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New Results on Coefficient Estimates for Different Classes of Bi-Univalent Functions

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ABSTRACT

In the current work, we investigate two new subclasses $G_{\sum}(\delta,\alpha,\tau)$ and $G_{\sum}^{*}(\delta,\alpha,\beta)$ of class $\sum$ of bi-univalent functions found within the open unit disk $U=\{z:|z|<1\}$. We derive the normalized forms of functions that belong to the two classes described above. Furthermore, we obtain estimates of the starting coefficients $|a_2|$ and $|a_3|$ for these functions. Multiple classifications are also taken into consideration, and connections to previously established findings are established.

MSC. 30C45, 30C50, 30C80

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1. Introduction

Let $A(U)$ as a class of all analytic functions f in the open unit disk $U=\{z:|z|<1\}$ normalized by $f(0)=0$ and $f'(z)=1$ having the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \qquad (z\in U). \tag{1}$$

The class S includes all functions A that are univalent in the U. The classes $S^{*}(\tau)$ and $K(\tau)$ of starlike and convex functions of order $\tau,(0\le\tau<1)$ are defined by:

$$Re\left(\frac{zf'(z)}{f(z)}\right)>\tau, \quad (z\in U) \tag{2}$$

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and

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \tau, \quad (z \in U) \tag{3}$$

Since they provide information about the geometric features of these functions, determining the bounds for coefficients and is a crucial topic in geometric function theory. As an illustration, the bound for the second coefficient a_2 of functions in S provides covering theorems, growth, and distortion bounds. It is commonly known that for any $f \in A(U)$, the n-th coefficient a_n is bounded by n.

In this work, we estimate the initiat coefficient problems ($|a_2|$ and $|a_3|$) for a certain subclass of bi- univalent functions.

The disk of radius $\frac{1}{4}$ is contained in the image of U under all univalent function $f \in S$, as demonstrated by the Keobe one-quarter theorem [1]. As a result, each function $f \in S$ has an inverse function $f = f^{-1}$ that satisfies $f^{-1}f(z) = z\ (z \in U)$ and:

$$f(f^{-1}(z)) = w, \left(\lceil w \rceil < r_0(z) \quad , r_0(z) \geq \frac{1}{4} \right)$$

and is really provide by:

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{4}$$

If the functions f and f^{-1} are univalent in an open unit disk U, then a function $f \in A$ is called to the bi-univalent in U. We symbolize by $\sum$ the class of bi-univalent functions in the unit disk U known by Equation 1. Example of functions in the class $\sum$ are:

$$l_1 = \frac{z}{1-z}, l_2 = -log(1-z), l_3 = \frac{1}{2} log\left(\frac{1+z}{1-z}\right), z \in U.$$

But the well-known Koebe function is not included in $\sum$. Other functions, examples like $z - \frac{z^2}{2}$ and $\frac{z}{1-z^2}$, that are frequently used as examples of functions in U are not included in $\sum$.

Lewin in 1967 [2] examined the class $\sum$ and proved that $|a_2| < 1.51$ for element of class $\sum$. Brannan and Clunie [3] hypothesized in 1967 that $|a_2| \leq \sqrt{2}$. Brannan and Taha [4] presented a certain subclass of the bi-univalent function class $\sum$ comparable to the well-known subclass $S^*(\tau)$ and $K(\tau)$ [3, 5-11]. Therefore, if all of the specified conditions are satisfied, as stated by Brannan and Taha [4], then a function $f \in A$ belong to the class $S_{\sum}^*(\tau)$ of strongly bi-starlike functions of order τ, $(0 < \tau \leq 1)$, $f \in \sum$, and:

$$\left| arg\left(\frac{zf'(z)}{f(z)}\right) \right| < \frac{\tau\pi}{2} \quad (0 < \tau \leq 1, z \in U)$$

and

$$\left| arg\left(\frac{wg'(w)}{g(w)}\right) \right| < \frac{\tau\pi}{2} \quad (0 < \tau \leq 1, w \in U)$$

where g expansion of f^{-1} $to\ U$. The classes $S_{\sum}^*(\tau)$ and $k_{\sum}(\tau)$ of bi-starlike functions of order τ, and biconvex functions of order τ. Analogously, they were also introduced, corresponding to the function classes defined by Equations 2 and 3. They discovered non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ [12-18], for each of the function classes $S_{\sum}^*(\tau)$ and $k_{\sum}(\tau)$.

This study draws inspiration from the previous research conducted by several authors such [1, 19-28] and others [29-38]. In this work, we present two new subclasses $G_{\sum}(\delta, \alpha, \tau)$ and $G_{\sum}^*(\delta, \alpha, \beta)$ of the function class $\sum$ and for functions in these new subclasses of the function class Σ, determine estimates of the coefficients $|a_2|$ and $|a_3|$.

Lemma 1.1 [17]:

Let $h(z) = 1 + h_1 z + h_2 z^2 + \cdots \in P$, where P is the family of every analytic function h in U, and for which $Re\{h(z)\} > 0, (z \in U)$, then $|h_i| \leq 2$ for $i = 1,2,3,\cdots$.

2. Coefficient bounds of the subclass $G_{\sum}(\delta,\alpha,\tau)$

Definition 2.1:

A function f as supplied by Equation 5 is said to be in the class $G_{\sum}(\delta,\alpha,\tau)$, if the following conditions are satisfied $0 < \tau \leq 1$:

$$f \in \sum \text{ and } \left|arg\left(\frac{zf'(z)}{f(z)}\right)^{\delta}\left[\alpha\frac{(f(z))^{1-\delta}}{z^{1-\delta}} + (1-\alpha)f'(z) - 1\right]\right| < \frac{\tau\pi}{2}, \quad (5)$$

and

$$g \in \sum \text{ and } \left|arg\left(\frac{wg'(w)}{g(w)}\right)^{\delta}\left[\alpha\frac{(g(w))^{1-\delta}}{w^{1-\delta}} + (1-\alpha)g'(w) - 1\right]\right| < \frac{\tau\pi}{2}, \quad (6)$$

where $\delta, \alpha \geq 0,\ z, w \in U, and\ g = f^{-1}$.

Theorem 2.2

Let a function $f(z)$ provided by Equation 5, be in the class $G_{\sum}(\delta,\alpha,\tau)$ $0 < \tau \leq 1$. Then:

$$|a_2| \leq \frac{2\tau}{\sqrt{2\tau[(3+2\delta-\alpha(2+\delta)) + \delta[(\delta-1)(1-\alpha) - 2\alpha(1+\delta) + 2]] + (1-\tau)(\delta-\alpha(1+\delta)+2)^2}}$$

and

$$|a_3| \leq \frac{4\tau^2}{(\delta-\alpha(1+\delta)+2)^2} + \frac{2\tau}{(2\delta-\alpha(2+\delta)+3)}.$$

Proof

It derives from Equations 5 and 6:

$$\left(\frac{zf'(z)}{f(z)}\right)^{\delta}\left[\alpha\frac{(f(z))^{1-\delta}}{z^{1-\delta}} + (1-\alpha)f'(z) - 1\right] = [u(z)]^{\tau} \quad (7)$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\delta}\left[\alpha\frac{(g(w))^{1-\delta}}{w^{1-\delta}} + (1-\alpha)g'(w) - 1\right] = [v(w)]^{\tau}, \quad (8)$$

where $u(z)$ and $v(w)$ in P and have the form:

$$u(z) = 1 + u_1 z + u_2 z^2 + \cdots \quad (9)$$

$$v(w) = 1 + v_1 w + v_2 w^2 + \cdots \quad (10)$$

We achieve this by equal the coefficients in Equation 7 and 8.

$$(\delta + 2 - \alpha(1+\delta))a_2 = \tau u_1 \quad (11)$$

$$\delta\left[\frac{\delta-1}{2}(1-\alpha)-\alpha(1+\delta)+1\right]a_2^2+(2\delta-\alpha(2+\delta)+3)a_3=\tau u_2+\frac{\tau(\tau-1)}{2}u_1^2 \qquad (12)$$

and

$$-(\delta+2-\alpha(1+\delta))a_2=\tau v_1 \qquad (13)$$

$$\left[(6+4\delta-2\alpha(2+\delta))+\delta\left[\frac{\delta-1}{2}(1-\alpha)+(1-\alpha(1+\delta)\right]\right]a_2^2-(2\delta+3-\alpha(2+\delta))a_3=\tau v_2+\frac{\tau(\tau-1)}{2}v_1^2\,. \qquad (14)$$

Form Equation 11 and 12 we get:

$$u_1=-v_1. \qquad (15)$$

and

$$2(\delta+2-\alpha(1+\delta))^2a_2^2=\tau^2(u_1^2+v_1^2)\,, \qquad (16)$$

now by adding Equation 12 to Equation 14:

$$[2(3+2\delta-\alpha(2+\delta))+\delta[(1-\alpha)(\delta-1)+2(1-\alpha(1+\delta)]]a_2^2=\tau(u_2+v_2)+\frac{\tau(\tau-1)}{2}(u_1^2+v_1^2).$$

By using Equation 16:

$$[2(3+2\delta-\alpha(2+\delta))+\delta[(\delta-1)(1-\alpha)+2(1-\alpha(1+\delta)]]a_2^2=\tau(u_2+v_2)+\frac{\tau(\tau-1)}{2}\frac{2(\delta-\alpha(1+\delta)+2)^2a_2^2}{\tau^2}$$

Therefore, we have:

$$a_2^2=\frac{\tau^2(u_2+v_2)}{2\tau[(3+2\delta-\alpha(2+\delta))+\delta[(1-\alpha)(\delta-1)+2-2\alpha(1+\delta)]]-(\tau-1)(\delta-\alpha(1+\delta)+2)^2}$$

Applying Lemma 1.2 for the coefficient u_2 and v_2,we have:

$$|a_2|\leq\frac{2\tau}{\sqrt{2\tau[(3+2\delta-\alpha(2+\delta))+\delta[(1-\alpha)(\delta-1)+2-2\alpha(1+\delta)]]-(\tau-1)(\delta-\alpha(1+\delta)+2)^2}}$$

Next, by subtracting Equation 14 from Equation 12 to get the bound on $|a_3|$, we obtain:

$$2(2\delta-\alpha(2+\delta)+3)a_3-(6+4\delta-2\alpha(2+\delta))a_2^2=\tau(u_2+v_2)+\frac{\tau(\tau-1)}{2}(u_1^2-v_1^2).$$

Or equivalent:

$$a_3=\frac{\tau^2(u_1^2+v_1^2)}{2(\delta-\alpha(1+\delta)+2)^2}+\frac{\tau(u_2-v_2)}{2(2\delta+3-\alpha(2+\delta))}.$$

Applying Lemma 1.2 for coefficient u_1,u_2,v_1 and v_2 ,we have:

$$|a_3|\leq\frac{4\tau^2}{(\delta-\alpha(1+\delta)+2)^2}+\frac{2\tau}{(2\delta+3-\alpha(2+\delta))}.$$

This completes the proof.

Corollary 2.3

Let a function $f(z)$ as stated by Equation 5, be in the class $G_{\Sigma}(\delta, 1, \tau)$, $0 < \tau \leq 1$. Then:

$$|a_2| \leq \frac{2\tau}{\sqrt{2\tau[(1+\delta)-2\delta^2]+(1-\tau)(\delta+2-(1+\delta))^2}}$$

and

$$|a_3| \leq \frac{4\tau^2}{(\delta+2-(1+\delta))^2}+\frac{2\tau}{(2\delta+3-(2+\delta))}$$

Corollary 2.4

Let a function $f(z)$ presented by Equation 5, be in the class $G_{\Sigma}(1, \alpha, \tau)$, $0 < \tau \leq 1$. Then:

$$|a_2| \leq \frac{2\tau}{\sqrt{2\tau\,[(5-3\alpha)+2(1-2\alpha)]+(1-\tau)(3-2\alpha))^2}}$$

and

$$|a_3| \leq \frac{4\tau^2}{(3-2\alpha)^2}+\frac{2\tau}{(5-3\alpha)}.$$

Definition 2.5

A function $f(z)$ defined by Equation 5 is considered to belong to the class $G^*_{\Sigma}(\delta, \alpha, \beta)$ if the following conditions are satisfied, so that $\delta, \alpha \geq 0, 0 \leq \beta < 1$:

$$f \in \sum \text{ and } Re\left[\left(\frac{zf'(z)}{f(z)}\right)^{\delta}\left[\alpha\frac{(f(z))^{1-\delta}}{z^{1-\delta}}+(1-\alpha)f'(z)-1\right]\right] > \beta, \tag{17}$$

and

$$f \in \sum \text{ and } Re\left[\left(\frac{wg'(w)}{g(w)}\right)^{\delta}\left[\alpha\frac{(g(w))^{1-\delta}}{w^{1-\delta}}+(1-\alpha)g'(w)-1\right]\right] > \beta, \tag{18}$$

where $z, w \in U$, and $g = f^{-1}$.

Theorem 2.6

Let a function $f(z)$ as defined in Equation 5, which belongs to the class $G^*_{\Sigma}(\delta, \alpha, \beta)$, $0 \leq \beta < 1$ and $\delta, \alpha \geq 0$. Then

$$|a_2| \leq \frac{2\sqrt{(1-\beta)}}{\sqrt{[2(3+2\delta-\alpha(2+\delta))+\delta[(1-\alpha)(\delta-1)-2\alpha(1+\delta)+2]]}}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(\delta-\alpha(1+\delta)+2)^2}+\frac{2(1-\beta)}{(2\delta-\alpha(2+\delta)+3)} .$$

Proof

It derives from Equations 17 and 18:

$$\left(\frac{zf'(z)}{f(z)}\right)^{\delta}\left[\alpha\frac{(f(z))^{1-\delta}}{z^{1-\delta}}+(1-\alpha)f'(z)-1\right]=\beta+(1-\beta)u(z), \quad (19)$$

$$\left(\frac{wg'(w)}{g(w)}\right)^{\delta}\left[\alpha\frac{(g(w))^{1-\delta}}{w^{1-\delta}}+(1-\alpha)g'(w)-1\right]=\beta+(1-\beta)v(w). \quad (20)$$

Where $u(z)$ and $v(w)$ possess the form of Equations 9 and 10, according. Now, by equating the coefficients in Equations 7 and 8, respectively, by equating the coefficients in equations 19 and 20, we can derive the following result:

$$(\delta-\alpha(1+\delta)+2)a_2=(1-\beta)u_1 \quad (21)$$

$$\left[\frac{\delta-1}{2}(1-\alpha)-\alpha(1+\delta)+1\right]a_2^2+(2\delta-\alpha(2+\delta)+3)a_3=(1-\beta)u_2 \quad (22)$$

$$-(\delta+2-\alpha(1+\delta))a_2=(1-\beta)v_1 \quad (23)$$

$$\left[(4\delta-2\alpha(2+\delta)+6)+\delta\left[\frac{\delta-1}{2}(1-\alpha)-\alpha(1+\delta)+1\right]\right]a_2^2-(2\delta-\alpha(2+\delta)+3)a_3=(1-\beta)v_2. \quad (24)$$

Form Equations 21 and 23 we get:

$$u_1=-v_1, \quad (25)$$

and

$$2(\delta-\alpha(1+\delta)+2)^2a_2^2=(1-\beta)^2(u_1^2+v_1^2). \quad (26)$$

Now by adding Equation 22 to Equation 24:

$$[2(3+2\delta-\alpha(2+\delta))+\delta[(\delta-1)(1-\alpha)+2(1-\alpha(1+\delta))]]a_2^2=(1-\beta)(u_2+v_2).$$

Therefore, we have

$$a_2^2=\frac{(1-\beta)(u_2+v_2)}{[2(3+2\delta-\alpha(2+\delta))+\delta[(\delta-1)(1-\alpha)+2(1-\alpha(1+\delta))]]}.$$

Applying Lemma 1.2 for the coefficient u_2 and v_2, we have:

$$|a_2|\le\frac{2\sqrt{(1-\beta)}}{\sqrt{\left[2\left(3+2\delta-\alpha(2+\delta)\right)+\delta[(1-\alpha)(\delta-1)+2-2\alpha(1+\delta)]\right]}}.$$

Next, by subtracting Equation 24 from Equation 22, we can find the bound on $|a_3|$, we get:

$$2(2\delta-\alpha(2+\delta)+3)a_3-(4\delta-2\alpha(2+\delta)+6)a_2^2=(1-\beta)(u_2-v_2).$$

Or equivalent:

$$a_3=\frac{(1-\beta)^2(u_1^2+v_1^2)}{2(\delta-\alpha(1+\delta)+2)^2}+\frac{(1-\beta)(u_2-v_2)}{2(2\delta-\alpha(2+\delta)+3)}.$$

Applying Lemma 1.2 for coefficient u_1, u_2, v_1 and v_2, we have:

$$|a_3|\le\frac{4(1-\beta)^2}{\left(\delta+2-\alpha(1+\delta)\right)^2}+\frac{2(1-\beta)}{\left(2\delta+3-\alpha(2+\delta)\right)}.$$

This completes the proof.

Corollary 2.7

Let a function $f(z)$ as defined in Equation 5, which belongs to the class $G_{\Sigma}^{*}(1, \alpha, \beta)$, $0 \leq \beta < 1$. Then:

$$|a_2| \leq \frac{2\sqrt{(1-\beta)}}{\sqrt{[2(5-3\alpha)-2(1-2\alpha)]}}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(3-2\alpha)^2} + \frac{2(1-\beta)}{(5-3\alpha)}.$$

Corollary 2.8

Let a function $f(z)$ as stated by Equation 5, which belongs to the class $G_{\Sigma}^{*}(\delta, 1, \beta)$, $0 \leq \beta < 1$. Then:

$$|a_2| \leq \frac{2\sqrt{(1-\beta)}}{\sqrt{[2(1+\delta)-2\delta^2]}}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{\left(\delta+2-(1+\delta)\right)^2} + \frac{2(1-\beta)}{(\delta+1)}.$$

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