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On APP-Quasi prime Submodules

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ABSTRACT

Let R to be a commutative ring with identity and G is a unitary left R-module. In this paper we introduce the concept of approximaitly-quasi prime submodules (for short APP-quasi prime submodule) as a generalizations of prime submodule and quasi prime submodule, where a proper submodule F of R-module G is said to be an APP-quasi prime submodule of G, if whenever $rsm \in G$, where $r, s \in R$ and $m \in G$ implies that either $rm \in F + Soc(G)$ or $sm \in F + Soc(G)$[5]. Many new examples, characterization and basic properties of this concepts are introduce. Furthermore new characterizations of APP-quasi prime submodules in some types of modules are given.

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1. Introduction

Let G be an R-module and F be a proper submodule of an R-module G then F is called prime submodule of G if $rm \in F$, where $r \in R$, $m \in G$, implies that either $m \in F$ or $r \in [F:_R G]$ where $[F:_R G] = \{r \in R: rG \subseteq F\}$ this concept was first introduce in 1978 by Dauns see [1]. This concept"was generalizations to (primary, semi prime , quasi prime) submodules in 1989, 1996, and 1999 respectively see [2,3,4] , the concepts of prime submodules recently several generalization to APP-quasi prime submodules by Ali Sh. Haibat K. in 2019 see [5]. In this study, we introduce new examples, characterizations and basic properties of these concepts. A nonzero submodule F of an R-module G is said to be essential if $F \cap K \neq (0)$ for all nonzero submodule K of G [6] and the socle of an R-module G denoted by Soc(G) is the intersection all"essential submodules F of G [7. P.212]. It is well"known that if a submodule F of an R-module G be an essential"in G, therefore $Soc(F) = Soc(G)$ [6, P. 29]. An R-module G is a semi simple if each submodule of an R-module G is a direct"summand of G, that is if F is a submodule of G, then $G = F \oplus B$ for"some submodule B of G [7]. Equivalently G is a semi simple if and"only if $Soc(G) = G$ [6]. It is well known that an R-module G is a semi simple if and"only if for each submodule F of G , $Soc(\frac{G}{F}) = \frac{Soc(G)+F}{F}$ [7 , P. 239]. An R-module G is called multiplication if every

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submodule F of G written in the form $F = IG$ for certain ideal I of R. Equivalent to G is multiplication if $F = [F:_R G]G$ [8]. For any submodules K, L of a multiplication"R-module G with $K = IG$, $L = JG$ for"some ideals I and J of R, the product of two submodule of a multiplication R-module is $KL = IGJG = IJG$, that"is $KL = IL$. To be specific $KG = IGRG = IRG = IG = K$ [9]. An R-module G is said to be a Z-regular if $\forall\, m \in G, \exists\, f \in M^* = Hom_R(G,R)$ in such a way that $m = f(m)m$ [11]. It is well know that if G is a Z- regular R-module, then $Soc(G) = Soc(R)G$ [12, prop. 3 −25]."An R -module G is a projective if for every R-epimorphism Ψ from R-module V into an R-module V' and for any R-homomorphism Φ from G into V' then there exists an R-homomorphism θ from G into V such that $\Phi o \theta = \Psi$ [7]. It is well know that if G be a projective R-module, then $Soc(G) = Soc(R)G$ [12,prop. 3.24]. An R-module G is weak cancellation if $IG = JG$, where I, J are ideals of R then $I + ann_R(G) = J + ann_R(G)$ [14]. It is well-know that if G is a multiplication R-module. Then G is finitely generated if and only if G is a weak"cancellation [13 , prop. (3 − 9)].

2. Some New Results Of An APP-Quasi Prime Submodules.

In this part of we recall that a proper submodule F of an R-module G is quasi prime if $rsm \in F$, where $r, s \in R$, $m \in G$,"this means either $rm \in F$ or $sm \in F$.

Definition 2.1:[5]

A proper submodule F of an R-module G is called an APP-quasi prime submodule of G, if $rsm \in F$, for $r, s \in R$, $m \in G$, "implies that either $rm \in F + Soc(G)$ or $sm \in F + Soc(G)$. And an ideal I of a ring R is called an APP-quasi prime ideal of R if I is an APP-quasi prime submodule of"an R-Module R.

Examples and Remark 2.2:

1. It's clear that since $Soc(Z_{48})$ is the intersection of all essential submodules of Z_{48} that is $Soc(Z_{48}) =<\bar{2}> \cap <\bar{4}> \cap <\bar{8}> \cap Z_{48} =<\bar{8}>$.

2. It can be checked in the Z -module Z_{48} , the submodules $<\bar{6}>$ and $<\overline{16}>$ are APP-quasi prime submodules.

3. It can be checked in the Z-module Z_{48}, the submodules $<\bar{4}>, <\bar{8}>, <\overline{12}>$, and $<\overline{24}>$ are not an APP-quasi prime submodules of Z_{48}.

4. It's clear that each quasi prime submodule of R-module G is an APP-quasi prime submodule of an R-module G, however not conversely. The following example explain that: Consider the Z-Module Z_{48}, the submodule $N =<\overline{16}>$ is not a quasi prime submodule of Z_{48} since $2.2.\bar{4} = \overline{16} \in <\overline{16}>$ for $2 \in Z$ $\bar{4} \in Z_{48}$, but $2.\bar{4} = \bar{8} \notin <\overline{16}>$. However $<\overline{16}>$ is an APP-quasi prime submodule of Z_{48} by part (2).

5. It's clear that each prime submodule of an R-module G is an APP-quasi prime submodule of an R-module G. However the converse is not true in general the following example shows this: The submodule $<\bar{6}>$ of Z-module Z_{48} is an APP-quasi prime submodule by part (2). but $<\bar{6}>$ is not prime submodule, since $2.\bar{3} = \bar{6} \in <\bar{6}>$ for $2 \in Z, \bar{3} \in Z_{48}$,however $\bar{3} \notin <\bar{6}>$ and $2 \notin [<\bar{6}>:_Z Z_{48}] = 6Z$.

6. It's clear that The submodules $<\bar{2}>$, $<\bar{3}>$ of the Z- module Z_{48} are prime submodules of Z_{48}. Thus they are APP-quasi prime submodules by part (5).

7. If F is an APP-quasi prime submodule of an R-module G, then $[F:_R G]$ not necessary be an APP-quasi prime ideal of R.The following example shows that:

It is shown that in part (2) the submodule $<\bar{6}>$ of the Z- module Z_{48} is an APP-quasi prime submodule. However $[<\bar{6}> :_Z Z_{48}] = 6Z$ is not APP-quasi prime ideal of of Z since $2.3.1 = 6 \in 6Z$, for $2,3,1 \in Z$ however $2.1 = 2 \notin 6Z + Soc(Z) = 6Z$, and $3.1 = 3 \notin 6Z + Soc(Z) = 6Z$.

Before we introduce the first characterizations of APP-quasi prime submodules, we need to recall the following lemma which a pear in [5, Coro. 2.7].

Lemma 2.3:

If G be an R-module, and F be a proper submodule of G.Then F is an APP-quasi prime submodule of G if and only if when $IJm \subseteq F$, for I, J are ideals of R, $m \in G$, this means either $Im \subseteq F + Soc(G)$ or $Jm \subseteq F + Soc(G)$.

Proposition 2.4 :

Let F be a proper submodule of an R-module G.Then F is an APP-quasi prime submodule of G if and only if $[F:_G IJ] \subseteq [F + Soc(G):_G I] \cup [F + Soc(G):_M J]$ for all ideals I and J of R.

proof:

($\Longrightarrow$) Suppose that F is an APP-quasi prime submodule of G, and let $m \in [F:_M IJ]$,for $m \in G$ and ideals I and J of R implies that $IJm \subseteq F$. However F is an APP-quasi prime submodule of G, so by lemma (2.3) either $Im \subseteq F + Soc(G)$ or $Jm \subseteq F + Soc(G)$. Therefore either $m \in [F + Soc(G):_G I]$ or $m \in [F + Soc(G):_G J]$. Hence $m \in [F + Soc(G):_G I] \cup [F + Soc(G):_G J]$.Thus $[F:_G IJ] \subseteq [F + Soc(G):_G I] \cup [F + Soc(G):_G J]$.
($\Longleftarrow$) Suppose that $[F:_G IJ] \subseteq [F + Soc(G):_G I] \cup [F + Soc(G):_G J]$, and let $IJm \subseteq F$, for I, J are an ideal of R, $m \in G$, implies that $m \in [F:_R IJ]$. And since $[F:_G IJ] \subseteq [F + Soc(G):_G I] \cup [F + Soc(G):_G J]$. Thus $m \in [F + Soc(G):_G I] \cup [F + Soc(G):_G J]$, so either $m \in [F + Soc(G):_G I]$ or $m \in [F + Soc(G):_G J]$. Hence $Im \subseteq F + Soc(G)$ or $Jm \subseteq F + Soc(G)$. therefore F is an APP-quasi prime submodule of G by lemma (2.3) .

Before we introduce the first characterizations of APP-quasi prime submodules, we need to recall the following lemma which appear in [5, Prop. 2.4].

Lemma 2.5:

Let G be an R-module, and F be a proper submodule of an R-module G.Then F is an APP-quasi prime sub Module of G if and only if when $IJL \subseteq F$, for I, J are ideal of R, L is a submodule of G, then either $IL \subseteq F + Soc(G)$ or $JL \subseteq F + Soc(G)$.

Proposition 2.6:

Let G be an R-module, and F be a proper submodule of an R-module G. Then F is an APP-quasi prime submodule of G if and only if when $rJD \subseteq F$, where $r \in R$, J is an ideal of R, and D is a submodule of G, therefore either $rD \subseteq F + Soc(G)$ or $JD \subseteq F + Soc(G)$.

Proof:

($\Longrightarrow$) Assume that F is an APP-quasi prime submodule of G, and $rJD \subseteq N$, where $r \in R$, J is a ideal of R, D is a submodule of G. Now $rJD = < r > JD$. Since$G$ is an APP-quasi prime submodule of G, thus with lemma (2.5) either $< r > D \subseteq F + Soc(G)$ or $JD \subseteq F + Soc(G)$ that is either $rD \subseteq F + Soc(G)$ or $JD \subseteq F + Soc(G)$.
($\Longleftarrow$) Assume that $rsd \in N$, for $r, s \in R$, $d \in M$ implies that $< r >< s >< d > \subseteq N$, so with hypothesis either $< r >< d > \subseteq N + Soc(M)$ or $< s >< d > \subseteq N + Soc(M)$. That is either $rd \in N + Soc(M)$ or $sd \in N + Soc(M)$. Therefore F is an APP-quasi prime submodule of G .

The following is a Characterization of APP-quasi prime submodule of G.

Proposition: 2.7:

If F is a proper submodule of R-module G. Then F is an APP-quasi prime submodule of G if and only if $[F:_G rJ] \subseteq [F + Soc(G):_G r] \cup [F + Soc(G):_G J]$, where $r \in R$, and J is an ideal of R.

Proof:

($\Longrightarrow$)Suppose that F is an APP-quasi prime submodule of G, and let $D \subseteq [F:_M rJ]$, for D is a submodule of G, $r \in R$, and J is an ideal of R implies that $rJD \subseteq N$. However F is an APP-quasi prime submodule of G. Hence with proposition (2.6) $rD \subseteq F + Soc(G)$ or $JD \subseteq F + Soc(G)$. that is $D \subseteq [F + Soc(G):_G r]$ or $D \subseteq [F + Soc(G):_G J]$. Hence $D \subseteq [F + Soc(G):_G r] \cup [F + Soc(G):_G J]$. Thus $[F:_G rJ] \subseteq [F + Soc(G):_G r] \cup [F + Soc(G):_G J]$.
($\Longleftarrow$)Suppose that $[F:_G rJ] \subseteq [F + Soc(G):_G r] \cup [F + Soc(G):_G J]$, and let $rJD \subseteq N$, for $r \in R$, J is an ideal of R, and D is a submodule of G, implies that $D \subseteq [F:_R rJ]$. And since $[N:_M rJ] \subseteq [N + Soc(M):_M r] \cup [N + Soc(M):_M J]$. Thus $D \subseteq [N + Soc(M):_M r] \cup [N + Soc(M):_M J]$, implies that $D \subseteq [N + Soc(M):_M r]$ or $D \subseteq [N + Soc(M):_M J]$. Hence $rD \subseteq N + Soc(M)$ or $JD \subseteq N + Soc(M)$. Hence with proposition (2.6) F is an APP-quasi prime submodule of G.

Proposition 2. 8:

If F is a proper submodule of R-module G. Then F is an APP-quasi prime submodule of G if and only if whenever $Ism \subseteq F$, for I is an ideal of R, $s \in R$ and $m \in G$, implies that either $Im \subseteq F + Soc(G)$ or $sm \in F + Soc(G)$.

Proof:

The prove with lemma (2.5).

Corollary 2. 9:

If F is a proper submodule of an R-module G. Then F is an APP-quasi prime submodule of G if and only if whenever $IsD \subseteq F$, for I is an ideal of R, $s \in R$, and D is a submodule of G, implies that either $ID \subseteq F + Soc(G)$ or $sD \subseteq F + Soc(G)$.

Proposition 2. 10:

Let F be a proper submodule of an R-module G.Then F is an APP-quasi prime submodule of G if and only if $[F{:_G}\, Is] \subseteq [F + Soc(G){:_G}\, I] \cup [F + Soc(G){:_G}\, s]$ for all I is an ideal of R,$s \in R$.

Proof:

($\Longrightarrow$)Assume that F is an APP-quasi prime submodule of G, and let $m \in [F{:_G}\, Is]$, for $m \in G$, I is an ideal of R, and $s \in R$ then $Ism \subseteq F$. However F is an APP-quasi prime submodule of G. Hence by proposition (2.8) either $Im \subseteq F + Soc(G)$ or $sm \subseteq F + Soc(G)$. Therefore either $m \in [F + Soc(G){:_G}\, I]$ or $m \in [F + Soc(G){:_G}\, s]$. Hens $m \in [F + Soc(G){:_G}\, I] \cup [F + Soc(G){:_G}\, s]$ .Thus $[F{:_G}\, Is] \subseteq [F + Soc(G){:_G}\, I] \cup [F + Soc(G){:_G}\, s]$.
($\Longleftarrow$)Suppose that $[F{:_G}\, Is] \subseteq [F + Soc(G){:_G}\, I] \cup [F + Soc(G){:_G}\, s]$, and let $Ism \subseteq F$, where I is an ideal of $R, s \in R$, and $m \in G$, implies that $m \in [F{:_R}\, Is]$. And since $[F{:_G}\, Is] \subseteq [F + Soc(G){:_G}\, I] \cup [F + Soc(G){:_G}\, s]$. Thus $m \in [F + Soc(G){:_G}\, I] \cup [F + Soc(G){:_G}\, s]$, implies that $m \in [F + Soc(G){:_G}\, I]$ or $m \in [F + Soc(G){:_G}\, s]$. Hence $Im \subseteq F + Soc(G)$ or $sm \subseteq F + Soc(G)$. Thus by proposition (2.8) F is an APP-quasi prime submodule of G.

Proposition 2. 11:

If F is a proper submodule of a cyclic R-module G. Then F is an APP-quasi prime submodule of G if and only if $[F{:_R}\, IJD] \subseteq [F + Soc(G){:_R}\, ID] \cup [F + Soc(G){:_R}\, JD]$ for each I ,J are an ideal of R, and every submodule D of an R-module G.

Proof:

($\Longrightarrow$) Assume that F is an APP-quasi prime submodule of an R-module G, and let $m \in [F{:_R}\, IJD]$, for $m \in G$, I ,J ideals of R, and D is a submodule of R-module G, then $IJ(mD) \subseteq N$. However F is an APP-quasi prime submodule of G, then by lemma (2.5) $I(mD) \subseteq F + Soc(G)$ or $J(mD) \subseteq F + Soc(G)$, therefore $m \in [F + Soc(G){:_R}\, ID]$ or $m \in [F + Soc(G){:_R}\, JD]$. That is $m \in [F + Soc(G){:_R}\, IB] \cup [F + Soc(G){:_R}\, JB]$ .Thus $[F{:_R}\, IJD] \subseteq [F + Soc(G){:_R}\, ID] \cup [F + Soc(G){:_R}\, JD]$.
($\Longleftarrow$)Suppose that $[F{:_G}\, IJD] \subseteq [F + Soc(G){:_G}\, ID] \cup [F + Soc(G){:_G}\, JD]$, and let $rsm \in F$, for $r, s \in R$, and $m \in G$. Since G is a cyclic R-module, then $m = tm'$, for $t \in R$, $m' \in G$ implies that $rsm = rstm'$. Thus $rstm' \in F$ then $t \in [F{:_R}\, rsm']$, so by hypothesis $t \in [F{:_R}\, rsm'] \subseteq [F + Soc(G){:_R}\, rm'] \cup [F + \mathrm{Soc(G)}{:_R}\, sm']$, implies that $t \in [F + Soc(G){:_R}\, rm']$ or $t \in [F + Soc(G){:_R}\, sm']$ therefore $rtm' \in F + Soc(G)$ or $stm' \in F + Soc(G)$. Hence $rm \in F + Soc(G)$ or $sm \in F + Soc(G)$, implies that F is an APP-quasi prime submodule of an R-module G.

Remark 2. 12 [6, P. 29]:

it is well known that, if a submodule F of an R-module G is an essential in G, then $Soc(F) = Soc(G)$.

Proposition 2. 13:

Let F, K are submodules of an R-module G with F is a proper submodule of K and K is an essential submodule of G such that F is an APP-quasi prime submodule of G. Then F is an APP-quasi prime submodule of K.

Proof:

Suppose that F is an APP-quasi prime submodule of G, and let I$JA \subseteq F$, where I, J are ideals of R, A is a submodule of K. Since K is a submodule of G that is A is a submodule of G. However F is an APP-quasi prime submodule of G , then by lemma (2.5) $IA \subseteq F + Soc(G)$ or $JA \subseteq F + Soc(G)$ .However K is an essential submodule of G then $Soc(G) = Soc(K)$.

Thus, it follows that either $IA \subseteq F + Soc(K)$ or $JA \subseteq F + Soc(K)$. Therefore by lemma (2.5) F is an APP-quasi prime submodule of K.

Proposition 2.14:

If F, K are submodules of an R-module G such that $L \subseteq F$, and F be a proper submodule of G. If F is an APP-quasi prime submodule of G ,then $\frac{F}{L}$ is an APP-quasi prime submodule of $\frac{G}{L}$.

Proof:

Assume that F is an APP-quasi prime sub Module of G, and let $rs(m+L) \in \frac{F}{L}$ then $rsm + L \in \frac{F}{L}$, for $r, s \in R$,$m + L \in \frac{G}{L}$, $m \in G$. Then $rsm \in F$ and since F is an APP-quasi prime submodule of G, implies that $rm \in F + Soc(G)$ or $sm \in F + Soc(G)$. It follows that $rm + L \in \frac{F+Soc(G)}{L}$ or $sm + L \in \frac{F+Soc(G)}{L}$,thus $rm + L \in \frac{F}{L} + \frac{F+Soc(G)}{L} \subseteq \frac{F}{L} + Soc(\frac{G}{L})$ or $sm + L \subseteq \frac{F}{L} + \frac{F+Soc(G)}{L} \subseteq \frac{F}{L} + Soc(\frac{G}{L})$, implies that either $rm + L \in \frac{F}{L} + Soc(\frac{G}{L})$ or $sm + L \in \frac{F}{L} + Soc(\frac{G}{L})$ therefore either $r(m + L) \in \frac{F}{L} + Soc(\frac{G}{L})$ or $s(m + L) \in \frac{F}{L} + Soc(\frac{G}{L})$. Thus $\frac{F}{L}$ is an APP-quasi prime submodule of $\frac{G}{L}$.

Now, the following is the converse of proposition (2.14)

Proposition 2.15:

Let F , L are submodules of semi simple R-module G, such that $L \subseteq F$, and F be a proper submodule of G . If L , $\frac{F}{L}$ are APP-quasi prime submodules of G and $\frac{G}{L}$ respectively, then F is an APP-quasi prime submodule of G.

Proof :

Assume that L , $\frac{F}{L}$ are an APP-quasi prime submodules of G and $\frac{G}{L}$ respectively and let $rsm \in F$ for $r, s \in R$,$m \in G$. So $rs(m + L) \in \frac{F}{L}$ then $rsm + L \in \frac{F}{L}$. If $rsm \in L$ and L is an APP-quasi prime submodule of G, implies that either $rm \in L + Soc(G)$ or $sm \in L + Soc(G)$, and since $L \subseteq F$, it follows that $L + Soc(G) \subseteq F + Soc(G)$, then either $rm \in L + Soc(G) \subseteq F + Soc(G)$ or $sm \in L + Soc(G) \subseteq F + Soc(G)$, implies that either $rm \in F + Soc(G)$ or $sm \in F + Soc(G)$. Hence F is an APP-quasi prime submodule of G. So, we may suppose that $rsm \notin L$. It follows that $rs(m + L) \in \frac{F}{L}$, However $\frac{F}{L}$ is an APP-quasi prime submodule of $\frac{G}{L}$, implies that $r(m + L) \in \frac{F}{L} + Soc(\frac{G}{L})$ or $s(m + L) \in \frac{F}{L} + Soc(\frac{G}{L})$. since G is a semi simple ,then $Soc(\frac{G}{L}) = \frac{L+Soc(G)}{L}$, hence either $r(m + L) \in \frac{F}{L} + \frac{L+Soc(G)}{L}$ or $s(m + L) \in \frac{F}{L} + \frac{L+Soc(G)}{L}$. Since $L \subseteq F$, it follows that $L + Soc(G) \subseteq F + Soc(G)$, hence $\frac{F}{L} + \frac{L+Soc(G)}{L} \subseteq \frac{F}{L} + \frac{F+Soc(G)}{L}$, and since $\frac{F}{L} \subseteq \frac{F+Soc(G)}{L}$, implies that $\frac{F}{L} + \frac{F+Soc(G)}{L} = \frac{F+Soc(G)}{L}$. Thus either $r(m + L) \in \frac{F+Soc(G)}{L}$ or $s(m + L) \in \frac{F+Soc(G)}{L}$, then either $rm + L \in \frac{F+Soc(G)}{L}$ or $sm + L \in \frac{F+Soc(G)}{L}$, therefore either $rm \in F + Soc(G)$ or $sm \in F + Soc(G)$. Hence F is an APP-quasi prime submodule of G.

3.Some Characterizations Of An APP-quasi prime Submodules In Some Types Of Modules.

In this part of paper we introduce several characterizations of APP-quasi prime submodules in class of multiplication modules before we introduce first characterization we need to recall the following.

Lemma 3.1[5, Prop. 2.37]

Let G be a multiplication R-module, and F be a proper submodule of G. Then F is an APP-quasi prime submodule of G if and only if, when $K_1K_2K_3 \subseteq F$, where K_1, K_2, K_3 are submodules of G, this means either $K_1K_3 \subseteq F + Soc(G)$ or $K_2K_3 \subseteq F + Soc(G)$.

Remark 3.2 [13]:

Let G be a multiplication R-module, then for every elements $m_1, m_2 \in G$, by m_1m_2 mean the product of two submodules Rm_1 and Rm_2, that is $m_1m_2 = Rm_1Rm_2$ is a submodule of G

Proposition 3.3 :

Let G be a multiplication R-module, and F be a proper submodule of G. Then F is an App-quasi prime submodule of G if and only if whenever $m_1m_2m_3 \subseteq F$ for $m_1, m_2, m_3 \in G$, this means either $m_1m_3 \subseteq F + Soc(G)$ or $m_2m_3 \subseteq F + Soc(G)$.

Proof:

($\Longrightarrow$) Suppose that F is an APP-quasi prime submodule of G, and let $m_1m_2m_3 \subseteq F$ for $m_1, m_2, m_3 \in G$. Since G is a multiplication, then $m_1 = Rm_1$, $m_2 = Rm_2$, $m_3 = Rm_3$ are submodules of G, therefore $m_1 = I_1G, m_2 = I_2G, m_3 = I_3G$, for I_1, I_2, and I_3 are an ideal of R, thus $m_1m_2m_3 = I_1I_2(I_3G) \subseteq F$. However F is an APP-quasi prime submodule of G, then with lemma (3.1), $I_1I_3G \subseteq F + Soc(G)$ or $I_2I_3G \subseteq F + Soc(G)$. Therefore $m_1m_3 \subseteq F + Soc(G)$ or $m_2m_3 \subseteq F + Soc(G)$.

($\Longleftarrow$) Suppose that $m_1m_2m_3 \subseteq F$ for $m_1, m_2, m_3 \in G$, implies that either $m_1m_3 \subseteq F + Soc(G)$ or $m_2m_3 \subseteq F + Soc(G)$, and let $I_1I_2m_3 \subseteq F$, for $m_3 \in G$ and I_1,I_2 are an ideal of R Since G is a multiplication R-module, and $m_3 = I_3G$, then $I_1I_2m_3 = I_1I_2I_3G = (I_1G)(I_2G)(I_3G)$, implies that $I_1I_2m_3 = m_1m_2m_3$, so either $m_1m_3 \subseteq F + Soc(G)$ or $m_2m_3 \subseteq F + Soc(G)$ by hypothesis. That is $I_1I_3G \subseteq F + Soc(G)$ or $I_2I_3G \subseteq F + Soc(G)$. Thus $I_1m_3 \subseteq F + Soc(G)$ or $I_2m_3 \subseteq F + Soc(G)$, therefore with lemma (3.1) F is an APP-quasi prime submodule of G.

The corollaries follow from lemma (3.1) and proposition (3.3).

Corollary 3.4:

Let G be a multiplication R-module, and F be a proper submodule of G. Then F is an APP-quasi prime submodule of G if and only if whenever $KmL \subseteq G$ where L, K are a submodules of G and $m \in G$, that is either $KL \subseteq F + Soc(G)$ or $mL \subseteq F + Soc(G)$.

Corollary 3.5:

Let G be a multiplication R-Module, and F be a proper submodule of G. Then F is an APP-quasi prime submodule of G if and only if whenever $Km_1m_2 \subseteq F$ where K is a submodules of G and $m_1, m_2 \in G$, that is $Km_2 \subseteq F + Soc(G)$ or $m_1m_2 \subseteq F + Soc(G)$.

Corollary 3.6:

Let G be a multiplication R-Module, and F be a proper submodule of G. Then F is an APP-quasi prime submodule of G if and only if whenever $m_1Km_2 \subseteq F$ where K be a submodule of G and $m_1, m_2 \in G$, that is either $m_1m_2 \subseteq N + \mathrm{Soc(M)}$ or $Km_2 \subseteq N + Soc(G)$.

Corollary 3.7:

Let G be a multiplication R-module, and F be a proper submodule of G. Then F is an APP-quasi prime submodule of G if and only if whenever $m_1m_2K \subseteq F$ where K is a submodule of G and $m_1, m_2 \in G$, that is $m_1K \subseteq N + Soc(G)$ or $m_2K \subseteq F + Soc(G)$.

The following proposition are characterizations of an APP-quasi prime submodules by its results.

Proposition 3.8:

If F is a proper submodule of Z-regular multiplication R-module G. Then F is an APP-quasi prime submodule of G if and only if $[F:_R G]$ is an APP-quasi prime ideal of R.

Proof:

($\Longrightarrow$) Assume that F is an APP-quasi prime submodule of G, and let $J_1J_2J_3 \subseteq [F:_R G]$,for J_1,J_2,J_3 are ideals of R , that is $J_1J_2J_3G \subseteq F$. Since G is a multiplication , then $K_1K_2K_3 \subseteq F$ where $K_1 = J_1G$, $K_2 = J_2G$ and $K_3 = J_3G$ .However F is an APP-quasi prime submodule of G , then with lemma (3.1) $K_1K_3 \subseteq F + Soc(G)$ or $K_2K_3 \subseteq F + Soc(G)$, thus $J_1J_3G \subseteq F + Soc(G)$ or $J_2J_3G \subseteq F + Soc(G)$ .Since G is Z-regular multiplication R-module, then $Soc(G) = Soc(R)G$, and also $F = [F:_R G]G$. Thus either $J_1J_3G \subseteq [F:_R G]G + Soc(R)G$ or $J_2J_3G \subseteq [F:_R G]G + Soc(R)G$. That is either $J_1J_3 \subseteq [F:_R G] + Soc(R)$ or $J_2J_3 \subseteq [F:_R G] + Soc(R)$.Therefore by lemma (3.1) $[F:_R G]$ is an APP-quasi prime ideal of R.

($\Longleftarrow$)Assume that $[F:_R G]$ is an APP-quasi prime ideal of R, and let $rIB \subseteq F$, where $r \in R$, I is an ideal of R, and B is a submodule of R. Since G is a multiplication, then $B = JG$ for certain ideal J of R, thus it follows that $rIB = rIJG \subseteq F$, that is $rIJ \subseteq [F:_R G]$. However $[F:_R G]$ is an APP-quasi prime ideal of R, next, by proposition (2.6) either $rJ \subseteq [F:_R G] + Soc(R)$ or $IJ \subseteq [F:_R G] + Soc(R)$, thus $rJG \subseteq [F:_R G]G + Soc(R)G$ or $IJG \subseteq [F:_R G]G + Soc(R)G$. Again since G is a multiplication , then $F = [F:_R F]F$, and G is a Z-regular, then $Soc(G) = Soc(R)G$. Hence either $rB \subseteq F + Soc(G)$ or $IB \subseteq F + Soc(G)$. Therefore by proposition (2,6) F is an APP-quasi prime submodule of G.

Proposition 3.9:

If G be a multiplication projective R-module, and F be a proper submodule of G. Then F is an APP-quasi prime submodule of G if and only if $[F:_R G]$ is an APP-quasi prime ideal of R.

Proof:

($\Longrightarrow$)Assume that G is APP-quasi prime submodule of G, and let $I_1I_2I_3 \subseteq [F:_R G]$,for I_1, I_2, I_3 are ideals of R , that is $I_1I_2I_3G \subseteq F$. Since G is a multiplication , then $L_1L_2L_3 \subseteq F$ where $L_1 = I_1G$, $L_2 = I_2G$ and $L_3 = I_3G$ .However F is an APP-quasi prime submodule of G, then by lemma (3.1) $L_1L_3 \subseteq F + Soc(G)$ or $L_2L_3 \subseteq F + Soc(G)$, that is either $I_1I_3G \subseteq F + Soc(G)$ or $I_2I_3G \subseteq F + Soc(G)$. since G is projective multiplication R-module, then $Soc(G) = Soc(R)G$, and also $F = [F:_R G]G$. Thus either $I_1I_3G \subseteq [F:_R G]G + Soc(R)G$ or $I_2I_3G \subseteq [F:_R G]G + Soc(R)G$. That is either $I_1I_3 \subseteq [F:_R G] + Soc(R)$ or $I_2I_3 \subseteq [F:_R G] + Soc(R)$.Therefore by lemma (3.1) $[F:_R G]$ is an APP-quasi prime ideal of R. ($\Longleftarrow$)Assume that $[F:_R G]$ is an APP-quasi prime ideal of R, and let $IsD \subseteq F$, where $s \in R$, I is an ideal of R, and D is a sub Module of R. Since G is a multiplication, then $D = JG$ for certain ideal J of R, it follows that $IsD = IsJG \subseteq F$, that is $IsJ \subseteq [F:_R G]$. However $[F:_R G]$ is an APP-quasi prime ideal of R, then with corollary (2.9) either $IJ \subseteq [F:_R G] + Soc(R)$ or $sJ \subseteq [F:_R G] + Soc(R)$, hence either $IJG \subseteq [F:_R G]G + Soc(R)G$ or $sJG \subseteq [F:_R G]G + Soc(R)G$. Again since G is a multiplication , then $F = [F:_R G]G$, and G is projective, then $Soc(G) = Soc(R)G$. Hence either $IJ \subseteq F + Soc(G)$ or $sJ \subseteq F + Soc(G)$.Therefore with corollary(2.9) F is an APP-quasi prime sub Module of G.

Lemma 3.10: [14, coro of Theo(9)]

Let I ,J are two ideals of R, and G be a multiplication finitely generated R-Module. Then $IG \subseteq JG$ if and only if $I \subseteq J + ann_R(G)$.

Proposition 3.11:

If G is a finitely generated Z-regular multiplication R-module, and I is an ideal of R with $ann_R(G) \subseteq I$. Then I is an APP-quasi prime ideal of R if and only if IG is an APP-quasi prime submodule of G.

Proof:

($\Longrightarrow$) Assume that I is an APP-quasi prime ideal of R with $ann_R(G) \subseteq I$, and let $rJB \subseteq IG$, where $r \in R$, J is an ideals of R, B is a submodule of G. Since G is a multiplication, this means $B = AG$ for certain ideal A of R, that is $rJB = rJAG \subseteq IG$ .However G is a finitely generated, then by lemma(3.10) $rJA \subseteq I + ann_R(G)$. since $ann_R(G) \subseteq I$, then $I + ann_R(G) = I$,thus $rJA \subseteq I$. since I is an APP-quasi prime ideal of R, implies that by proposition (2.6) $rA \subseteq I + Soc(R)$ or $JA \subseteq I + Soc(R)$, thus $rAG \subseteq IG + Soc(R)G$ or $JAG \subseteq IG + Soc(R)G$. However G is a Z-regular, then $Soc(R)G = Soc(G)$. that is either $rB \subseteq IG + Soc(G)$ or $JB \subseteq IG + Soc(G)$. Therefore with proposition (2.6) IG is an APP-quasi prime submodule of G. **($\Longleftarrow$)**Suppose that IG is an APP-quasi prime submodule of G and let $rJB \subseteq I$ for $r \in R$, J is an ideal of R, implies that $rJBG \subseteq IG$, thus $rJ(BG) \subseteq IG$. However IG is an APP-quasi prime submodule of M, then with proposition (2.6), $rBG \subseteq IG + Soc(G)$ or $JBG \subseteq IG + Soc(G)$. Since G is a Z-regular ,then $Soc(R)G = Soc(G)$. Hence $rBG \subseteq IG + Soc(R)G$ or $JBG \subseteq IG + Soc(R)G$. Thus $rB \subseteq I + Soc(R)$ or $JB \subseteq I + Soc(R)$, therefore by proposition (2.6) I is an APP-quasi prime submodule of G.

Lemma 3.12[5, Coro.(2.5)]

Let G be an R-module, and F is a proper submodule of G.Then F is an APP-quasi prime submodule of G if and only if whenever $rsD \subseteq G$, where, $r, s \in R$, and, D is a submodule of G, implies that either $rD \subsetneq F + Soc(G)$ or sD ⊆ F + Soc(G).

Proposition 3.13:

If G be a finitely generated multiplication projective R-module, and I is an APP-quasi prime ideal of R with $ann_R(G) \subseteq I$. Then I is an APP-quasi prime ideal of R if and only if IG is an APP-quasi prime submodule of G.

Proof:

($\Longrightarrow$) Assume that I is an APP-quasi prime ideal of R with $ann_R(G) \subseteq I$, and let $rsB \subseteq IG$, for $r, s \in R$, , and B is a submodule of G. Since G is a multiplication, thus $B = JG$ for J is an ideal J of R, that is $rsB = rsJG \subseteq IG$ .However G is a finitely generated then with lemma (3.10) $rsJ \subseteq I + ann_R(G)$. Since $ann_R(G) \subseteq I$,then $I + ann_R(G) = I$,thus $rsJ \subseteq I$. Since I is an APP-quasi prime ideal of R, therefore by lemma (3.12) either $rJ \subseteq I + Soc(R)$ or $sJ \subseteq I + Soc(R)$, thus either $rJG \subseteq IG + Soc(R)G$ or $JsG \subseteq IG + Soc(R)G$. However G is a projective ,then by $Soc(R)G = Soc(G)$, and $B = JG$. Thus either $rB \subseteq IG + Soc(G)$ or $sB \subseteq IG + Soc(G)$. Therefore by lemma (3.12) IG is an APP-quasi prime submodule of G. **($\Longleftarrow$)**Suppose that IG is an APP-quasi prime submodule of G, and let $rJB \subseteq I$ for $r \in R$, J is an ideal of R and B is a submodule of G, implies that $rJBG \subseteq IG$, thus $rJ(BG) \subseteq IG$. However IG is an APP-quasi prime sub Module of G, then by proposition (2.6) , either $rBG \subseteq IG + Soc(G)$ or $JBG \subseteq IG + Soc(G)$. Since G is a projective ,then $Soc(R)G = Soc(G)$. Thus $rBG \subseteq IG + Soc(R)G$ or

$JBG \subseteq IG + Soc(R)G$. implies that $rB \subseteq I + Soc(R)$ or $JB \subseteq I + Soc(R)$. Therefore by proposition (2.6) I is an APP-quasi prime ideal of R.

Proposition 3. 14:

If F is a proper submodule of finitely generated multiplication Z-regular R-module G although $ann_R(G) \subseteq [F:_R G]$.Then the following statements are equivalent :

1. F is an APP-quasi prime submodule of G.
2. $[F:_R G]$ is an APP-quasi prime ideal of R.
3. $F = JG$ for certain APP-quasi prime ideal J of R with $ann_R(G) \subseteq J$.

Proof (1) ⟺ (2) Follows by proposition (3.7)
(2) ⟹ (3) Suppose that $[F:_R G]$ is an APP-quasi prime ideal of R. Since G is a multiplication R-module, then $F = [F:_R G]G$, Put $[F:_R G] = J$,then $F = JG$ and since $[F:_R G]$ is an APP-quasi prime ideal of R, thus J is an APP-quasi prime ideal of R with $ann_R(G) = [0:_R G] \subseteq [F:_R G] = J$,implies that $ann_R(G) \subseteq J$.
(3) ⟹ (2) Assume that $F = JG$ for certain APP-quasi prime ideal J of R with $ann_R(G) \subseteq J$. However G is a multiplication, then $F = [F:_R G]\, G = JG$,and since G is a finitely generated multiplication, then G is a weak cancellation, thus $[F:_R G] + ann_R(G) = J + ann_R(G)$,however $ann_R(G) \subseteq J$,and $ann_R(G) \subseteq [F:_R G]$, implies that $ann_R(G) + J = J$ and $[F:_R G] + ann_R(G) = [F:_R G]$.Thus $[F:_R G] = J$, however J is an APP-quasi prime ideal of R, thus $[F:_R G]$ is an APP-quasi prime ideal of R.

Proposition 3. 15:

If F is a proper submodule of finitely generated multiplication projective R-module G although $ann_R(G) \subseteq [F:_R G]$. then the following statements are equivalent :

1. F is an APP-quasi prime submodule of G.
2. $[F:_R G]$ is an APP-quasi prime ideal of R.
3. $F = JG$ for certain APP-quasi prime ideal J of R, with $ann_R(G) \subseteq J$.

Proof:
(1) ⟺ (2) Follows by proposition (3.8).
(2) ⟺ (3) Similar as in proposition (3.13).

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