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Numerical Solution of Volterra's Integral Equations by Collocation and Taylor Method

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ABSTRACT

In this article, two numerical methods (collocation method and Taylor method) are introduced to solve Volterra integral equation system. In the collocation method, using Bessel polynomials and collocation points, we convert the device into a matrix form and solve the device using the matrix form and obtain an approximate solution for the device. This answer is such that the larger the N becomes, the approximate answer is closer to the exact answer of the device. In Taylor method, With the use of the Taylor series, the system of integral equations is transformed into a matrix equation, and by integrating the output of this new system, a system of algebraic equations is created. By working through this device, we can obtain a rough solution for the Integral Equations device. Using these two methods when the answers are polynomial, we can get real answers.

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1. Introduction

The theory of integral equations is one of the most important branches of mathematical science. In principle, its importance in terms of boundary value issues is in the theory of equations with partial derivatives. Integral equations have many applications in mathematical sciences, physics, chemistry, technical sciences, etc. Integral equations have appeared in mathematics for many years. Because its origin dates back to the integral theory of February (1811) [1-3]. In fact, the development of integral equations began in the late 19th century؛ because it was around the years 1900 to 1903 that an Italian mathematician named Volterra worked on it and also a Swedish mathematician named Fred Helm proposed a new method in the same years to solve Dirichle problem. Since then, until the present day, integral equations it has been the subject of many mathematicians' research. Single integral equations were introduced by two people earlier this century in connection with two completely different problems. One of them was Hlbert, who, while researching some of the boundary value issues of analytical theory, and the other was Poincare, who faced these issues in general tidal theory [4-7].

Fred Helm's theorems are fundamental theorems of integral equations. Of course, Riesz generalized the above in a broader way؛ because he considered an integral equation as an operator and generalized Fred helm's theorems for

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this operator. So far, several numerical methods have been presented to solve Volterra system of integral equations, such as homotopy perturbation method, Haar function method, Tau method, Legendre matrix method, Adomian method and Galerkin method [4-9].

Numerical integral methods, such as the Euler-Chebyshev and Runge-Kutta processes, are also the basis for solving the many integral equations that come up in engineering, biology, and physics. The Taylor method for resolving the Volterra integral issue is initially presented by Kanwal and Liu [10]. Sezar then expanded this approach to include differential equations and the Volterra integral equation. Sezar and Yalcinbas have also solved Volterra-Fredhelm linear integral-differential equations using the previously discussed technique [11-14]. The approximate solution to the integral equations and the device of these equations may be found using the highly efficient Taylor technique of expansion [15]. The first time, the Bessel matrix method for solving Volterra-Differential integral equations and the systems of these equations was presented by Caesar, and then the same researcher introduced the Bessel collocation method for the approximate solution of this system of equations. In this article, we will introduce this method [16-18].

2. Taylor and Bessel polynomials

Definition 2.1

The series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ where x_0 and a_n are fixed numbers, we call a power series and $\{a_n\}$ the sequence of coefficients of the power series. Now, if we take the derivative from the above-mentioned series term by term, by calculating the derivatives of f at the point $x = a$, we get:

$$y(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

$$y'(x) = \sum_{n=1}^{\infty} n\ a_n (x-a)^{n-1}, \qquad y'(a) = a_1$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)\ a_n (x-a)^{n-2}, \qquad y''(a) = 1 \times 2a_2 \ \rightarrow a_2 = \frac{y''(a)}{2!}$$

$$y'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)\ a_n (x-a)^{n-3}, \qquad y'''(a) = 1 \times 2 \times 3a_3 \ \rightarrow a_3 = \frac{y'''(a)}{3!}$$

Likewise, we get a_n as follows:

$$a_n = \frac{y^n(a)}{n!}$$

If we put these values in the initial series, assuming $y^{(0)}(a) = y(a) = a_0$, we get:

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(a)}{n!} (x-a)^n$$

In the above equation, the series on the right side is called the Taylor series y around the point a.

Theorem 2.2 (Taylor theorem)

Suppose $[a, b]$ on $f^{(n+1)}$ and $f \in c^n[a, b]$. Also $x_0 \in [a, b]$ in this case, for each $x \in [a, b]$, there exists a point such that $\xi(x) \in (a, b)$such that:

$$f(x) = p_n(x) + R_n(x)$$

Where in

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

$$R_n(x) = \frac{f^{(n+1)}(\zeta_0)}{(n+1)!}(x - x_0)^{n+1}$$

Here $p_n(x)$ is the Taylor polynomial of the nth degree around the point x_0 and $R_n(x)$ is called the residual term or shear error depending on $p_n(x)$.

Theorem 2.3 (Weirastras approximation)

If f is defined on $[a, b]$ and is continuous and $\varepsilon > 0$ is also assumed, then there exists a polynomial such as p, which is defined on $[a, b]$ such that:

$$|f(x) - p(x)| < \varepsilon; \quad \forall x \in [a, b]$$

The result is that for each function defined and continuous over a closed interval, there are a few sentences that are as close to the assumed function as we want.

3. Bessel polynomials

Differential equation:

$$x^2y'' + xy' + (x^2 - p^2)y = 0 \qquad (1)$$

Where P is a non-negative fixed number, it is called Bessel equation and its solutions are known as Bessel function (Bessel polynomials). These functions appeared for the first time in Daniel Bernoulli's investigations on hanging chain oscillations and then again in Euler's theory for circular membrane vibrations and Bessel's studies on planetary motion. Recently, Bessel functions have many applications in physics and engineering in relation to the propagation of elastic waves, the motion of planets, and especially in many problems related to potential theory and diffusion, which have cylindrical symmetry [19]. These functions appear even in some interesting problems of pure mathematics [20].

The Bessel equation has two answers. One of them is called the Bessel function of the first type and the other is called the Bessel function of the second type. Here we explain the method of obtaining the Bessel function of the first type of order p ie $J_p(x)$ and the method of obtaining the Bessel function of the second type ie $Y_p(x)$ refer to [21]. By obtaining the answers, the general answer of (1) is given as follows:

$$y = c_1J_p(x) + c_2Y_p(x)$$

4. Numerical solution of Voltrey linear integral equations using collocation method

We consider a system of linear Volterra integral equations as follows:

$$\sum_{j=1}^{k} p_{i.j}(x)\,{}_{y_j}(x) = g_i(x) + \int_a^x \sum_{j=1}^{k} k_{i.j}(x.t)y_j(x)dt. \quad ii = 11.2. \dots . ik.\ 0 \le ia \le x.\ it \le ib \qquad (2)$$

Where $y_j(x)$ is an unknown function and $p_{i.j}(x)$, $g_i(x)$ and $k_{i.j}(xi.t)$ are a continuous functions defined on the interval $a \le x.t \le b$ and the function $k_{i.j}(x.t)$ for $(i.j = 1.2. \dots . k)$ has MacLauren expansion.

"The goal is to find an approximate solution for" (2), "for this purpose we use the finite Bessel series":

$$iy_i(ix) = \sum_{n=0}^{N} a_{i.ni}\, iJ_n(ix).\;\; i = 1.2. \dots .\, ik.\, 0 \le a \le ix.t \le ib \tag{3}$$

Sore that's in (3) $a_{i.n}$ for $(n = 0,1,2,\cdots,N)$ are unknown Bessel coefficients and $J_n(x)$ force $(n = 0,1,2,\cdots,N)$ areapolynomials bellowsare the first types of bases. (N is chosen as desired).

$$J_n(xx) = \sum_{k=01}^{\left[\!\left[\frac{N1-n1}{2}\right]\!\right]} \frac{(-1)^k}{k!\,(k+n)!}\left(\frac{x1}{21}\right)^{2k1+n} \tag{4}$$

5. Basic matrix relations

First, we can write $J_n(x)$ in the following matrix form:

$$J^T(X) = DX^T(\,xx) \leftrightarrow J(xx\,) = Xi(\,x\,)iD^T \tag{5}$$

Which in (5) $J_{(x)}$ and $X_{(x)}$ are defined as follows:

$$J(x) = [J_0(x)\; J_1(x)\; \dots\; J_N(x)]\,,\quad X(x) = [1iX^1 \quad iX^2 \;\; \dots \;\; iX^N]$$

And in (5), D is defined in the following two ways:

If N is odd, D is as follows:

$$D = \begin{bmatrix} \frac{1}{0!\,0!\,2^0} & 0 & \frac{-1}{1!\,1!\,2^2} \; 0 & \dots & \frac{(-1)^{\frac{N-1}{2}}}{\left(\frac{N-1}{2}\right)!\left(\frac{N-1}{2}\right)!\,2^{N-1}} & 0 \\ 0 & \frac{1}{0!\,1!\,2^1} & 0 & \dots & 0 & \frac{(-1)^{\frac{N-1}{2}}}{\left(\frac{N-1}{2}\right)!\left(\frac{N+1}{2}\right)!\,2^{N}} \\ 0 & 0 & \frac{1}{0!\,2!\,2^2} & \dots & \frac{(-1)^{\frac{N-3}{2}}}{\left(\frac{N-3}{2}\right)!\left(\frac{N+1}{2}\right)!\,2^{N-1}} & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{0!\,(N-1)!\,2^{N-1}} & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{0!\,N!\,2^N} \end{bmatrix}_{(N+1)\times(N+1)}$$

And if N is even, then D is as follows:

$$D = \begin{bmatrix} \frac{1}{0!\,0!\,2^0} & 0 & \frac{-1}{1!\,1!\,2^2} & \dots & 0 & \frac{(-1)^{\frac{N}{2}}}{\left(\frac{N}{2}\right)!\left(\frac{N}{2}\right)!\,2^{N}} \\ 0 & \frac{1}{0!\,1!\,2^1} & 0 & \dots & \frac{(-1)^{\frac{N-2}{2}}}{\left(\frac{N-2}{2}\right)!\left(\frac{N}{2}\right)!\,2^{N-1}} & 0 \\ 0 & 0 & \frac{1}{0!\,2!\,2^2} & \dots & 0 & \frac{(-1)^{\frac{N-2}{2}}}{\left(\frac{N-2}{2}\right)!\left(\frac{N+2}{2}\right)!\,2^{N}} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{0!\,(N-1)!\,2^{N-1}} & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{0!\,N!\,2^N} \end{bmatrix}_{(N+1)\times(N+1)}$$

We can also write the relationship (3) as a matrix:

$$[iy_j(x)] = iJ(x)iA_j.\quad ij = 1.2. \dots .\, ik \tag{6}$$

Where $A_j = [a_{j.o} \quad a_{j.1} \quad \dots \quad a_{j.N}]^T$ and by substituting relation (4) in (5) we have:

$$[y_j(x)] = X(x)D^T A_j, \quad j = 1.2. \dots . k \tag{7}$$

The matrix $y_{(x)}$ can be expressed as follows:

$$y(x) = \overline{X}(x)\overline{D}A \tag{8}$$

As in (8):

$$y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_k(x) \end{bmatrix} . \overline{X}(x) = \begin{bmatrix} X(x) & 0 & \dots & 0 \\ 0 & X(x) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & X(x) \end{bmatrix}_{k\times k} . \ \overline{D} = \begin{bmatrix} D^T & 0 & \dots & 0 \\ 0 & D^T & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D^T \end{bmatrix} . \ A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix}$$

6. Numerical solution of the linear Volterra integral equation device using Taylor's method

We consider a system of linear Volterra integral equations in the following general form:

$$a_{mj}(x)y_j(x) = f_m(x) + \int_a^x k_{mj}(x.t)y_j(t)\,dt. \quad m.j = 1.2. \dots . s \qquad a \le t \le x$$

Or

$$\sum_{j=1}^{s} a_{mj}(x)y_j(x) = f_m(x) \int_a^x \sum_{j=1}^{s} k_{mj}(x.t)y_j(t)dt \tag{9}$$

Where $a_{mj}(x). f_m(x), k_{mj}(x.t)(m.j = 1.2. \dots . s)$ are defined in the interval $a \le c \le x$ and can be Taylor series in the interval $x = c.$ $a \le c \le x$ is expanded.

We assume that (9) has a unique answer. The purpose of this article is to find the answer to the device (9) in the figure below:

$$y_m(x) = \sum_{n=0}^{N} \frac{1}{n!} y_m^{(n)}(c)(x-c)^n \quad m = 1.2. \dots . s \quad a \le c \le x \tag{10}$$

Which is a Taylor polynomial of degree N at $x = c$ and $y_m^{(n)}(c)$ for $n = 0,1, \cdots, N$ are unknown coefficients that must be determined.

6.1. Fundamental relations

First, we write the systems of linear Volterra integrals given in (9) as follows:

$$E_m(x) = f_m(x) + V_m(x) \quad \text{or} \quad E_m(x) = I_m(x) \quad m = 1.2. \dots . s \tag{11}$$

So that:

$$E_m(x) = \sum_{j=1}^{s} a_{mj}(x)y_j(x) \quad m = 1.2. \dots . s$$

$$I_m(x) = f_m(x) + \int_a^x \sum_{j=1}^{s} k_{mj}(x.t)y_j(t)dt \quad m = 1.2. \dots . s$$

Here, the terms $E_m(x)$ and $I_m(x)$ are called the first and second parts of the integral equations, respectively, from (11). To get the answer to the problem given in the form (10), we first take the derivative from (9) with respect to x up to order n and we have:

$$E_m^{(n)}(x) = f_m^{(n)}(x) + V_m^{(n)}(x) \quad \text{or } E_m^{(n)}(x) = I_m^{(n)}(x) \quad m = 1,2,\dots,s \tag{12}$$

And then we analyze the expressions $E_m(x)$ and $I_m(x)$:

6.2. Matrix display for the first part

The expression $E_m^{(n)}(x)$ can be clearly written in the following form:

$$E_m^{(n)}(x) = \left[\sum_{j=1}^{s} a_{mj}(x)y_j(x)\right]^{(n)} \quad m = 1,2,\dots,s \qquad n = 0,1,2,\dots \tag{13}$$

When we deal with the derivation of the product of functions, we use Leibniz's command, which we refer to:

$$[p(x).y(x)]^{(n)}|_{x=c} = \sum_{i=0}^{n} \binom{n}{i} p^{(n-i)}(c)y^{(i)}(c)$$

And in the final solution of relation (13) by putting $x = c$, we have the following form:

$$E_m^{(n)}(x) = \left[\sum_{j=1}^{s} a_{mj}(x)y_j(x)\right]_{x=c}^{(n)} = \sum_{j=1}^{s}\sum_{i=1}^{N} \binom{n}{i} a_{mj}^{(n-i)}(c)y_j^{i}(c) \tag{14}$$

${y_j}^{(0)}(c), {y_j}^{(1)}(c), \dots, {y_j}^{(N)}(c), j = 1,2,\dots,s$. The coefficients are unknown and their number is N+1 $a_{mj}^{(i)}(c)$ for $m, j = 1,2,\dots,s$ are definite Taylor coefficients, which is the value of the i-th derivative of the function $a_{(m,j)}(x)$ at $x = c$. Now we display the matrix form of expression (14) as follows:

$$E = WY \tag{15}$$

So that:

$$Y = [y_1^{(0)} \quad y_1^{(1)} \dots y_1^{(N)} \dots y_s^{(0)} \quad y_s^{(1)} \dots y_s^{(N)}]^T = \begin{bmatrix} Y_1 \\ \vdots \\ Y_s \end{bmatrix}$$

$$[w] = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1s} \\ w_{21} & w_{22} & \cdots & w_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ w_{s1} & w_{s2} & \cdots & w_{ss} \end{bmatrix}$$

The elements of w are defined as follows:

$$w_{11} = \begin{bmatrix} (w_{11})_{00} & (w_{11})_{01} & \cdots & (w_{11})_{0s} \\ (w_{11})_{10} & (w_{11})_{11} & \cdots & (w_{11})_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ (w_{11})_{s0} & (w_{11})_{s1} & \cdots & (w_{11})_{ss} \end{bmatrix}, \dots, w_{1s} = \begin{bmatrix} (w_{1s})_{00} & (w_{1s})_{01} & \cdots & (w_{1s})_{0s} \\ (w_{1s})_{10} & (w_{1s})_{11} & \cdots & (w_{1s})_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ (w_{1s})_{s0} & (w_{1s})_{s1} & \cdots & (w_{1s})_{ss} \end{bmatrix}$$

$$w_{21} = \begin{bmatrix} (w_{21})_{00} & (w_{21})_{01} & \cdots & (w_{21})_{0s} \\ (w_{21})_{10} & (w_{21})_{11} & \cdots & (w_{21})_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ (w_{21})_{s0} & (w_{21})_{s1} & \cdots & (w_{21})_{ss} \end{bmatrix}, \dots, w_{2s} = \begin{bmatrix} (w_{2s})_{00} & (w_{2s})_{01} & \cdots & (w_{2s})_{0s} \\ (w_{2s})_{10} & (w_{2s})_{11} & \cdots & (w_{2s})_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ (w_{2s})_{s0} & (w_{2s})_{s1} & \cdots & (w_{2s})_{ss} \end{bmatrix} \tag{16}$$

$$\vdots \qquad\qquad \ddots \qquad\qquad \vdots$$

$$w_{s1}=\begin{bmatrix}(w_{s1})_{00} & (w_{s1})_{01} & \cdots & (w_{s1})_{0s}\\ (w_{s1})_{10} & (w_{s1})_{11} & \cdots & (w_{s1})_{1s}\\ \vdots & \vdots & \ddots & \vdots\\ (w_{s1})_{s0} & (w_{s1})_{s1} & \cdots & (w_{s1})_{ss}\end{bmatrix},\ldots,w_{ss}=\begin{bmatrix}(w_{ss})_{00} & (w_{ss})_{01} & \cdots & (w_{ss})_{0s}\\ (w_{ss})_{10} & (w_{ss})_{11} & \cdots & (w_{ss})_{1s}\\ \vdots & \vdots & \ddots & \vdots\\ (w_{ss})_{s0} & (w_{ss})_{s1} & \cdots & (w_{ss})_{ss}\end{bmatrix}$$

The values $(n,m=1,2,\ldots,s,i,j=0,1,2,\ldots,N)$ $(W_{nm})_{ij}$ are described as follows:

$$(W_{11})_{ij}=\binom{i}{j}a_{11}^{(i-j)}(c),(W_{12})_{ij}=\binom{i}{j}a_{12}^{(i-j)}(c),\ldots,(W_{1s})_{ij}=\binom{i}{j}a_{1s}^{(i-j)}(c)$$

$$(W_{21})_{ij}=\binom{i}{j}a_{21}^{(i-j)}(c),(W_{22})_{ij}=\binom{i}{j}a_{22}^{(i-j)}(c),\ldots,(W_{2s})_{ij}=\binom{i}{j}a_{2s}^{(i-j)}(c)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \ddots \qquad\qquad \vdots$$

$$(W_{s1})_{ij}=\binom{i}{j}a_{s1}^{(i-j)}(c),(W_{s2})_{ij}=\binom{i}{j}a_{s2}^{(i-j)}(c),\ldots,(W_{ss})_{ij}=\binom{i}{j}a_{ss}^{(i-j)}(c) \qquad (17)$$

Note the relation (18). It is clear that for:

$$a_{ij}^{(L)}=0,\quad i,j=0,1,2,\ldots,s,\quad l<0$$

and for

$$\binom{i}{j}=0 \quad i,j\in z, \qquad j>i\ ,j<0$$

In this case, in relation (18) for $m=n+1,n+2,\ldots,N;n=1,2,3,\ldots N-1$ leads to $(W_{nm})_{ij}=0$. So, the matrix becomes:

$$w=\begin{bmatrix}(w_{11})_{00} & (w_{11})_{01} & \cdots & (w_{11})_{0s} & & (w_{1s})_{00} & (w_{1s})_{01} & \cdots & (w_{1s})_{0s}\\ (w_{11})_{10} & (w_{11})_{11} & \cdots & (w_{11})_{1s} & & (w_{1s})_{10} & (w_{1s})_{11} & \cdots & (w_{1s})_{1s}\\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots\\ (w_{11})_{s0} & (w_{11})_{s1} & \cdots & (w_{11})_{ss} & & (w_{1s})_{s0} & (w_{1s})_{s1} & \cdots & (w_{1s})_{ss}\\ & \vdots & & & \ddots & & \vdots & & \\ (w_{s1})_{00} & (w_{s1})_{01} & \cdots & (w_{s1})_{0s} & & (w_{ss})_{00} & (w_{ss})_{01} & \cdots & (w_{ss})_{0s}\\ (w_{s1})_{10} & (w_{s1})_{11} & \cdots & (w_{s1})_{1s} & & (w_{ss})_{10} & (w_{ss})_{11} & \cdots & (w_{ss})_{1s}\\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots\\ (w_{s1})_{s0} & (w_{s1})_{s1} & \cdots & (w_{s1})_{ss} & & (w_{ss})_{s0} & (w_{ss})_{s1} & \cdots & (w_{ss})_{ss}\end{bmatrix} \qquad (18)$$

7. Numerical examples

In this section, we will examine numerical examples. A numerical example is provided, and then an example is provided at the end to compare the accuracy of these two methods.

Example 7.1

Solve the following system of linear Voltre integral equations considering $y_i(x)$ as a finite Bessel series.

$$\begin{cases} y_1(x)+xy_2(x)=\sin(x)+x\cos(x)+\int_0^x [ix^2\ i\cos(it)\ iy_1(it)-ix^2 i\sin(t)\ iy_2(ti)]idt \\ y_2(x)-2xy_1(x)=\cos(x)-2x\sin(x)+\int_0^x [\sin(x)\ i\cos(it)\ iy_1(it)-i\sin(xi)\ i\sin(ti)\ iy_2(it)]idt \end{cases} ;0\le x\le 1 \qquad (19)$$

Solution:

The exact solution of the above equations is $y_2(x)=\cos x\ ,y_1(x)=\sin(x)$, now by using the finite Bessel series and choosing $N=2$, we get the approximate solutions.

$$y_i(x) = \sum_{n=0}^{2} a_{i.n} \quad J_n(x) \qquad i = 1.2$$

By comparing the device (19) with the device of (2), we have:

$Ki = 2.$

$ig_1(x) = i\sin(x) + x\,i\cos(x) \qquad . \qquad ig_2(x) = i\cos(x) - 2xi\,i\sin(ix)\,.$

$P_{1.1}(x) = 11 \qquad .\, P_{1.21}(x) = x. \quad P_{21.1}(x) = -2xi. \quad P_{z2.2}(zxz) = 11.$

$Kz_{1.11}(x.t) = x^2\cos(t). \quad iK_{1.21}(ix.it) = -ix^2\, i\sin(it)$

$Ki_{2.11}(xi.ti) = \sin i(xi)\cos(t). \qquad K_{12.12}(xi.ti) = -\,i\sin(ix)\, i\sin(it)$

Using they relation (19), the sets off collocations points are obtained as follows:

$$x_0 = 0 + \frac{1-0}{2}0 = 0$$

$$x_1 = 0 + \frac{1-0}{2} \times 1 = \frac{1}{2}$$

$$x_2 = 0 + \frac{1-0}{2} \times 2 = 1$$

So $\{x_0 = 0, x_1 = \frac{1}{2}.x_2 = 1\}$ is the set of collocation points.

Using relation (18), the basic matrix equation is as follows:

$\{PX\overline{D} - \overline{\overline{X}}\,\overline{\overline{D}}\,\overline{K_v}\,\overline{\overline{Q}}\}A = G$

so that we have:

$$P(x) = \begin{bmatrix} 1 & x \\ -2x & 1 \end{bmatrix}, \quad p = \begin{bmatrix} p(0) & 0 & 0 \\ 0 & p(\frac{1}{2}) & 0 \\ 0 & 0 & p(1) \end{bmatrix},$$

$$X = \begin{bmatrix} \overline{X} & (0) \\ \overline{X} & (\frac{1}{2}) \\ \overline{X} & (1) \end{bmatrix} \quad . \quad \overline{X}(0) = \begin{bmatrix} X(0) & 10 \\ 0 & X(101) \end{bmatrix} \;.\, \overline{X}\left(\frac{1}{21}\right) = \begin{bmatrix} X(\frac{1}{2}) & 0 \\ 0 & X(\frac{1}{12}) \end{bmatrix} \;.\; \overline{X}(11) = \begin{bmatrix} X(1) & 0 \\ 0 & X(1) \end{bmatrix}$$

which using relation (19) we have:

$$x(01) = [11 \quad 01 \quad 01] \;.\; x\left(\frac{1}{2}\right) = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} \end{bmatrix} \;.\; x(11) = [11 \quad 11 \quad 11]$$

And we also have:

$$\overline{\overline{X}} = \begin{bmatrix} \overline{X}(01) & 10 & 10 \\ 10 & \overline{X}(\frac{1}{2}) & 10 \\ 10 & 10 & \overline{X}(11) \end{bmatrix}$$

Because the value of $N = 2$ is even, then:

$$D^T=\begin{bmatrix}11 & 110 & 10\\ 10 & 1\frac{11}{12} & 10\\ \frac{11}{14} & 110 & \frac{11}{18}\end{bmatrix}\quad .\ \overline{D}=\begin{bmatrix}D^T & 0\\ 0 & D^T\end{bmatrix}\quad .\ \overline{\overline{D}}=\begin{bmatrix}\overline{D} & 0 & 0\\ 0 & \overline{D} & 0\\ 0 & 0 & \overline{D}\end{bmatrix}$$

Now we get the K_v matrix:

$$K_v=\begin{bmatrix}K_b^{11} & K_b^{12}\\ K_b^{21} & K_b^{22}\end{bmatrix}$$

Now we want to get the K_b^{11} matrix from the following equation:

$$K_b^{11}=(D^T)^{-1}\ \ K_t^{ij}\ \ \ D^{-1}$$

So first, it is necessary to obtain the matrix K_t^{ij}.

On the other hand, according to the definition of ${}^tK_{mn}^{ij}$ we have:

$${}^tK_{mn}^{ij}=\frac{1}{m!\,n!}\frac{\partial^{m+n}\ \ K(0.0)}{\partial x^m\ \ \partial t^n}$$

$$K_t^{ij}=\left[{}^tK_{mn}^{ij}\right]\qquad m.n=0.1.2$$

So, we have:

$$K_t^{11}=\begin{bmatrix}K_{00}^{11} & K_{01}^{11} & K_{02}^{11}\\ K_{10}^{11} & K_{11}^{11} & K_{12}^{11}\\ K_{20}^{11} & K_{21}^{11} & K_{22}^{11}\end{bmatrix}\qquad .\ \ K_{11}(x.t)=x^2\cos t$$

$$K_{00}^{11}=\frac{1}{0!\ \ 0!}\frac{\partial^0\ K(0.0)}{\partial x^0\ \partial t^0}=x^2\cos t\,\Big|_{(0.0)}=0$$

$$K_{01}^{11}=\frac{1}{0!\ \ 1!}\frac{\partial^1\ K(0.0)}{\partial x^0\ \partial t^1}=-x^2\sin t\,\Big|_{(0.0)}=0$$

$$K_{02}^{11}=\frac{1}{0!\ \ 2!}\frac{\partial^2\ K(0.0)}{\partial x^0\ \partial t^2}=-x^2\cos t\,\Big|_{(0.0)}=0$$

$$K_{10}^{11}=\frac{1}{1!\ \ 0!}\frac{\partial^1\ K(0.0)}{\partial x^1\ \partial t^0}=2x\cos t\,\Big|_{(0.0)}=0$$

$$K_{11}^{11}=\frac{1}{1!\ \ 1!}\frac{\partial^2\ K(0.0)}{\partial x^1\ \partial t^1}=-2x\sin t\,\Big|_{(0.0)}=0$$

$$K_{12}^{11}=\frac{1}{1!\ \ 2!}\frac{\partial^3\ K(0.0)}{\partial x^1\ \partial t^2}=-2x\cos t\,\Big|_{(0.0)}=0$$

$$K_{20}^{11}=\frac{1}{2!\ \ 0!}\frac{\partial^2\ K(0.0)}{\partial x^2\ \partial t^0}=\frac{1}{2}(2\cos t)\,\Big|_{(0.0)}=1$$

$$K_{21}^{11}=\frac{1}{2!\ \ 1!}\frac{\partial^3\ K(0.0)}{\partial x^2\ \partial t^1}=\frac{1}{2}(-2\sin t)\,\Big|_{(0.0)}=0$$

$$K_{22}^{11}=\frac{1}{2!\ \ 2!}\frac{\partial^4\ K(0.0)}{\partial x^2\ \partial t^2}=\frac{1}{4}(-2\cos t)\,\Big|_{(0.0)}=\frac{-1}{2}$$

So, we have:

$$K_t^{11}=\begin{bmatrix}0&0&0\\0&0&0\\1&0&\frac{-1}{2}\end{bmatrix}$$

Now we have:

$$K_b^{11}=(D^T)^{-1}\ K_t^{11}\ \ D^{-1}=\begin{bmatrix}1&0&0\\0&2&0\\2&0&8\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\1&0&\frac{-1}{2}\end{bmatrix}\begin{bmatrix}1&0&2\\0&2&0\\0&0&8\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\8&0&-16\end{bmatrix}$$

In the same way, each of the $K_b^{22}, K_b^{21}, K_b^{12}$ matrices can be obtained and finally the K_v matrix will be as follows:

$$K_v=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\8&0&-16&0&-16&0\\0&0&0&0&0&0\\-2&0&4&0&4&0\\0&0&0&0&0&0\end{bmatrix}\quad,\ \overline{K}_v=\begin{bmatrix}K_v&0&0\\0&K_v&0\\0&0&K_v\end{bmatrix}$$

Now we want to get the matrix $Q(x)$ according to the relation $Q(x)=DH(x)D^T$ on the other hand

$$H(x)=[h_{rs}(x)]\qquad ,h_{rs}(x)=\frac{x^{r+s+1}-a^{r+s+1}}{r+s+1}\qquad r.s=0.1.\dots.N$$

So, we can write:

$$H(x)=\begin{bmatrix}h_{00}&h_{01}&h_{02}\\h_{10}&h_{11}&h_{12}\\h_{20}&h_{21}&h_{22}\end{bmatrix}$$

$$h_{00}=\frac{x^1}{1}\quad .h_{01}=\frac{x^2}{2}\quad .h_{02}=\frac{x^3}{3}$$

$$h_{10}=\frac{x^2}{2}\quad .h_{11}=\frac{x^3}{3}\quad .h_{12}=\frac{x^4}{4}$$

$$h_{20}=\frac{x^3}{3}\quad .h_{21}=\frac{x^4}{4}\quad .h_{22}=\frac{x^5}{5}$$

As a result, the $H(x)$ matrix will be as follows:

$$H(x)=\begin{bmatrix}x&\frac{x^2}{2}&\frac{x^3}{3}\\\frac{x^2}{2}&\frac{x^3}{3}&\frac{x^4}{4}\\\frac{x^3}{3}&\frac{x^4}{4}&\frac{x^5}{5}\end{bmatrix}$$

So, the $Q(x)$ matrix will be as follows:

$$Q(x)=\begin{bmatrix}1 & 0 & \frac{1}{4}\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & \frac{1}{8}\end{bmatrix}\begin{bmatrix}x & \frac{x^2}{2} & \frac{x^3}{3}\\ \frac{x^2}{2} & \frac{x^3}{3} & \frac{x^4}{4}\\ \frac{x^3}{3} & \frac{x^4}{4} & \frac{x^5}{5}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & 0\\ \frac{1}{4} & 0 & \frac{1}{8}\end{bmatrix}$$

$$=\begin{bmatrix}x-\frac{x^3}{6}+\frac{x^5}{80} & \frac{x^2}{4}-\frac{x^4}{32} & \frac{x^3}{24}-\frac{x^5}{160}\\ \frac{x^2}{4}-\frac{x^4}{32} & \frac{x^3}{12} & \frac{x^4}{64}\\ \frac{x^3}{24}-\frac{x^5}{160} & \frac{x^4}{64} & \frac{x^5}{320}\end{bmatrix}$$

And so, we have:

$$\bar{Q}(x_s)=\begin{bmatrix}Q(x_s) & 0\\ 0 & Q(x_s)\end{bmatrix} \quad .\bar{\bar{Q}}=\begin{bmatrix}\bar{Q}(0)\\ \bar{Q}(\frac{1}{2})\\ \bar{Q}(1)\end{bmatrix}$$

$$G=\begin{bmatrix}g(0)\\ g(\frac{1}{2})\\ g(1)\end{bmatrix} \quad . \; g(0)=\begin{bmatrix}0\\ 1\end{bmatrix} \quad .g\left(\frac{1}{2}\right)=\begin{bmatrix}2863/3118\\ 432/1085\end{bmatrix} \quad .g(1)=\begin{bmatrix}561/406\\ -753/659\end{bmatrix}$$

$$A=\begin{bmatrix}A_1\\ A_2\end{bmatrix} \quad . \quad A_1=\begin{bmatrix}a_{1.0}\\ a_{1.1}\\ a_{1.2}\end{bmatrix} \quad . \quad A_2=\begin{bmatrix}a_{2.0}\\ a_{2.1}\\ a_{2.2}\end{bmatrix}$$

By replacing the matrices obtained in relation (19) and performing equation calculations, we have the following matrix:

$$[W;G]=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & ; & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & ; & 1\\ 1541/1879 & 241/1024 & 145/4826 & 511/1024 & 25/192 & 33/2048 & ; & 2863/3118\\ -591/841 & -113/512 & -443/15360 & 449/512 & 23/96 & 31/1024 & ; & 432/1085\\ -1/40 & 5/16 & 23/240 & 19/16 & 2/3 & 5/32 & ; & 561/406\\ -29/40 & -13/16 & -53/240 & 5/16 & 1/3 & 3/32 & ; & -753/659\end{bmatrix}$$

Bye solving this device, the unknown Bessel coefficients are obtained:

$$A=[0 \quad 5909/2686 \quad -771/343 \quad 1 \quad -195/5038 \quad -508/315]^T$$

By replacing these numbers in relation (16), approximate answers are obtained and according to relation (18) we have:

$$J_0(x)=\frac{(-1)^0}{0!\,0!}\left(\frac{x}{2}\right)^0+\frac{(-1)^1}{1!\,1!}\left(\frac{x}{2}\right)^2=1-\frac{x^2}{4}$$

$$J_1(x)=\frac{(-1)^0}{0!\,1!}\left(\frac{x}{2}\right)^1=\frac{x}{2}$$

$$J_2(x)=\frac{(-1)^0}{0!\,2!}\left(\frac{x}{2}\right)^2=\frac{x^2}{8}$$

$y_{1.2}(x) = a_{1.0}\ J_0(x) + a_{1.1}\ J_1(x) + a_{1.2}\ J_2(x)$

$y_{2.2}(x) = a_{2.0}\ J_0(x) + a_{2.1}\ J_1(x) + a_{2.2}\ J_2(x)$

$y_{1.2}(x) = 11.9996277580x - 10.280976732383x^{20}$

$iy_{12.21}(ixi) = 11 - (10.1193529200065e - 11)ix - 10.4515872448221xi^{20}$

In a similar way, we can get approximates answers witch N =10, N =5 using this's methods:

$$y_{1.5}(x) = 0.9999976422151x + (10.297609798352e - 41)x^{20} - 10.16808109183x^3 + (10.319794928355e - 31)x^{40} + (10.801544277479e - 21)x^{50}.$$

$$y_{2.5}(x) = 1 - (10.184800992035e - 14)x - 110.499763047743x^{20} - (10.11465739065e - 20)ix^3 + (10.443423532282e - 1)x^{40} - (10.305939800922e - 12)ix^{50}$$

And

$$y_{1.101}(x) = x - (10.921757315336e - 110)x^{20} - 10.1666666655071xi^3 - (10.812838264920e - 8)xi^{40} + (10.833336902465e - 2)xi^{50} - (10.103698169354e - 6)xi^6 - (10.198208865170e - 31)xi^{70} - (10.269896643927e - 61)xi^{80} + (10.298752982861e - 5)xi^9 - (10.116415087416e - 16)x^{101}$$

$$iy_{12.10}(xi) = 1 - (10.752255497617ei - 11)xi - 10.4999999997851xi^2 - (10.248570007402ei - 8)xi^3 + (10.416666822694ei - 11)x^4 - (10.59270029354e - 17)x^5 - (0.1388746246671ei - 2)x^6 - (10.24171821569e - 9)x^7 - (10.50109112079e - 4)x^8 - (10.114893666515e - 16)xi^9 - (0.21921123313e - 6)ix^{100}.$$

In Tables 1-4, numerical results, exact answers and approximate answers for N = 2,5,10 are shown.

Table 1 - Comparison between exact and approximate solutions of $y_1(x)$ for N = 2, 5, 10.

	Exact answer	Approximate answer		
x_i	$y_1(x_i) = \sin(x_i)$	$N = 2. y_{1.2}(x_i)$	$N = 5. y_{1.5}(x_i)$	$N = 10. y_{1.10}(x_i)$
0	0	0	0	0
0.2	0.198669330795061	0.2087534858646680	0.198669330622303	0.193669330794415
0.4	0.389418342308651	0.395028833138720	0.389418364539241	0.389413342307331
0.6	0.564642473395035	0.558826041822120	0.564643473951095	0.564642473391378
0.8	0.717356090899523	0.700145111914880	0.717358897188496	0.717356090835151
1	0.841470984307896	0.818986043417000	0.341554531714980	0.841470983950814

Table 2 - Comparison between exact and approximate solutions of $y_2(x)$ for N = 2, 5, 10.

	Exact answer	Approximate answer		
x_i	$y_2(x_{ii}) = i\sin(ix_{ii})$	$Ni = 2iy_{2.2}(x_{ii})$	$Ni = 5. iy_{2.5}(x_i i)$	$Ni = 10. iy_{2.10}(x_i i)$
00	10	10	10	10
0.2	0.9800665778412420	0.9780659262058200	0.9800665782369890	0.9800665778412330
0.4	0.9210609940028850	0.9200048728258800	0.9210609755934460	0.9210609940028550
0.6	0.8253356149096780	0.8258163898601800	0.8253354249783590	0.8253356149099080
0.8	0.6977067093471650	0.6955018273087200	0.6967099438675680	0.6967067093317130
1	0.5403023058681400	0.5290598351715000	0.5403548534702170	0.5403023053309720

Tables 3 - Maximums absolutes errors force $y_{10}(x)$.

N1	12	51	71	101	121
$ie_{1.Ni}$	$2.2126\times 110^{-2}1$	16.2855×110^{-3}	11.2191×10^{-6}	18.6444×10^{-10}	$14.7\times 110^{-12}1$

Table 4 - Maximum absolute error for $iy_2(x)$.

iN	21	51	71	101	121
$ie_{2.N}$	11.1242×101^{-2}	15.2548×101^{-5}	117.67×10^{-7}	17.516×10^{-10}	03.2868×110^{-12}

We define the maximum absolute error as follows:

$$ee_{i.Ne} = \|ey_{i.Ne}(exe) - ey_i(exe)\|_\infty = e\max\{|iy_{i.Ni}(xi) - iy_i(xi)|.\, i0 \le ix \le 11\}$$

In this example, it can be seen that the error decreases as N increases.

Example 7.2

In this example, Volterra's integral equation system is presented in the following form:

$$\begin{cases} y_1(x) = e^{2x}\left(-\frac{1}{2}x^2 + \frac{1}{4}x + 1\right) + e^{-2x}\left(x + \frac{1}{4}\right) - \frac{3}{4}x - \frac{1}{4} + \int_0^x [xty_1(t) + (x+t)y_2(t)]dt \\ iy_{20}(x) = ei^{-2xi}\left(2xi^2 + ix + \frac{5}{4}\right) - \frac{1}{4}ie^{2x} - \frac{1}{2}ix^{21} + \int_0^x [(ix - it)ity_1(it) + (ix + it)^2 iy_2(it)]idt \;.\; 01 \le x \le 11 \end{cases}$$

The real answer of this device is $y_1(x) = e^{2x}$ and $y_2(x) = e^{-2x}$.

Based on the collocation method, we have:

$$ip_{11.1}(xi) = ip_{2.2}(x) = 1 \quad . \quad ip_{1.2}(x) = ip_{2.1}(ix) = 0.$$

$$ig_1(xi) = ei^{2xi}\left(-\frac{1}{2}ix^2 + \frac{1}{4}xi + 10\right) + e^{-2xi}\left(xi + \frac{1}{4}\right) - \frac{3}{4}x - \frac{1}{4}.$$

$$ig_2(xi) = ie^{-2xi}\left(2ix^2 + ix + \frac{5}{4}\right) - \frac{1}{4}ie^{2xi} - \frac{1}{2}x^2$$

By choosing $N = 5,7,10$ and with a process similar to Example 7.1, we can get the approximate answers of this device using the collocation method. Also, by using Taylor's method and with a process similar to Example 7.2, we can get the approximate answers of this device. In the Tables 5 and 6, a numerical comparison between the absolute error of each method and the actual answer value of the device is shown. Based on these tables, it can be seen that the collocation method has higher accuracy, meaning less error than the Taylor method.

Table 5 – A numerical comparison between the absolute error of each method and the actual ansvalue of the device

	Taylor method		Collocation method		
x_i	$e_{1.5}(x_i)$	$e_{1.7}(x_i)$	$e_{1.5}(x_i)$	$e_{1.7}(x_i)$	$e_{1.10}(x_i)$
011	011	011	101	011	011
110.21	6.031ee-161	1.7e-181	2.1190ee-10061	3.6977ee-0771	2.0139ee-0131
0.411	4.10261ee-141	4.562ee-161	2.4829ee-10061	1.0646ee-0071	1.4766ee-0121
0.611	4.98092ee-131	1.22774ee-141	1.0519ee-10051	1.3253ee-0071	5.0400ee-0121
0.811	2.99178ee-121	1.289997ee-131	1.5004ee-10051	3.1732ee-0071	1.1426ee-0111
111	1.22389ee-111	8.10372ee-131	9.3000ee-10051	4.9275ee-0071	1.8997ee-0111

Table 6 - A numerical comparison between the absolute error of each method and the actual ansvalue of the device

	Taylor method		Collocation method		
x_i	$e_{2.5}(x_i)$	$e_{2.7}(x_i)$	$e_{2.5}(x_i)$	$e_{2.7}(x_i)$	$e_{2.10}(x_i)$
01	011	011	011	110	011
0.21	5.3793e-611	10.55e-81	3.8983ee-0061	3.7623ee-081	3.7779ee-0121
0.41	3.26297ee-411	3.8187ee-61	9.6593ee-0061	6.8382ee-0081	8.6704ee-0121
0.61	1353021e-31	9.39605ee-41	2.0735ee-0051	1.1996ee-0071	1.6507ee-0121
0.81	1.88779ee-211	9.02263ee-31	3.9343ee-0051	1.7505ee-0071	3.4660ee-0111
11	16.86686ee-11	5.17655ee-31	1.2025ee-0051	6.4723ee-0071	1.0121ee-0111

Conclusions

In this article, the aim is to numerically solve the linear Volterra integral equation device. Two numerical methods, the same location method and the Taylor method, have been proposed to solve these devices. Based on the collocation method, the unknown functions i.e. $y_i(x)$ can be approximated by finite Bessel polynomials. In this method, when N becomes larger, the dimensions of the matrices become larger and the calculations increase, but it gives us a better answer. Because when N is larger, a greater number of Bessel polynomial sentences, $J_n(x)$ for $n = 0,1,2,\ldots,N$ are used. This makes the approximate answer closer to the exact answer and the error decreases. By using the methods presented in this article, we can get approximate answers and exact answers in some problems. When the real answers are in the form of polynomials, we can find these answers using these two methods. In Taylor's method, after calculating Taylor's coefficients and placing in Taylor's expansion, for every value of x, the solutions to $y(x)$ are simply determined. Additionally, the precise solution can be obtained when the integral device has a linear independent polynomial solution of degree n or fewer.

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