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Stability Result in Lattice Random Normed Space

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ABSTRACT

We use the direct method to study the generalized Hyers-Ulam stability of a mixed type(cubic-quartic) functional equation

$$f(\varkappa - 2y) + f(\varkappa + 2y) + 6f(\varkappa) = 4[f(\varkappa - y) + f(\varkappa + y)] + 3f(2y) - 24f(y)$$

for all $\varkappa, y \in W$, in LRN-space.

MSC..

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1. Introduction

A functional equation Ƒ is deemed stable if any solution ƒ of Ƒ is approximately close to a precise solution. The last several decades a number of mathematicians have dug deep into issues related to the stability of functional equation. This solves a lot of difficulties, including the optimization and the approximation theory problems, and it also handles the mistake that happens when the approximate solutions to an equation are substituted with their a precise functions. It was Ulam [12] who initially brought this issue to light. After that scientist Hyers [10] provided affirmative answers to Ulam's inquiries regarding additive groups. Subsequently, both scientists, T. Aoki [4] and Rassias [11], extended Hyers's theory. Aoki focused on additive mappings, while Rassias investigated linear mapping's, specifically examining the indefinite Cauchy difference regulated by the logical operator "$\|\varkappa\|^s + \|y\|^s$ " when $s \in [0,1)$. Similarly to Rassias, Gajda provided a favorable answer when $p \geq 1$. Gavruta [9] in 1994 introduced a broader interpretation of Rassias's theorem encompassing all earlier findings. In this formulation "$\|\varkappa\|^s + \|y\|^s$" was substituted by a control function of $\delta(\varkappa, y)$. We addressed the stability of the functional equation

$$f(\varkappa - 2y) + f(\varkappa + 2y) + 6f(\varkappa) = 4[f(\varkappa - y) + f(\varkappa + y)] + 3f(2y) - 24f(y)$$

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for all $\varkappa, y \in W$. (1.1)

In LRN-space using the direct method, which is generally related to the amount of difference between the a precise and approximate solutions, this is what we worked on in this research. It is possible for functional equation (1.1) to be either a quartic or a cubic function, depending on whether it is an even or odd function, respectively . This makes it a mixed-functional equation.

2. Preliminaries

2.1. Definition [5]

An order set $l = (Ł, \geq_Ł)$ ($\geq_Ł$ relation on Ł) is called complete lattice, if every anon-empty subset A of Ł admits supremum and infimum, alsoinf $Ł = 0_l$, sup$Ł = 1_l$.

$\Delta_Ł^+$ is the set of distribution function

$$\Delta_Ł^+ = \{g | g: \mathbb{R} \cup \{-\infty, \infty\} \to Ł, g(0) = 0_l, g(+\infty) = 1_l, \text{g is anon} - \text{decreasing, left continuous on } \mathbb{R}\}.$$

The subset $D_Ł^+$ of $\Delta_Ł^+$ defined as $D_Ł^+ = \{P \in \Delta_Ł^+ : \lim_{\varkappa \to +\infty} P(\varkappa) = 1_l\}$.

The function $Є_\circ(t) = \begin{cases} 0_l & \text{, if } t \leq 0 \\ 1_l & \text{, if } t > 0 \end{cases}$ is the maximal element for $D_Ł^+$.

2.2. Definition [5]

Let $₦: Ł \to Ł$ be function and denoted by a negation function, iff

(1) $₦(0_l) = 1_l, ₦(1_l) = 0_l$.

(2) $₦(\beta) \leq ₦(\alpha)$ if $\beta \geq \alpha$.

This function involutive, iff $₦\big(₦(\alpha)\big) = \alpha, \forall \alpha \in Ł$.

2.3. Definition [5]

The mapping $T: Ł^2 \to Ł$ (Ł is complete lattice), is Triangular norm (t-norm) on Ł, if the following are achieving:

1) $T(\varkappa, 1_l) = \varkappa, \forall \varkappa \in Ł$ (Boundary condition).

2) $T(\varkappa, y) = T(y, \varkappa), \forall (\varkappa, y) \in Ł^2$ (Commutativity condition).

3) $T\big(\varkappa, T(y, z)\big) = T(T(\varkappa, y), z), \forall (\varkappa, y, z) \in Ł^3$ (Associativity condition).

4) $y \leq_Ł y' \Longrightarrow T(\varkappa, y) \leq_Ł T(\varkappa, y'), \forall (\varkappa, y, y') \in Ł^3$ (Monotonicity condition).

The t-norm has four basic examples given by :

1) $T_m(\varkappa, y) = \min\{\varkappa, y\}$.

2) $T_L(\varkappa, y) = \max\{\varkappa + y - 1_l, 0\}$.

3) $T_p(\varkappa, y) = \varkappa \cdot y$.

4) $T_D(\varkappa, y) = \begin{cases} \min\{\varkappa, y\} \text{ if } \max\{\varkappa, y\} = \sup Ł \\ 0 \qquad\qquad\qquad \text{o.w.} \end{cases}$.

2.4. Definition [5]

The mapping $Ć: Ł^2 \to Ł$ triangular co-norm (t-conorm) on Ł (Ł is complete lattice), if the following are achieving:

(1) $\acute{C}(\varkappa, 0_l) = \varkappa, \forall \varkappa \in Ł$ (Boundary condition).

(2) $\acute{C}(\varkappa, y) = \acute{C}(y, \varkappa), \forall (\varkappa, y) \in Ł^2$ (Commutativity condition).

(3) $\acute{C}(\varkappa, \acute{C}(y, z)) = \acute{C}(\acute{C}(\varkappa, y), z), \forall (\varkappa, y, z) \in Ł^2$ (Associativity condition).

(4) $y \geq_L y' \Longrightarrow \acute{C}(\varkappa, y) \geq_L \acute{C}(\varkappa, y'), \forall (\varkappa, y, y') \in Ł^3$ (Monotonicity condition).

2.5. Definition [5]

Assume that $Ł = [0,1]^2$ and $\circ$, $\diamond$ are continuous t-norm and continuous t-conorm, respectively . Then the continuous t-norm is called continuous t-representable, i.e. for all $\varkappa = (\varkappa_1, \varkappa_2)$, $y = (y_1, y_2) \in Ł^2$, $Ŧ(\varkappa, y) = (\varkappa_1 \circ y_1, \varkappa_2 \diamond y_2)$

2.6. Definition [5]

A lattice random normed space (LRN-space), is $(W, P, Ŧ)$, W is a vector space and $P: W \times [0, \infty) \to D_Ł^+$ (mapping), Ŧ is a t-norm on lattice set Ł. So that the following is achieved:

1) $P(a, t) = \mathcal{E}_\circ(t), \forall t > 0$ iff $a = 0$.

2) $P(\alpha a, t) = P\left(a, \frac{t}{|\alpha|}\right), \forall\ 0 \neq a \in W, t \geq 0$.

3) $P(a + b, t + s) \geq_L Ŧ\big(P(a, t), P(b, s)\big), \forall a, b \in W$ and $t, s \geq 0$.

2.7. Example [5]

If $(Ł, \leq_Ł)$ defined by

$$Ł = \{(\varkappa_1, \varkappa_2): (\varkappa_1, \varkappa_2) \in [0,1]^2, x_1 + \varkappa_2 \leq 1\}$$

$(\varkappa_1, \varkappa_2) \leq_Ł (y_1, y_2)$ if and only if $\varkappa_1 \leq_Ł y_1, y_2 \leq_Ł \varkappa_2$.

$\forall \varkappa = (\varkappa_1, \varkappa_2), y = (y_1, y_2) \in Ł^2$.

Then $(Ł, \leq_Ł)$ complete lattice.

We denoted its unite by $0_Ł = (0,1), 1_Ł = (1,0)$.

Assume that $(W, \|\cdot\|)$ is normed space, $Ŧ(\varkappa, y) = (\min\{\varkappa_1, y_1\}, \max\{\varkappa_2, y_2\})$, $\forall \varkappa = (\varkappa_1, \varkappa_2), y = (y_1, y_2) \in Ł^2$.

$$P(\varkappa, t) = \left(\frac{t}{t + \|\varkappa\|}, \frac{\|\varkappa\|}{t + \|\varkappa\|}\right), \forall t \in \mathbb{R}^+.$$

Hence $(W, P, Ŧ)$ is an LRN-space.

2.8. Definition [5]

If $(W, P, Ŧ)$ is LRN-space,

a) Let $\{\varkappa_n\}$ be any sequence in W is called convergent to $\varkappa \in W$, if for any $\varepsilon > 0, r \in Ł/\{0_l, 1_l\}, \exists N \in \mathbb{Z}^+$ such that $P(\varkappa_n - \varkappa, \varepsilon) >_Ł ₦(r), \forall n \geq N$.

b) Let $\{\varkappa_n\}$ be sequence in W is called Cauchy sequence, if for any $\varepsilon > 0$ and $r \in Ł/\{0_l, 1_l\}, \exists N \in \mathbb{Z}^+$ such that $P(\varkappa_n - \varkappa_m, \varepsilon) >_Ł ₦(r), \forall n \geq m \geq N$.

c) Let $\{\varkappa_n\}$ is Cauchy sequence in LRN-space $(W, P, Ŧ)$, then W is called complete, if $\varkappa_n \to \varkappa, \forall \varkappa \in W$.

3. "Stability of functional equation(1.1) in LRN-space if ƒ is even mapping".

3.1. Lemma [7]

Assume that W, Y are linear space and $f: W \to Y$ mapping satisfy (1.1), then (1.1) is quartic functional equation if ƒ is even function.

3.2. Theorem

Assume that $l = (Ł, \geq_Ł)$ is complete lattice and $f: W \to Y$ is even mapping from real linear space W to complete LRN-space (Y, P, T) with $f(0) = 0$ for which $Ͼ: W \times W \to D_Ł^+$ such that

$P(f(\varkappa - 2y) + f(\varkappa + 2y) + 6f(\varkappa) - 4[f(\varkappa - y) + f(\varkappa + y)] - 24f(y), t) \geq_Ł Ͼ(\varkappa, y, t)$

for all $\varkappa, y \in X, t > 0$. (3.1)

for all $\varkappa, y \in X, t > 0$.

If $\lim_{n\to\infty} T_{i=1}^{\infty}\left(Ͼ(0, 2^{n+i-1}y, 2^{4n+3i}t)\right) = 1_l$ (3.2)

and $\lim_{n\to\infty} Ͼ(2^n\varkappa, 2^n y, 2^{4n}t) = 1_l$ (3.3)

for all $\varkappa, y \in W, t > 0$, then there exists a unique quartic mapping $K: W \to Y$ suchthat

$P(f(y) - K(y), t) \geq_Ł T_{i=1}^{\infty}\left(Ͼ(0, 2^{i-1}y, 2^{3i+1}t)\right) \quad \forall y \in W, t > 0$ (3.4)

Note: $1_l = \sup Ł > ₦(k), \forall k \in Ł/\{0_l, 1_l\}$.

Proof: Putting $\varkappa = 0$ in (3.1) we obtaining

$$P(2f(2y) - 32f(y), t) \geq_Ł Ͼ(0, y, t)$$

$$P\left(2\left(f(2y) - 16f(y)\right), t\right) \geq_Ł Ͼ(0, y. t)$$

$$P\left(\frac{f(2y)}{2^4} - f(y), t\right) \geq_Ł Ͼ(0, y, 2^5 t) \geq_Ł Ͼ(0, y, 2^4 t) \quad (3.5)$$

$\forall y \in W, t > 0, z \in \mathbb{N}$.

$$P\left(\frac{f(2^{z+1}y)}{2^{4(z+1)}} - \frac{f(2^z y)}{2^{4z}}, t\right) \geq_Ł Ͼ\left(0, 2^z y, 2^{4(z+1)}t\right) \quad (3.6)$$

$$P\left(\frac{f(2^{z+1}y)}{2^{4(z+1)}} - \frac{f(2^z y)}{2^{4z}}, \frac{t}{2^{z+1}}\right) \geq_Ł Ͼ\left(0, 2^z y, 2^{3(z+1)}t\right) \quad (3.7)$$

$\forall y \in W, t > 0, z \in \mathbb{N}$, since $\sum_{z=1}^{n} \frac{1}{2^z} < 1$, by the tringle inequality, we have

$$P\left(\frac{f(2^n y)}{2^{4n}} - f(y), t\right) \geq_Ł P\left(\frac{f(2^n y)}{2^{4n}} - f(y), \sum_{z=1}^{n} \frac{t}{2^z}\right)$$

$$\geq_Ł T_{z=0}^{n-1}\left(P\left(\frac{f(2^{z+1}y)}{2^{4(z+1)}} - \frac{f(2^z y)}{2^{4z}}, \frac{t}{2^{z+1}}\right)\right)$$

$$\geq_Ł T_{z=0}^{n-1}\left(Ͼ\left(0, 2^z y, 2^{3(z+1)}t\right)\right)$$

$$= T_{i=1}^{n}\left(Ͼ(0, 2^{i-1}y, 2^{3i}t)\right) \quad (3.8)$$

We should prove the sequence $\left\{\frac{f(2^n y)}{2^{4n}}\right\}$ is convergence, we exchange y with $2^m y$ in (3.8) for all $n, m \in \mathbb{N}$.

$$P\left(\frac{f(2^{n+m}y)}{2^{4(n+m)}}-\frac{f(2^m y)}{2^{4m}},t\right)\geq_L T_{i=1}^{n}\left(C(0,2^{i+m-1}y,2^{3i+4m}t)\right) \quad (3.9)$$

by (3.2) when $n,m\to\infty$ the sequence $\left\{\frac{f(2^n y)}{2^{4n}}\right\}$ is Cauchy sequence. Therefore we have $K(y)=\lim_{n\to\infty}\frac{f(2^n y)}{2^{4n}}$ for all $y\in W$.

Now exchanging $\varkappa, y$ with $2^m\varkappa, 2^m y$ respectively, in (3.1) we have

$$P\left(\frac{f(2^m\varkappa-2^m y)}{2^{4m}}+\frac{f(2^m\varkappa+2^m y)}{2^{4m}}+\frac{6f(2^m\varkappa)}{2^{4m}}-\frac{4[f(2^m\varkappa-2^m y)+f(2^m\varkappa+2^m y)]}{2^{4m}}-\frac{24f(2^m y)}{2^{4m}},t\right)\geq_L C(2^m\varkappa,2^m y,2^{4m}t) \quad \forall\varkappa,y\in W, m\in N, t>0 \quad (3.10)$$

Taking $m\to\infty$, we show that $K(y)$ satisfying (3.1), for all $\varkappa,y\in W$, hence $K(y)$ is quartic mapping.

To prove (3.4) taking $n\to\infty$ in (3.8) we get (3.4).

Finally, to prove the uniqueness of the quartic mapping K, suppose that there exist quartic mapping K' which satisfies (3.7), since $K(2^n y)=2^{4n}K(y)$ then $K'(2^n y)=2^{4n}K'(y)$, for $y\in X, n\geq 1$.

$$P\left(K(y)-K'(y),t\right)=P\left(K(2^n y)-K'(2^n y),2^{4n}t\right)$$

$$\geq_L T\left(P(K(2^n y)-f(2^n y),2^{4n-1}t),P\left(f(2^n y)-K'(2^n y),2^{4n-1}t\right)\right)$$
$$\geq_L T\left(T_{i=1}^{\infty}\left(C(0,2^{n+i-1}y,2^{4n+3i}t)\right),T_{i=1}^{\infty}\left(C(0,2^{n+i-1}y,2^{4n+3i}t)\right)\right) \quad (3.11)$$

$\forall y\in W, n\in\mathbb{N}, t>0$, by taking $n\to\infty$ in (3.4), we have $K=K'$.

3.3. Corollary

Let $l=(Ł,\geq_Ł)$ be complete lattice, $f: W\to Y$, even mapping from W (real linear space), to (Y,P,T) (complete LRN space), $f(0)=0$ and

$$P(f(\varkappa-2y)+f(\varkappa+2y)+6f(x)-4[f(\varkappa-y)+f(\varkappa+y)]-24f(y),t)\geq_Ł \frac{t}{t+\delta\|\varkappa_\circ\|}$$

$\forall\varkappa,y\in W, t>0$.

Then there exists a unique quartic mapping $K: W\to Y$ such that

$$P(f(y)-K(y),t)\geq_Ł (T_m)_{i=1}^{\infty}\left(\frac{2^{3i+1}t}{2^{3i+1}t+\delta\|\varkappa_\circ\|}\right) \ \forall y\in W, t>0.$$

Proof This corollary can easily proving by using theorem (3.2) and replacing $C(\varkappa,y,t)$ with $\frac{t}{t+\delta\|x_\circ\|}$ which is in $D_Ł^+$.

If $T=T_m$ or $T=T_p$, Ł must be equal [0,1]. Also if $T=T_L$ or $T=T_D$, all the condition of theorem (3.2) are met. Since

$$\lim_{n\to\infty}(T_L)_{i=1}^{\infty}\left(C(0,2^{n+i-1}y,2^{4n+3i}t)\right)=\lim_{n\to\infty}\max\left\{\left(\sum_{i=1}^{\infty}C(0,2^{n+i-1}y,2^{4n+3i}t)-1_l\right)+1_l,0_l\right\}$$

$$=\lim_{n\to\infty}\max\left\{\left(\sum_{i=1}^{\infty}\frac{2^{4n+3i}t}{2^{4n+3i}t+\delta\|\varkappa_\circ\|}-1_l\right)+1_l,0_l\right\}$$

$$=\lim_{n\to\infty}\max\left\{\left(\sum_{i=1}^{\infty}\frac{-\delta\|\varkappa_\circ\|}{2^{4n+3i}t+\delta\|\varkappa_\circ\|}\right)+1_l,0_l\right\}$$

$$=\lim_{n\to\infty}\max\left\{1_l-\frac{\delta\|\varkappa_\circ\|}{7\cdot 2^{4n}t+\delta\|\varkappa_\circ\|},0_l\right\}$$

$= \max\{1_l, 0_l\} = 1_l.$

$$\lim_{n\to\infty}(T_D)_{i=1}^{\infty}\left(\mathrm{C}(0, 2^{n+i-1}y, 2^{4n+3i}t)\right) = \lim_{n\to\infty}(T_D)_{i=1}^{\infty}\left(\frac{2^{4n+3i}t}{2^{4n+3i}t+\delta\|x_\circ\|}\right)$$

$$= \lim_{n\to\infty}\begin{cases}\min\left(\dfrac{2^{4n+3i}t}{2^{4n+3i}t+\delta\|x_\circ\|}\right)_{i=1}^{\infty} & \text{if } \max\left(\dfrac{2^{4n+3i}t}{2^{4n+3i}t+\delta\|x_\circ\|}\right)_{i=1}^{\infty} = 1_l \\ 0_l & \text{o.w}\end{cases}$$

$= \lim_{n\to\infty}\left(\frac{2^{4n+3i}t}{2^{4n+3i}t+\delta\|x_\circ\|}\right) = 1_l.$

And for all t-norms

$$\lim_{n\to\infty}\mathrm{C}(2^n\varkappa, 2^n y, 2^{4n}t) = \lim_{n\to\infty}\left(\frac{2^{4n}t}{2^{4n}t+\delta\|x_\circ\|}\right) = 1_l.$$

3.4. Corollary

Let $(Ł = [0,1], \geq_Ł)$ be complete lattice, $f: W \to Y$, even mapping from W (real linear space), to (Y, P, T) (complete LRN space), $f(0) = 0$. If $T = T_m$ or $T = T_p$.

$$P\big(f(\varkappa-2y)+f(\varkappa+2y)+6f(\varkappa)-4[f(\varkappa-y)+f(\varkappa+y)]-24f(y)\big) \geq_Ł \frac{t}{t+\alpha(\|\varkappa\|^s+\|y\|^s)}, \forall\varkappa, y \in W, t > 0.$$

Then there exists a unique quartic function $K: W \to Y$ such that

$$P(f(y)-K(y), t) >_Ł T_{i=1}^{\infty}\left(\frac{2^{4n+3i}t}{2^{4n+3i}t+\alpha(\|2^{n+i-1}y\|^s)}\right), \forall y \in W, t > 0.$$

Proof This corollary can easily proving by using theorem (3.2) and replacing $\mathrm{C}(x, y, t)$ with $\frac{t}{t+\alpha(\|\varkappa\|^s+\|y\|^s)}$, which is in $D_Ł^+$, and put $0 < s < 4$.

3.5. Corollary

Assume that $(Ł, \geq_Ł)$ is complete lattice, $f: W \to Y$, even mapping from W (real linear space), to (Y, P, T) (complete LRN space), $f(0) = 0$. If $T = T_m$ and $T = T_p$, and

$$P(f(\varkappa-2y)+f(\varkappa+2y)+6f(\varkappa)-4[f(\varkappa-y)+f(\varkappa+y)]-24f(y), t) \geq_L 1_l - \frac{\|\varkappa\|}{t+\|\varkappa\|}, \forall\varkappa, y \in W, t > 0.$$

Then there exists a unique quartic function $K: W \to Y$ such that

$$P(f(y)-K(y), t) \geq_L \sup Ł.$$

Proof This corollary can easily proving by using theorem (3.2) and replacing $\mathrm{C}(\varkappa, y, t)$ with $1_l - \frac{\|\varkappa\|}{t+\|\varkappa\|}$, which is in $D_Ł^+$. If $T = T_m$ the prove is direct. If $T = T_p$ then L must be equal [0,1].

$$\lim_{n\to\infty}(T_m)_{i=1}^{\infty}\left(\mathrm{C}(0, 2^{n+i-1}y, 2^{4n-3i}t)\right) = \lim_{n\to\infty}(T_m)_{i=1}^{\infty}(1_l - 0)$$

$= 1_l.$

$$\lim_{n\to\infty}(\mathrm{T_p})_{i=1}^{\infty}\left(\mathrm{C}(0,2^{n+i-1}y,2^{4n+3i}t)\right)=\lim_{n\to\infty}\left((1_l-0_l)\cdot(1_l-0_l)\ldots\ldots\right)$$

$= 1_l.$

And for all t-norm.

$$\lim_{n\to\infty}\mathrm{C}(2^n\varkappa,2^n y,2^{4n}t)=\lim_{n\to\infty}\left(1_l-\frac{2^n\|\varkappa\|}{2^{4n}t+2^n\|\varkappa\|}\right)$$

$= 1_l.$

Then the result is achieved.

4) “Stability of functional equation (1.1) in LRN-space iff is odd mapping”

4.1. Lemma [7]

Let W, Y be linear space and $f: W \to Y$ mapping satisfy (1.1), then (1.1) is cubic functional equation if f is odd function.

4.2. Theorem

Let $l = (Ł, \geq_Ł)$ be complete lattice and $f: W \to Y$, even mapping from real linear space W to complete lattice random normed space (Y, P, T) with $f(0) = 0$ for which $ß: W \times W \to D_Ł^+$ such that

$P(f(\varkappa-2y)+f(\varkappa+2y)+6f(\varkappa)-4[f(\varkappa-y)+f(\varkappa+y)]) \geq_Ł ß(\varkappa,y,t)$

$\forall \varkappa, y \in W, t > 0.$ (4.1)

If $\lim_{n\to\infty} T_{i=1}^{\infty}\left(ß(3^{n+i-1}\varkappa,3^{n+i-1}\varkappa,3^{3n+2i}t)\right)=1_l.$ (4.2)

And $\lim_{n\to\infty} ß(3^n\varkappa,3^n y,3^{3n}t)=1_l.$ (4.3)

For all $x \in W, t > 0, n \in \mathbb{N}$, there exists a unique cubic mapping $K: W \to Y$ such that

$$P(K(x)-f(x),t) \geq_Ł T_{i=1}^{\infty}\left(ß(3^n\varkappa,3^n\varkappa,3^{2i+1}t)\right) \quad (4.4)$$

Note: $1_l = \sup Ł \geq_Ł ₦(\rho), \forall \rho \in Ł/\{0_l, 1_l\}.$

Proof Suppose that $\varkappa = y$ we are getting

$$P(f(3\varkappa)-27f(\varkappa),t) \geq_L ß(,x,t)$$

$$P\left(\frac{f(3\varkappa)}{3^3}-f(x),\frac{t}{|3^3|}\right) \geq_L ß(\varkappa,\varkappa,t)$$

$$P\left(\frac{f(3\varkappa)}{3^3}-f(\varkappa),t\right) \geq_L ß(\varkappa,\varkappa,3^3t) \quad (4.5)$$

$\forall \varkappa \in W, t > 0, k \in \mathbb{N}.$

$$P\left(\frac{f(3^{k+1}\varkappa)}{3^{3(k+1)}}-\frac{f(3^k\varkappa)}{3^{3k}},\frac{t}{3^{3k}}\right) \geq_L ß(3^k\varkappa,3^k\varkappa,3^3t)$$

$$P\left(\frac{f(3^{k+1}x)}{3^{3(k+1)}}-\frac{f(3^kx)}{3^k},t\right) \geq_L ß\left(3^k\varkappa,3^k\varkappa,3^{3(k+1)}t\right)$$

$$P\left(\frac{f(3^{k+1}x)}{3^{3(k+1)}}-\frac{f(3^kx)}{3^k},\frac{t}{3^{k+1}}\right) \geq_L ß\left(3^k\varkappa,3^k\varkappa,3^{2(k+1)}t\right) \quad (4.6)$$

For all $\varkappa \in W, t > 0, n, k \in \mathbb{N}$. Since $\sum_{k=1}^{n} \frac{1}{3^k} < 1$ by tringle inquality.

Thus, $$P\left(\frac{f(3^n\varkappa)}{3^{3n}} - f(\varkappa), t\right) \geq_Ł P\left(\frac{f(3^n\varkappa)}{3^{3n}} - f(x), \sum_{k=1}^{n} \frac{t}{3^k}\right)$$

$$\geq_L T_{k=0}^{n-1}\left(P\left(\frac{f(3^{k+1}\varkappa)}{3^{3(k+1)}} - \frac{f(3^k\varkappa)}{3^k}, \frac{t}{3^{k+1}}\right)\right)$$

$$\geq_L T_{k=0}^{n-1}\left(ß\left(3^k\varkappa, 3^k\varkappa, 3^{2(k+1)}\right)\right)$$

$$= T_{i=1}^{n}\left(ß(3^{i-1}\varkappa, 3^{i-1}\varkappa, 3^{2i})\right). \quad (4.7)$$

We should prove the sequence $\left\{\frac{f(3^n\varkappa)}{3^{3n}}\right\}$ is convergence, we exchange $\varkappa$ with $2^n\varkappa$ in (4.7) for all $n, s \in \mathbb{N}$.

$$P\left(\frac{f(3^{n+s}\varkappa)}{3^{3(n+s)}} - \frac{f(3^s\varkappa)}{3^{3s}}, t\right) \geq_L T_{i=1}^{n}\left(ß(3^{s+i-1}\varkappa, 3^{s+i-1}\varkappa, 3^{3s+2i}t)\right) \quad (4.8)$$

By (4.2) when $n, s \to \infty$, the sequence $\left\{\frac{f(3^n\varkappa)}{3^{3n}}\right\}$ is Cauchy sequence .Therefore we have $K(\varkappa) = \lim_{n\to\infty} \frac{f(3^{3n}\varkappa)}{3^{3n}}$, for all $\varkappa \in W$.

Now exchanging $\varkappa, y$ with $3^s\varkappa, 3^s y$ respectively in (4.1) we have

$$P\left(\frac{f(3^s\varkappa - 2\cdot 3^s y)}{3^{3s}} + \frac{f(3^s\varkappa + 2\cdot 3^s y)}{3^{3s}} + \frac{6f(3^s y)}{3^{3s}} - \frac{4f(3^s(\varkappa - y))}{3^{3s}} + \frac{4f(3^s(\varkappa + y))}{3^{3s}}, t\right) \geq_L ß(3^s\varkappa, 3^s y, 3^{3s}t) \quad (4.9)$$

$\forall \varkappa, y \in W, t > 0, s \in \mathbb{N}$.

Taking $s \to \infty$, then $K(\varkappa)$ satisfying (4.1) for all $\varkappa, y \in W$, hence $K(\varkappa)$ is cubic function.

To prove (4.4) taking $n \to \infty$, in (4.7) we get (4.4).

Finally, to prove the uniqueness of the cubic mapping $K(\varkappa)$, suppose that there exists cubic mapping $K'(\varkappa)$ which satisfies (4.4), and since $K(3^n\varkappa) = 3^n K(\varkappa)$, then $K'(3^n\varkappa) = 3^n K'(\varkappa)$. For all $\varkappa \in W, n \geq 1$.

$$P(K(\varkappa) - K'(\varkappa), t) = P(K(3^n\varkappa) - K'(3^n\varkappa), 3^{3n}t)$$

$$\geq_L T^2\left(P(K(3^n\varkappa) - f(3^n\varkappa), 3^{3n-1}t), P(f(3^n\varkappa) - K'(3^n\varkappa), 3^{3n-1}t), P(K(3^n\varkappa) - K(3^n\varkappa), 3^{3n-1}t)\right).$$

$$= T\left(P(K(3^n x) - f(3^n x), 3^{3n-1}t), P(f(3^n x) - K'(3^n x), 3^{3n-1}t)\right)$$

$$\geq_L T\left(T_{i=1}^{\infty}\left(ß(3^{n+i-1}\varkappa, 3^{n+i-1}\varkappa, 3^{2i+3n}t)\right), T_{i=1}^{\infty}\left(ß(3^{n+i-1}\varkappa, 3^{n+i-1}\varkappa, 3^{2i+3n}t)\right)\right)$$

$\forall \varkappa \in W, n \in \mathbb{N}, t > 0.$ (4.10)

By taking $n \to \infty$, in (4.4) we have $K = K'$.

4.3. Corollary

Let $l = (Ł, \geq_Ł)$ be complete lattice, $f: W \to Y$, even mapping from W (real linear space), to (Y, P, T) (complete LRN space), And

$$P(f(\varkappa - 2y) + f(\varkappa + 2y) + 6f(\varkappa) - 4[f(\varkappa - y) + f(\varkappa + y)], t) \geq_L \frac{t}{t + \delta\|x_\circ\|} \; \forall \varkappa, y \in W, t > 0.$$

Then there exists a unique cubic mapping $K: W \to Y$ such that

$$P(\mathrm{f}(\varkappa)-\mathrm{K}(\varkappa),t)\geq_L \mathrm{T}_{i=1}^{\infty}\left(\frac{3^{2i+1}t}{3^{2i+1}t+\delta\|\varkappa_\circ\|}\right),\forall\varkappa\in X,t>0.$$

Proof This corollary can easily proving by using theorem (4.2) and replacing $\beta(\varkappa,y,t)$ with $\frac{t}{t+\delta\|\varkappa_\circ\|}$, $\forall t>0$ which is in $D_{Ł}^{+}$. If $\mathrm{T}=\mathrm{T}_m$ or $\mathrm{T}=\mathrm{T}_L$ the prove is direct, in case $\mathrm{T}=\mathrm{T}_p$ or T_D Ł must be equal [0,1].

4.4. Corollary

Assume that $(Ł=[0,1],\geq_Ł)$ is complete lattice, and $\mathrm{f}: W\to Y$, W (real linear space), to (Y,P,T) (complete LRN-space). If $\mathrm{T}=\mathrm{T}_m$, $\mathrm{T}=\mathrm{T}_p$.

$$P(\mathrm{f}(\varkappa-2y)+\mathrm{f}(\varkappa+2y)+6\mathrm{f}(\varkappa)-4[\mathrm{f}(\varkappa-y)+\mathrm{f}(\varkappa+y)],t)\geq_Ł \frac{t}{t+\alpha(\|\varkappa\|^z+\|y\|^z)}$$

For all $\varkappa,y\in W,t>0$.

Then there exists a unique cubic mapping $\mathrm{K}: W\to Y$ such that

$$P(\mathrm{f}(\varkappa)-\mathrm{K}(\varkappa),t)\geq_Ł \mathrm{T}_{i=1}^{\infty}\left(\frac{3^{2i+1}t}{3^{2i+1}t+\alpha(\|3^n\varkappa\|^z+\|3^n\varkappa\|^z)}\right)$$

$\forall\varkappa\in W,t>0,n\in\mathbb{N}$.

Proof This corollary can easily proving by using theorem (4.2) and replacing $\beta(\varkappa,y,t)$ with $\frac{t}{t+\alpha(\|\varkappa\|^z+\|y\|^z)}$, $\forall\varkappa,y\in W,t>0$ which is in $D_{Ł}^{+}$. And put $0<z<3$.

4.5. Corollary

Assume that $(Ł=[0,1],\geq_Ł)$ is complete lattice, $\mathrm{f}: W\to Y$, even mapping from W (real linear space), to (Y,P,T) (complete LRN-space). If $\mathrm{T}=\mathrm{T}_m$ or T_p and

$$P(\mathrm{f}(\varkappa-2y)+\mathrm{f}(\varkappa+2y)+6\mathrm{f}(\varkappa)-4[\mathrm{f}(\varkappa-y)+\mathrm{f}(\varkappa+y)],t)\geq_Ł 1_l-\frac{\|\varkappa\|}{t+\|\varkappa\|}.$$

For all $\varkappa,y\in W,t>0$.

Then there exists a unique cubic mapping $\mathrm{K}: W\to Y$ such that

$$P(\mathrm{f}(\varkappa)-\mathrm{K}(\varkappa),t)\geq_L \mathrm{T}_{i=1}^{\infty}\left(1_l-\frac{\|3^n\varkappa\|}{3^{2i+1}t+\|3^n\varkappa\|}\right)$$

For all $\varkappa\in W,t>0$.

Proof This corollary can easily proven by using theorem (4.2) and replacing $\beta(\varkappa,y,t)$ with $1_l-\frac{\|x\|}{t+\|x\|}$, $\varkappa\in W,t>0$, which is in D_L^{+}.

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