X, if:

(i) $S(0) = I$, here, I denote the identity on the space X.

(ii) $S(n + s) = S(n)S(s), \forall\, n, s \geq 0$

some basic properties of semi group given by the following theorem

Theorem 1.4 [9] The uniformly continuous semigroup $T(n)$, $S(n)$ of bounded linear operators are equal for all $n > 0$ if $lim_{n\to o+}\frac{T(n)x-x}{n} = A = lim_{n\to o+}\frac{S(n)x-x}{n}$.

The following lemma provide a solution for the equation $AX - XB = C$

Lemma 1.5: Suppose that $f_1 \in D(B)$ and $X(t) = T(t)CS(t)$. Then for every $f_2 \in D(A)$

the function $\langle X(t)f_1, f_2\rangle$ is differentiable and

2. Main results

This section, will be showing some important theorems for the solution of unbounded operator equations. This is generalizations of some facts appeared in [4].

Lemma (2.1)

Given the operators B and C on H with B unbounded and generates *a* C_0*-semigroup of* $\{S(n)\}_{n\geq 0}$ C , and C bounded self-adjoint. Then to each $f_1, f_2 \in H$, $\langle X(n)f_1, f_2\rangle$ differentiable with the derivative given *by* $\frac{d}{dn}\langle X(n)f_1, f_2\rangle = \langle X^*(n)B^*f_1, f_2\rangle + \langle BX(n)f_1, f_2\rangle\rangle$. Here $X(n) = S(n)CS^*(n)$

Proof

$$\langle(\Delta n)^{-1}[S(n+\Delta n)\,C\,S^*(n+\Delta n)f_1 - S(n)\,CS^*(n)f_1], f_2\rangle - \langle(S(n)C\,S^*(n))^*\,B^*f_1, f_2\rangle - \langle BS(n)\,C\,S^*(n)f_1, f_2\rangle$$

$$= \langle(\Delta n)^{-1}S(n+\Delta n)\,C\,S^*(n+\Delta n)f_1, f_2\rangle - \langle BS(n)\,C\,S^*(n)f_1, f_2\rangle - \langle S(n)\,C\,S^*(n)B^*f_1, f_2\rangle + \langle-(\Delta n)^{-1}S(n)\,C\,S^*(n)f_1, f_2\rangle$$

$$= \langle(\Delta n)^{-1}CS^*(n+\Delta n)f_1, S^*(n+\Delta n)f_2\rangle + \langle-(\Delta n)^{-1}CS^*(n)f_1, S^*(n)f_2\rangle - \langle CS^*(n)B^*f_1, S^*(n)f_2\rangle - \langle BCS^*(n)f_1, S^*(n)f_2\rangle$$

$$= \langle(\Delta n)^{-1}CS^*(n+\Delta n)f_1, S^*(n+\Delta n)f_2\rangle + \langle-(\Delta n)^{-1}CS^*(n)f_1, S^*(n)f_2\rangle - \langle BCS^*(n)f_1, S^*(n)f_2\rangle - \langle CS^*(n)B^*f_1, S^*(n)f_2\rangle + \langle CS^*(n)B^*f_1, S^*(n+\Delta n)f_2\rangle - \langle CS^*(n)B^*f_1, S^*(n+\Delta n)f_2\rangle + \langle(\Delta n)^{-1}CS^*(n)f_1, S^*(n+\Delta n)f_2\rangle - \langle(\Delta n)^{-1}CS^*(n)f_1, S^*(n+\Delta n)f_2\rangle$$

$$= \langle (\Delta n)^{-1}[CS^*(n+\Delta n)f_1 - S^*(n)f_2] - CS^*(n)B^*f_1, S^*(n+\Delta n)f_2\rangle$$
$$+ \langle CS^*(n)f_1, (\Delta n)^{-1}[S^*(n+\Delta n)f_2 - S^*(n)f_2] - S^*(n)B^*f_2\rangle + \langle CS^*(n)B^*f_1, S^*(n+\Delta n)f_2 - S^*(n)f_2\rangle$$

$$\lim_{\Delta n\to 0}\|(\Delta n)^{-1}[CS^*(n+\Delta n)f_1 - S^*(n)f_1] - CS^*(n)B^*f_1\| = 0$$

$$\lim_{\Delta n\to 0}\langle CS^*(n)f_1, (\Delta n)^{-1}[S^*(n+\Delta n)f_2 - S^*(n)f_2] - S^*(n)B^*f_2\rangle = 0$$

$$\lim_{\Delta n\to 0}\|S^*(n+\Delta n)f_2 - S^*(n)f_2\| = 0$$

then

$$\lim_{\Delta n\to 0}\langle (\Delta n)^{-1}\,[T(n+\Delta n)\,C\,S^*\,(n+\Delta n)f_1 - S(n)\,C\,S^*\,(n)f_1], f_2\rangle - \langle BS(n)CS^*\,(n)f_1, f_2\rangle - \langle S(n)\,C\,S^*\,(n)B^*\,f_1, f_2\rangle = 0$$

Thus

$$\lim_{\Delta n\to 0}\langle (\Delta n)^{-1}[S(n+\Delta n)CS^*(n+\Delta n)f_1 - S(n)CS^*(n)f_1], f_2\rangle = \langle S(n)CS^*(n)B^*f_1, f_2\rangle + \langle BS(n)CS^*(n)f_1, f_2\rangle$$

$$\frac{d}{dn}\langle X(n)f_1, f_2\rangle = \langle X^*(n)B^*f_1, f_2\rangle + \langle BX(n)f_1, f_2\rangle$$

From lemma

Lemma (2. 2)

Suppose further that $\{X(n)\}_{n\geq 0}$ has a local solution which is bounded and continuous operators and satisfy $X(0)=0$ and$X^*B^* + BX = C$.Then $X(n)\equiv 0 \quad \forall\, n\geq 0$

Proof:

From Lemma (2.1) we get

$$\frac{d}{dn}\langle S(n)\,XS^*(n)f_1, f_2\rangle = \langle S(n)\,XS^*(n)B^*f_1, f_2\rangle + \langle BS(n)\,XS^*(n)f_1, f_2\rangle$$

$$\frac{d}{dn}[e^{-\gamma n}\,\langle S(n)\,XS^*(n)f_1, f_2\rangle] = -\gamma e^{-\gamma n}\langle S(n)\,XS^*(n)f_1, f_2\rangle + e^{-\gamma n}\left[\frac{d}{dn}\langle S(n)\,XS^*(n)f_1, f_2\rangle\right]$$

$$\frac{d}{dn}[e^{-\gamma n}\,\langle S(n)\,XS^*(n)f_1, f_2\rangle] = -\gamma e^{-\gamma n}\langle\, S(n)\,XS^*(n)f_1, f_2\rangle + e^{-\gamma n}[\langle S(n)\,XS^*(n)B^*f_1, f_2\rangle + \langle BS(n)\,XS^*(n)f_1, f_2\rangle]$$

Taking the integral

$$\int_0^\infty \frac{d}{dn}[e^{-\gamma n}\,\langle S(n)\,XS^*(n)f_1, f_2\rangle]dn$$
$$= \int_0^\infty -\gamma e^{-\gamma n}\langle S(n)\,XS^*(n)f_1, f_2\rangle dn + \int_0^\infty e^{-\gamma n}\,[\langle S(n)\,XS^*(n)B^*f_1, f_2\rangle + \langle BS(n)\,XS^*(n)f_1, f_2\rangle]dn$$

$$-\langle X\,f_1, f_2\rangle = -\gamma\int_0^\infty e^{-\gamma n}\,\langle S(n)\,XS^*(n)f_1, f_2\rangle dn + \int_0^\infty e^{-\gamma n}\,[\langle S(n)\,XS^*(n)B^*f_1, f_2\rangle + \langle BS(n)\,XS^*(n)f_1, f_2\rangle]dn$$

We can define the operator

$$R_\gamma Y = \int_0^\infty e^{-\gamma n}\;S(n)\,YS^*(n)\,dn$$

$$-\langle X f_1, f_2\rangle = -\gamma \langle R_\gamma X f_1, f_2\rangle + \int_0^\infty e^{-\gamma n}\,[\langle S(n)\,X(n)S^*(n)B^* f_1, f_2\rangle + \langle BS(n)\,XS^*(n) f_1, f_2\rangle]dn$$

$$-\langle X(s) f_1, f_2\rangle = -\gamma\langle R_\gamma X(s) f_1, f_2\rangle + \int_0^\infty e^{-\gamma n}\left[\frac{d}{ds}\langle S(n)X(s)S^*(n) f_1, f_2\rangle\right]dn$$

$$\gamma\langle R_\gamma X(s) f_1, f_2\rangle = \langle X(s) f_1, f_2\rangle + \int_0^\infty e^{-\gamma n}\frac{d}{ds}\langle S(n)\,X(s)S^*(n) f_1, f_2\rangle dn$$

$$\gamma\langle R_\gamma X(s) f_1, f_2\rangle = \langle X(s) f_1, f_2\rangle\rangle + \frac{d}{ds}\int_0^\infty e^{-\gamma n}\langle S(n)\,X(s)S^*(n) f_1, f_2\rangle dn$$

$$\gamma\langle R_\gamma X(s) f_1, f_2\rangle = \langle X(s) f_1, f_2\rangle + \frac{d}{ds}\langle R_\gamma X(s) f_1, f_2\rangle$$

Multiplying by $e^{-\gamma s}$

$$e^{-\gamma s}\gamma\langle R_\gamma X(s) f_1, f_2\rangle = e^{-\gamma s}\langle X(s) f_1, f_2\rangle + e^{-\gamma s}\frac{d}{ds}\langle R_\gamma X(s) f_1, f_2\rangle$$

$$-e^{-\gamma s}\langle X(s) f_1, f_2\rangle = -e^{-\gamma s}\gamma\langle R_\gamma X(s) f_1, f_2\rangle + e^{-\gamma s}\frac{d}{ds}\langle R_\gamma X(s) f_1, f_2\rangle$$

Taking the integral

$$-\int_0^n e^{-\gamma}\langle X(s) f_1, f_2\rangle ds = \int_0^n -\gamma e^{-\gamma s}\langle R_\gamma X(s) f_1, f_2\rangle ds + \int_0^n e^{-\gamma s}\frac{d}{ds}\langle R_\gamma X(s) f_1, f_2\rangle ds$$

$$-\int_0^n e^{-\gamma s}\langle X(s) f_1, f_2\rangle ds = \int_0^n [-\gamma e^{-\gamma s}\langle R_\gamma X(s) f_1, f_2\rangle ds + e^{-\gamma s}\frac{d}{ds}\langle R_\gamma X(s) f_1, f_2\rangle]ds$$

$$-\int_0^n e^{-\gamma s}\langle X(s) f_1, f_2\rangle ds = \int_0^n \frac{d}{ds}\left[e^{-\gamma s}\langle R_\gamma X(s) f_1, f_2\rangle\right]ds$$

$$-\int_0^n e^{-\gamma s}\langle X(s) f_1, f_2\rangle ds = \left[e^{-\gamma n}\langle R_\gamma X(n) f_1, f_2\rangle\right] - \left[e^{-\gamma(0)}\langle R_\gamma X(0) f_1, f_2\rangle\right]$$

$$-\int_0^n e^{-\gamma s}\langle X(s) f_1, f_2\rangle ds = \left[e^{-\gamma n}\langle R_\gamma X(n) f_1, f_2\rangle\right]$$

$$e^{-\gamma n}\langle R_\gamma X(n) f_1, f_2\rangle = -\int_0^n e^{-\gamma s}\langle X(s) f_1, f_2\rangle ds$$

$$\langle R_\gamma X(n) f_1, f_2\rangle = -\int_0^n e^{\gamma(n-s)}\langle X(s) f_1, f_2\rangle\, ds$$

$$\|R_\gamma X(s)\| \le \int_0^\infty e^{-\gamma n} M\|X(s)\| \to 0 \quad as \quad \gamma \text{ approaches } 0$$

$$\lim_{\gamma\to 0}\int_0^n e^{\gamma(n-s)}\langle X(s) f_1, f_2\rangle ds = 0$$

Theorem (2.3)

For B and C defined on H for which:

I. C bounded and self-adjoint.
II. B is unbounded and a generator of a strongly semigroup, the C_0-semigroup $\{S(n)\}_{n\ge 0}$
III. 0 is contained in the closure of $\{S(n)CS^*(n)\}$

Then $\{Q_n : n \geq 0\}$ is relatively compact, where $Q_n f_1 = -\int_0^n S(s)CS^*(n)f_1 ds, \quad \forall f_1 \in H$, and there is an unbounded solution for the equation $X^*B^* + BX = C$.

Proof:

From the conditions above, we can get $n_\alpha \to \infty$ s.t. $S(n_\alpha)CS^*(n_\alpha)$ is weakly convergent to 0 and Q_{n_α} converge to a bounded operator Q

setting $X(n) = S(n)CS^*(n)$ and using lemma (2.1)

$\Rightarrow \frac{d}{dn}\langle X(n)f_1, f_2\rangle = \langle X^*(n)B^*f_1, f_2\rangle + \langle BX(n)f_1, f_2\rangle$

Since a net $X(n_\alpha)$ converges to 0 weakly then

$$-\langle Ch, f_2\rangle = \lim_{\alpha\to\infty}\int_0^{n_\alpha}\frac{d}{ds}\langle X(s)f_1, f_2\rangle ds$$

$$= \lim_{\alpha\to\infty}\int_0^{n_\alpha}\{\langle X^*(n)B^*f_1, f_2\rangle + \langle BX(n)f_1, f_2\rangle\}ds$$

$$= \lim_{\alpha\to\infty}\left\{\langle\int_0^{n_\alpha} BS(s)CS^*(n)\, hds, f_2\rangle + \langle\int_0^{n_\alpha} S(s)CS^*(n)B^*\, hds, f_2\rangle\right\}$$

$$= \lim_{\alpha\to\infty}\left\{\langle\int_0^{n_\alpha} S(s)CS^*(n)B\, hds, f_2\rangle + \langle\int_0^{n_\alpha}\left(S(s)CS^*(n)\right)^* B^*\, hds, f_2\rangle\right\}$$

$$= \lim_{\alpha\to\infty}\left\{\langle -Q_{n_\alpha}^* B^* hds, f_2\rangle + \langle -BQ_{n_\alpha} hds, f_2\rangle\right\}$$

Then

$$-\langle Ch, f_2\rangle = \langle -B^*Q_{n_\alpha}^* hds, f_2\rangle + \langle -BQ_{n_\alpha} hds, f_2\rangle$$

Thus $\langle Ch, f_2\rangle = \langle B^*Q_{n_\alpha}^* B^* hds, f_2\rangle + \langle BQ_{n_\alpha} hds, f_2\rangle$

Therefore $B^*Q^*f_1 + BQh = Ch$ which indicate Q is a bounded solution ■

Uniqueness of $(BX + X^*B^* = C)$ is discussed next.

<u>Theorem (2.4)</u>

suppose additionally that all bounded operators $Y: H \to H$, such that $S(n)YS^*(n)$ is convergent to 0 as n goes to ∞. Then there is a unique solution for $X^*B^* + BX = C$.

Proof :

Start with the X as a solution for $X^*B^* + BX = C$ which is bounded, then

$\frac{d}{dn}\langle S(n)XS^*(n)f_1, f_2\rangle = \langle S(n)X^*S^*(n)B^*f_1, f_2\rangle + \langle S(n)BXS^*(n)f_1, f_2\rangle$

$\frac{d}{dn}\langle S(n)XS^*(n)f_1, f_2\rangle = \langle S(n)[X^*B^* + BX]S^*(n)f_1, f_2\rangle$

$= \langle S(n)CS^*(n)f_1, f_2\rangle$

According to lemma (2.1) one can have:

Therefore;

$$\langle Q_n f_1, f_2\rangle = -\int_0^n \langle S(s)CS^*(s)f_1, f_2\rangle\, ds$$

$$= -\int_0^n \frac{d}{dn}\langle S(n)XS^*(n)f_1, f_2\rangle\, ds$$

$$= -\langle S(n)XS^*(n)f_1, f_2\rangle + \langle S(0)XS^*(0)f_1, f_2\rangle$$

$$= -\langle (S(n)XS^*(n) - X)f_1, f_2\rangle \qquad \forall f_1, f_2 \in H$$

Implying

$Q_n = -S(n)XS^*(n) + X$

Limiting the above as n approaches ∞ we have

$\lim_{n\to\infty} Q_n = \lim_{n\to\infty} X - \lim_{n\to\infty} S(n)XS^*(n)$,

then $X = \lim_{n\to\infty} Q_n$.

3. Conclusion

A solution was found for the equation $X^*B^* + BX = C$. Where B and B^* are unbounded operators. This paper presents several definitions and theorems pertinent to semigroup theory. The solution was obtained by utilizing the semigroup of bounded linear operators.

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