On Some $FT_i$-space ; $i = 0, 1, 2$ in Topological Space

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1. Introduction

In the topological space $X$, a subset $B$ of a space $X$ is said to be a regularly-closed, called also closed domain if $B = cl(int(B))$. A subset $B$ of $X$ is said to be a regularly-open, called also open domain if $B = int(cl(B))$ [2].

2. Preliminaries

Definition (2.1) [4] Let $B$ open and subset of topological space $(X, \tau)$, then $cl(B) \setminus B$ is finite set and is denoted by $F_{open}$.
Definition (2.2) [4] Let B closed and subset of topological space \((X,\tau)\), then the \(B\\setminus\text{int}(B)\) is finite set and is denoted by \(F_\text{close}\).

Remarks (2.3) [4] Let \((X,\tau)\) is topological space, and \(U \subseteq X\).

(i) Let \(U\) is \(F_\text{open}\), the complement of \(U\) is \(F_\text{closed}\).

(ii) Let \(U\) is \(F_\text{closed}\), the complement of \(U\) is \(F_\text{open}\).

Remarks (2.4) Every \(F_\text{open}\) set is open set but not convers true.

Example (2.5) X infinite set, a \(\in X\), \(\tau = \{A \subseteq X : a \in A\} \cup \{\emptyset\}\), A open set, \(a \in A\), \(A = X\), \(b(A) = A = X\).

If \(A\) finite set, then \(b(A)\) infinite set, Then \(A\) not \(F_\text{open}\) set.

Definition (2.6) [4] \((X,\tau)\) is a topological space, a point in \(X\), a \(F_\text{open}\) neighbourhood of \(X\) is \(V F_\text{open}\), \(V \subseteq X\), which is containing a.

Theorem (2.7) [4] a topological space\((X,\tau)\), then

(i) every union finite \(F_\text{closed}\) subset of \(X\) is \(F_\text{closed}\).

(ii) every union finite \(F_\text{open}\) subset of \(X\) is \(F_\text{open}\).

(iii) every intersection finite \(F_\text{closed}\) subset of \(X\) is \(F_\text{closed}\).

(iv) every intersection finite \(F_\text{open}\) subset of \(X\) is \(F_\text{open}\).

Definition (2.8) [4] If \((X,\tau)\) a topological space, and \(H \subseteq X\) the intersection of all \(F_\text{closed}\) containing \(H\) is called \(F_\text{closure}\), denoted by \(\overline{H}^F\).

Theorem (2.9) [4] Let \(A\) be a subset of the topological space, \((X,\tau)\) then \(A \subseteq \overline{A} \subseteq \overline{A}^F\).

Corollary (2.10) [4] If \(U\) is \(F_\text{open}\) set and \(U \cap \overline{V}^F = \emptyset\), then \(U \cap \overline{V}^F = \emptyset\). In particular, if \(U\) and \(V\) are disjoint \(F_\text{open}\) set then, \(U \cap \overline{V}^F = \emptyset = (U)^F \cap V\).

Definition (2.11) [4] If \((X,\tau)\) a topological space, and \(H \subseteq X\), A point \(z \in X\) is called \(F_\text{limit points}\) of \(H\) if and only if for any \(F_\text{open}\) set \(U\) containing \(x\), we have \((U \setminus \{z\}) \cap H \neq \emptyset\).

Remark (2.12) [4] the set of all \(F_\text{limit points}\) of \(H\) is called the \(F_\text{derived set}\) and denoted by \(d_F(K)\).

Theorem (2.13) [4] If \((X,\tau)\) a topological space, and \(H, U \subseteq X\), then.

(i) \(d(H) \subseteq d_F(H)\), \(d(H)\) is the derived set of \(H\).

(ii) \(H \subseteq U\), then \(d_F(H) \subseteq d_F(U)\).

(iii) \(d_F(H) \cup d_F(U) = d_F(H \cup U)\) and \(d_F(H \cap U) \subseteq d_F(H) \cap d_F(U)\).

Theorem (2.14) [4] If \((X,\tau)\) a topological space, and \(H, U \subseteq X\), then.

(i) \(\overline{\emptyset}^F = \emptyset\).

(ii) \(H \subseteq \overline{H}^F\).

(iii) If \(H \subseteq U\), then \(\overline{H}^F \subseteq \overline{U}^F\).

(iv) If \(\overline{(H \cup U)}^F = \overline{H}^F \cup \overline{U}^F\).
Definition (2.15) [4] g: \( (X, \tau) \rightarrow (Y, \tau) \) a function \( g \) is called \( F \) _-continuous if \( g^{-1}(H) \) is \( F \)-open set in \( X \) for every open set \( H \) in \( Y \).

Definition (2.16) [4] g: \( (X, \tau) \rightarrow (Y, \tau) \) a function \( g \) is called \( F \)-open if \( g(H) \) is a \( F \)-open set in \( Y \) for every open sets \( H \) in \( X \).

Definition (2.17) [4] g: \( (X, \tau) \rightarrow (Y, \tau) \) a function \( g \) is called \( F \)-closed if \( g(H) \) is a \( F \)-closed set in \( Y \) for every closed sets \( H \) in \( X \).

Definition (2.18) [4] g: \((X, \tau) \rightarrow (Y, \tau)\) a function \( g \) is called \( F \)-homeomorphism if and only if \( h \) and \( h^{-1} \) are \( F \)-continuous one to one, onto.

Theorem (2.19) Let \((Y, \tau_Y)\) be \( F \)-open subspace of \((Y, \tau)\) if \( U \) \( F \)-open set in \( X \) then \((U \cap Y)\) \( F \)-open set in \( Y \).

Proof: Let \( U \) be \( F \)-open set in \( X \). Then \( U \) is open set in \( X \) (since every \( F \)-open is open). Then \( U \cap Y \) open set in \( Y \). To prove by \((U \cap Y)\) is finite, \( b_y \) \((U \cap Y) \subseteq b(U \cap Y) \cap Y = [U \cap Y] \cap Y = [(U \cup U) \cup Y \cap Y \subseteq [(U \cap Y) \cap Y \cup (U \cap Y) \cap Y \subseteq \{b(U) \cup b(Y)\} \cap Y[b(U) \cup b(Y)]\). Since \( b(U) \) and \( b(Y) \) are finite, then \( b_y(U \cup Y) \) is finite, then \( b_y(U \cup Y) \) is \( F \)-open set in \( Y \) is \( F \)-open, Therefore \( U \cap Y \) is \( F \)-open set in \( Y \).

3. The Main Results

3.1 On \( F \)-Separation Axiom

Definition (3.1.1) If \((X, \tau)\) be a topological space, then \( X \) is called \( FT_{0\_} \) space if and only for each \( x, y \in X \) such that \( x \neq y \) and there exists \( V \) \( F \)-open set, \( [x \in V \text{ and } y \notin V] \text{ or } [x \notin V \text{ and } y \in V] \).

Example (3.1.2) Let \( X = \{a, b, c\} \) and \( \tau = \{X, \emptyset, \{a\}, \{a, b\}\} \) then \((X, \tau)\) is \( FT_{0\_} \) space, since \( X \) is finite every open set is \( F \)-open set, the \( F \)-open set is \( X, \emptyset, \{a\}, \{a, b\} \), There exists \( F \)-open set \( \{a\} \); \( a \notin \{a\} \) \& \( b \notin \{a\}, a \neq c \), There exists \( F \)-open set \( \{a\}; a \notin \{a\} \) \& \( c \notin \{a\}, b \neq c \), There exists \( F \)-open set \( \{a\}; a \notin \{a\} \) \& \( c \notin \{a\}, b \neq c \), Then \( \{a\} \) is \( FT_{0\_} \) space.

Example (3.1.3) Let \( X = \{x, y\} \) is indiscrete space and \( \tau = \{X, \emptyset\}, \text{The } F \)-open set is \( X, \emptyset, x \neq y \); There is not exists \( V \) is \( F \)-open set such that \( [x \in V \text{ and } y \notin V] \text{ or } [x \notin V \text{ and } y \in V] \), so \((X, \tau)\) is not \( FT_{0\_} \) space.

Theorem (3.1.4) If \( X \) is \( FT_{0\_} \) space then \( X \) is \( T_{0\_} \) space.

Proof: There exists \( V \) \( F \)-open set such that \([x \in V \text{ and } y \notin V] \text{ or } [x \notin V \text{ and } y \in V] \). Every \( F \)-open set is open set (by Remark 2.4) So There exists \( U \) open set such that \([x \in U \text{ and } y \notin U] \text{ or } [x \notin U \text{ and } y \in U] \), so \( X \) is \( T_{0\_} \) space.

Example (3.1.5) Let \( X \) infinite set; \( a \in X \) and \( \tau = \{X \subseteq X: a \in X \} \cup \{\emptyset\}, \text{A open set}. \text{There exists } a, b \in X \text{ and } a \neq b \text{ such that } [a \notin A \land b \notin A]. \text{So } X \text{ is } T_{0\_} \text{ space if } A \text{ is open set and finite set. Then } A \text{ is not } F_{\_} \text{ open set. There is not exists } U \text{ is } F_{\_} \text{ open set such that } \neq b, [a \notin U \land b \notin U] \text{ or } [a \notin U \land b \in U], \text{So } X \text{ is not } FT_{0\_} \text{ space.}

Theorem (3.1.6) Let \( h: (X, \tau) \rightarrow (Y, \tau') \) be \( F_{\_} \)continuous, onto, one to one, then \( X \) is \( FT_{0\_} \) space if and only if \( Y \) is \( FT_{0\_} \) space.

Proof: Let \( X \) is \( FT_{0\_} \) space. Since \( h: (X, \tau) \rightarrow (Y, \tau') \), there exists \( h \) one to one and \( h \) is onto, \( h \) is \( F \)-continuous, \( h^{-1} \) is \( F \)-continuous, let \( y_1, y_2 \in Y \); \( y_1 \neq y_2 \) then \( h^{-1}(y_1), h^{-1}(y_2) \in X \) onto function so \( h^{-1}(y_1) \neq \emptyset, h^{-1}(y_2) \neq \emptyset, h \) one to one function, there exists \( x_1 \in X; h^{-1}(y_1) = x_1, \) and there exists \( x_2 \in X; h^{-1}(y_2) = x_2 \) \& \( x_1 \neq x_2 \) and \( x_1, x_2 \in X \). Since \( X \) is \( FT_{0\_} \) space there exists \( U \) \( F \)-open set, \( [x_1 \in U \land x_2 \in U] \lor [x_1 \notin U \land x_2 \notin U] \), \( h^{-1} \) is \( F \)-continuous then \( h(U) \) is \( F \)-open set \( [h(x_1) \in h(U) \land h(x_2) \in h(U)] \lor [h(x_1) \notin h(U) \land h(x_2) \notin h(U)] \). So \( Y \) is \( FT_{0\_} \) space.

Converse: Let \( Y \) is \( FT_{0\_} \) space. Since \( h: (X, \tau) \rightarrow (Y, \tau') \), there exists \( h \) one to one and \( h \) onto and \( h \) is \( F \)-continuous and \( h^{-1} \) \( F \)-continuous, let \( x_1, x_2 \in X; x_1 \neq x_2 \) then \( h^{-1}(x_1), h^{-1}(x_2) \in Y \), \( h \) is onto function then \( h^{-1}(x_1) \neq \emptyset, h^{-1}(x_2) \neq \emptyset, h \) is one to one function, there exists \( y_1 \in Y; h^{-1}(x_1) = y_1 \) and there exists \( y_2 \in Y; h^{-1}(x_2) = y_2; \)
Theorem 3.1.7 Let $(W, \tau_W)$ be an open subspace of a topological space $(X, \tau)$ if $(X, \tau)$ is $FT_0$-space, then $(W, \tau_W)$ is $FT_0$-space.

Proof: Let $x, y \in W$; $x \neq y, x, y \in X$ since $W \subseteq X$. There exists $U$ that is $F$-open in $X$, $(x \in U \land y \notin U) \lor (x \notin U \land y \in U)$, then $U \cap W$ are $F$-open in $W$ (by Theorem 2.19), $(x \in U \land W \land y \notin U \land W) \lor (x \notin U \land W \land y \in U \land W)$, so $(W, \tau_W)$ is $FT_0$-space.

Definition 3.1.8 $(X, \tau)$ a topological space is defined $FT_0$-space if and only if for each $a, b \in X$ such that $a \neq b$, there exists $U, V$ is $F$-open set such that, $[a \in U \land b \notin U \lor b \in V \land a \notin V]$.

Example 3.1.9 Let $X = \{a, b, c\}$, The $F$-open sets are $\emptyset, X, \{a\}, \{b\}$ such that $a \neq b$ there exists $U = \{a\}$ is $F$-open set, there exists $V = \{b\}$ is $F$-open set, $[a \in U \land b \notin U \lor b \in V \land a \notin V]$, $(X, \tau)$ is $FT_0$-space.

Lemma 3.1.10 If $X$ is $FT_1$-space then $X$ is $FT_0$-space.

Proof: Let $X$ is $FT_1$-space and $a, b \in X$ such that $a \neq b$, there exist $U, V$ is $F$-open set. Such that, $[a \in U \land b \notin U \lor b \in V \land a \notin V]$, (Since $X$ is $FT_1$-space), there exist $U$ is $F$-open set such that $[a \in U \land b \notin U] \lor [b \in V \land a \notin V]$, so $X$ is $FT_0$-space.

Example 3.1.11 Let $X = \{a, b\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}\}$, $X$ is $FT_0$-space; $\emptyset, X, \{a\}$, Let $a, b \in X$ such that $a \neq b$ there exists $U$ is $F$-open set such that $[a \in U \land b \notin U] \lor [b \in V \land a \notin V]$ but there is not exists $V$ is $F$-open set such that $[a \in U \land b \notin U]$ and $[b \in V \land a \notin V]$, $(X, \tau)$ is not $FT_1$-space.

Lemma 3.1.12 Every $FT_1$-space is $T_1$-space.

Proof: There exists $U, V$ is $F$-open set such that $[a \in U \land b \notin U] \lor [b \in V \land a \notin V]$, Since every $F$-open set is open set there exists $U, V$ open set such that $[a \in U \land b \notin U]$, so $X$ is $T_1$-space.

Lemma 3.1.13 Let $\{x\}$ is $F$-closed then $(X, \tau)$ is $FT_0$-space for each $x \in X$.

Proof: Let $x, y \in X$ such that $x \neq y$, let $U = X - \{x\}$, $(\{x\}$ $F$-closed), $V = X - \{y\}$, $(\{y\}$ $F$-closed), Such that, $U, V$ is $F$-open set, $[y \in U \land x \notin U] \lor [y \notin V \land x \in V]$, so $X$ is $FT_1$-space.

Theorem 3.1.14 Let $(W, \tau_W)$ be an open subspace of a topological space $(X, \tau)$ if $(X, \tau)$ is $FT_1$-space, then $(W, \tau_W)$ is $FT_1$-space.

Proof: Let $x, y \in W$; $x \neq y, x, y \in X$ since $W \subseteq X$ Since $X$ is $FT_1$-space There exists $U, V$ is $F$-open in $X$, $[x \in U \land y \notin U] \lor [x \notin U \land y \in V]$, then $U \cap W \land V \cap W$ are $F$-open (by Theorem 2.19) in $W$, $[x \in U \land W \land y \notin U \land W] \lor [x \notin U \land W \land y \in V \land W]$, so $(W, \tau_W)$ is $FT_1$-space.

Theorem 3.1.15 $h: (X, \tau) \rightarrow (Y, \tau')$ be $F$-continuous, onto, one to one then $X$ is $FT_1$-space if and only if $Y$ is $FT_1$-space.

Proof: let $h: (X, \tau) \rightarrow (Y, \tau')$ and suppose that $X$ is $FT_1$-space. Since $h: (X, \tau) \rightarrow (Y, \tau')$ there exists, $h$ is one to one and, onto, $F$-continuous, $h^{-1}$ is $F$-continuous let $y_1, y_2 \in Y$, $y_1 \neq y_2 \Rightarrow h^{-1}(y_1), h^{-1}(y_2) \in X$, $h$ onto function so $h^{-1}(y_1) \neq \emptyset$, $h^{-1}(y_2) \neq \emptyset$, $h$ one to one function, there exists $x_1 \in X$, $h^{-1}(y_1) = x_1$, and there exists $x_2 \in X$, $h^{-1}(y_2) = x_2$ and $x_1 \neq x_2$ and $x_1 \neq x_2 \in X$. Since $X$ is $FT_1$-space there exists $U, V$ is $F$-open set, $[x_1 \in U \land x_2 \notin U] \lor [x_1 \notin U \land x_2 \in V]$, $h^{-1}$ is $F$-continuous then $h(U), h(V)$ is $F$-open, $[h(x_1) \in h(U) \land h(x_2) \notin h(U)] \lor [h(x_1) \notin h(V) \land h(x_2) \in h(V)]$, so $Y$ is $FT_1$-space.

Converse: Let $Y$ is $FT_1$-space. Since $h: (X, \tau) \rightarrow (Y, \tau')$ then there exists, $h$ one to one and, onto, $F$-continuous and $h^{-1}$ is $F$-continuous, let $x_1, x_2 \in X$, $x_1 \neq x_2$ then $h^{-1}(x_1), h^{-1}(x_2) \in Y$ $h$ is onto function then $h^{-1}(x_1) \neq \emptyset$, $h^{-1}(x_2) \neq \emptyset$, $h$ is one to one function, there exists $y_1 \in Y$, $h^{-1}(x_1) = y_1$ and there exists $y_2 \in Y$, $h^{-1}(x_2) = y_2$ and $y_1 \neq y_2$ and $y_1, y_2 \in Y$, $Y$ is $FT_1$-space there exists $U, V$ $F$-open set, $[y_1 \in U \land y_2 \notin U] \lor [y_1 \notin U \land y_2 \in V]$, $h^{-1}$ is $F$-continuous then $h(U), h(V)$ is $F$-open set, $[h(y_1) \in h(U) \land h(y_2) \notin h(U)] \lor [h(y_1) \notin h(V) \land h(y_2) \in h(V)]$, so $X$ is $FT_1$-space.
**Definition (3.1.16)** Let \((X, \tau)\) topological space is called a \(FT_2\)-space if for each pair distinct points \(a, b \in X\), the exist \(F\)-open sets \(U, V\) and \(a \neq b\) such that \([a \in U, b \in V, \text{and } U \cap V = \emptyset]\).

**Example (3.1.17)** let \(X = \{a, b\}\). \(\tau = \{\emptyset, X, \{a\}, \{b\}\}\) The \(F\)-open set in \(X\) are \(\emptyset, X, \{a\}, \{b\}\), for each \(a \neq b\) there exists \(U, V\) are \(F\)-open such that, \([a \in U, b \in V, \text{and } U \cap V = \emptyset]\). So \((X, \tau)\) is \(FT_2\)-space.

**Theorem (3.1.18)** Every \(FT_2\)-space is \(FT_1\) _space._

**Proof:** Let \(X\) is \(FT_2\)-space and \(a \neq b\), there exists \(U, V\) is \(F\)-open set such that \([a \in U, b \in V, \text{and } U \cap V = \emptyset]\). There exists \(U, V\) is \(F\)-open sets and \(a \neq b\) such that \([a \in U \land b \notin U] \land [a \notin V \land a \in V]\), So \(X\) is \(FT_1\)-space.

**Theorem (3.1.19)** Leth: \((X, \tau) \rightarrow (Y, \tau')\) be \(F\)-continuous, onto, one to one then \(Y\) is \(FT_2\)-space, if and only if \(X\) is \(FT_2\)-space.

**Proof:** Let that \(Y\) is \(FT_2\)-space, there exists \(h: (X, \tau) \rightarrow (Y, \tau')\) h is one to one, onto, \(F\)-continuous and \(h^{-1}\) is \(F\) continuous let \(x_1, x_2 \in X\); \(x_1 \neq x \Rightarrow h(x_1), h(x_2) \in Y\), h onto function then \(h(x_1) \neq \emptyset, h(x_2) \neq \emptyset\), h one to one function, there exists \(y_1 \in Y\); \(h(x_1) = y_1\) and there exists \(y_2 \in Y\); \(h(x_2) = y_2\) and \(y_1 \neq y_2\) and \(y_1, y_2 \in Y\). Since \(Y\) is \(FT_2\)-space there exist \(V_1, V_2\) are \(F\)-open set, \(V_1 \cap V_2 = \emptyset, [y_1 \in V_1 \land y_2 \notin V_2]\), \(h\) is \(F\)-continuous then \(h^{-1}(V_1) = U_1, h^{-1}(V_2) = U_2\) are \(F\)-open, \(U_1 \cap U_2 = h^{-1}(V_1) \cap h^{-1}(V_2) = h^{-1}(\emptyset) = \emptyset, (x_1 \in U_1 \land x_2 \in U_2)\), So \(X\) is \(FT_2\)-space.

**Converse** let \(h: (X, \tau) \rightarrow (Y, \tau')\) and suppose that \(X\) is \(FT_2\)-space, Since \(h: (X, \tau) \rightarrow (Y, \tau')\) there exists h one to one, onto, \(F\)-continuous, \(h^{-1}\) is \(F\)-continuous let \(y_1, y_2 \in Y\); \(y_1 \neq y_2 \Rightarrow h^{-1}(y_1), h^{-1}(y_2) \in X\). h onto function then \(h(y_1) \neq \emptyset, h(y_2) \neq \emptyset\), h one to one function, there exists \(x_1 \in X\); \(h(y_1) = x_1\) and there exists \(x_2 \in X\); \(h(y_2) = x_2\) and \(x_1 \neq x_2\) and \(x_1, x_2 \in X\). Since \(X\) is \(FT_2\)-space there exist \(V_1, V_2\) are \(F\)-open set, \(V_1 \cap V_2 = \emptyset, [x_1 \in V_1 \land x_2 \notin V_2]\), \(h\) is \(F\)-continuous then \(h^{-1}(V_1) = U_1, h^{-1}(V_2) = U_2\) are \(F\)-open, \(U_1 \cap U_2 = h^{-1}(V_1) \cap h^{-1}(V_2) = h^{-1}(\emptyset) = \emptyset, (y_1 \in U_1 \land y_2 \in U_2)\), So \(Y\) is \(FT_2\)-space.

**Remark (3.1.20)** if \((X, \tau)\) is \(FT_2\)-space, then not necessary test that space is \(FT_1\)-space and every \(FT_1\)-space is \(FT_0\)-space.

**Theorem (3.1.22)** Let \((W, \tau_w)\) \(F\)-open subspace of topological space \((X, \tau)\) if \((X, \tau)\) is \(FT_2\)-space, then \((W, \tau_w)\) is \(FT_2\)-space.

**Proof:** Let \(x, y \in W; x \neq y, x, y \in X\) (since \(W \subseteq X\)), Since \(X\) is \(FT_2\)-space, There exist \(U, V\) are \(F\)-open in \(X\) and \(U \cap V = \emptyset, [x \in U \land y \in V]\) then \(U \cap W \land V \cap W = \emptyset\) are \(F\)-open in \(W\) (by theorem 2.19). \((U \cap W) \cap (U \cap V) = (U \cap V) \cap W = \emptyset \cap W = \emptyset\), and \((x \in U \cap W \land y \in V \cap W)\), So \((W, \tau_w)\) is \(FT_2\)-space.

**References**


