Journal of Al-Qadisiyah for Computer Science and Mathematics Vol.16.(2) 2024,pp.Math 126–130

Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

On Some FT_i-space ;$i = 0, 1, 2$ in Topological Space

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ARTICLE INFO

Article history:
Received: 27 /2/2024
Revised form: 13 /4/2024
Accepted : 24 /4/2024
Available online: 30 /6/2024

Keywords:

F_open, F_closed, *FT_0_space, FT_1 _space, FT_2 _space.*

ABSTRACT

In this paper, we introduce a new definitions of Separation Axiom which we called FT_0 _space, FT_1 _space, FT_2 _space, Then we set the characteristics for each definition and demonstrated the interconnection between the three definitions. We also demonstrated the genetic and partial characteristics in each definition.

MSC: 30C45

https://doi.org/10.29304/jqcsm.2024.16.21561

1. Introduction

In the topological space X, a subset B of a space X is said to be a regularly-closed, called also closed domain if B = cl(int(B)). A subset B of X is said to be a regularly-open, called also open domain if $B = int(cl(B))$ [2]. An open (resp., closed) subset B of a topological space (X, T) is called F −open (resp.,F-closed) set if $cl(B)\backslash B$ $(resp., B\backslash int(B))$ is finite set [4].They introduce a new type of semi-open sets which they call S_g-open sets[5]. An open (resp., closed) subset B of a topological space (X, T) is called C −open (resp.,C-closed) set if $cl(B)\backslash B$ $(resp., B\backslash int(B))$ is a countable set[6]. In this work, we are interested in studying the concepts of Separation Axiom which we call FT_0 _space, FT_1 _space, FT_2 _space, Then we set the characteristics for each definition and demonstrated the interconnection between the three definitions. We also demonstrated the genetic and partial characteristics in each definition. We have proven some theorems linking FT_space and T_space. We have also proven the transmission of genetic traits in our new topic, and we have also linked the relationships between FT_0 _space, FT_1 _space, FT_2 _space.

2. Preliminaries

Definition (2.1) [4] Let B open and subset of topological space (X,τ), then the cl(B)\B is finite set and is denoted by F_open .

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Definition (2.2) [4] Let B closed and subset of topological space (X,τ), then the B\int(B) is finite set and is denoted by F_close .

Remarks (2.3) [4]Let (X,τ) is topological space, and $U \subseteq X$.

(i) Let U is F_ open, the complement of U is F_closed .

(ii) Let U is F_closed, the complement of U is F_open .

Remarks (2.4) Every F_open set is open set but not convers true.

Example (2.5) X infinite set, $a \in X, \tau = \{A \subseteq X : a \in A\} \cup \{\emptyset\}$, A open set, $a \in A$, $\overline{A} = X$, $b(A) = \overline{A} - A = X - A$,
If A finite set, then b(A) infinite set**,** Then A not F_open set.

Definition (2.6) [4] (X, τ)is a topological space, a point in X, a F_open nieghbourhood of X is V F_open, $V \subseteq X$, which is containing a .

Theorem (2.7)[4] a topological space(X, τ), then

(i) every union finite F_closed subset of X is F_closed .

(ii) every union finite F_open subset of X is F_open.

(iii) every intersection finite F_closed subset of X is F_closed.

(iv) every intersection finite F_open subset of X is F_open.

Definition (2.8) [4] If (X, τ) a topological space, and $H \subseteq X$the intersection of all F_closed containing H is called F_closure, denoted by $\overline{H}^F$.

Theorem (2.9) [4] Let A be a subset of the topological space, (X,τ) then $A \subseteq \overline{A} \subseteq \overline{A}^F$.

Corollary(2.10)[4]If U is F_open set and $U \cap V = \emptyset$, then $U \cap \overline{V}^F = \emptyset$ In particular, if U and V are disjoint F _open set then, $U \cap \overline{V}^F = \emptyset = \overline{(U)}^F \cap V$.

Definition(2.11)[4] If (X, τ)a topological space, and $H \subseteq X$, A point $z \in X$ is called F_limit points of H if and only if for any F_open set U containing x ,we have $(U \setminus \{z\}) \cap H \neq \emptyset$.

Remark(2.12)[4]the set of all F_limit points of H is called the F_derived set and denoted by $d_F(K)$.

Theorem (2.13)[4] If (X, τ) a topological space, and $H, U \subseteq X$, Then .

(i)$d(H) \subset d_F(H)$, d(H) is the derived set of H .

(ii) $H \subseteq U$, then $d_F(H) \subseteq d_F(U)$.

(iii) $d_F(H) \cup d_F(U) = d_F(H \cup U)$ and $d_F(H \cap U) \subset d_F(H) \cap d_F(U)$.

Theorem(2.14)[4] If (X, τ)a topological space, and $H, U \subseteq X$,Then .

(i) $\overline{(\emptyset)}^F = \emptyset$.

(ii) $H \subseteq \overline{H}^F$.

(iii) If $H \subseteq U$, then $\overline{H}^F \subseteq \overline{U}^F$.

(iv) If $\overline{(H \cup U)}^F = \overline{(H)}^F \cup \overline{(U)}^F$) .

(v) $\overline{\overline{H}^{F}}^{F} = \overline{H}^{F}$.

Definition(2.15)[4] $g: (X, \tau) \longrightarrow (Y, \grave{\tau})$a function g is called F_continuous if $g^{-1}(H)$ is F_ open set in X for every open set H in Y.

Definition(2.16)[4] $g: (X, \tau) \longrightarrow (Y, \grave{\tau})$a function g is called F_open if g(H) is a F_open set in Y for every open sets H in X.

Definition(2.17)[4] $g: (X, \tau) \longrightarrow (Y, \grave{\tau})$ a function g is called F_closed if g(H) is a F_closed set in Y for every closed sets H in X.

Definition (2.18)[4] $g: (X, T) \rightarrow (Y, \grave{\tau})$ a function g is called F_ hmoeomrphism if and only if h and h^{-1} are F_continuous ,one to one, onto.

Theorem(2.19) Let(Y, τ_y) be F_open subspace of (Y, τ) if U F_open set in X then $(U \cap Y)$ F_open set in Y.

Proof: Let U be F_open set in X.Then U is open set in X (since every F_open is open). Then $U \cap Y$ open set in Y.To prove by $(U \cap Y)$ is finite, $b_y(U \cap Y) \subseteq b(U \cap Y) \cap Y = [\overline{U \cap Y} \cap (U \cap Y)^c] \cap Y \subseteq [(\overline{U} \cap \overline{U}) \cap U \cup Y^c] \cap Y \subseteq [(\overline{U} \cap \overline{Y} \cap U) \cup (\overline{U} \cap \overline{Y} \cap Y^c)] \cap Y \subseteq [b(U) \cup b(Y)] \cap Y[b(U) \cup b(Y)]$Since b(U) and b(Y) are finite, then $b_y(U \cup Y)$ is finite, then $b_y(U \cup Y)$ is F_open set in Y is F_open, Therefor $U \cap Y$ is F_open set in Y.

3. The Main Results

3.1 On F_Separation Axiom

Definition(3.1.1) If (X, τ) be a topological space, then X is called FT_0_ space if and only for each $x, y \in X$ such that $x \neq y$ and there exists V is F_open set, $[x \in V$ and $y \notin V]$or$[x \notin V$and $y \in V]$.

Example(3.1.2) Let $X = \{a, b, c\}$, and $\tau = \{X, \emptyset, \{a\}, \{a, b\}\}$ then (X, τ) is FT_0_space, since X is finite every open set is F_open set, the F _ open set is $X, \emptyset, \{a\}, \{a, b\}$,$a \neq b$, There exists F_ open set {a} ; $a \in \{a\} \wedge b \notin \{a\}$,$a \neq c$, There exists F_open set {a}; $a \in \{a\} \wedge c \notin \{a\}$, $b \neq c$, There exists F_ open set {a, b} ; $b \in \{a, b\} \wedge c \notin \{a, b\}$, so (X, τ) is FT_0_space.

Example(3.1.3) Let $X = \{x, y\}$ is indiscrete space and $\tau = \{X, \emptyset, \}$,The F_ open set is $X, \emptyset, x \neq y$; There is not exists V is F_ open set such that , $[x \in V$ and $y \notin V]$or$[x \notin V \wedge y \in V]$, so (X, τ) is not FT_0_space.

Theorem(3.1.4) If X is FT_0_space then X is T_0_space.

Proof: There exists V is F_open set such that, $[x \in V$ and $y \notin V]$ or$[x \notin V$ and $y \in V]$ .Every F_open set is open set (by Remark2.4) So There exists U is open set such that, $[x \in U \wedge y \notin U] \vee [x \notin U \wedge y \in U]$, so X is T_0_space.

Example(3.1.5) Let X infinite set; $a \in X$ and $\tau = \{A \subseteq X: a \in A\} \cup \{\emptyset\}$,A open set .There exists $a, b \in X$ and $a \neq b$ such that; $[a \in A \wedge b \notin A]$. So X is T_0_space If Ais open set and finite set. Then A is not F_ open set.There is not exists U is F_open set such that $a \neq b$, $[a \in U \wedge b \notin U] \vee [a \notin U \wedge b \in U]$, So X is not FT_0_space.

Theorem(3.1.6) Let $h: (X, \tau) \rightarrow (Y, \tau`)$ be F_continuous, onto, one to one, then X is FT_0_ space if and only if Y is FT_0 _ space.

Proof: Let X is FT_0 _ space. Since $h: (X, \tau) \rightarrow (Y, \tau`)$, there exists h one to one and h is onto, h is F_continuous, h^{-1} is F_continuous , let $y_1, y_2 \in Y$; $y_1 \neq y_2$ then $h^{-1}(y_1)$, $h^{-1}(y_2) \in X$, h onto function so $h^{-1}(y_1) \neq \emptyset$,$h^{-1}(y_2) \neq \emptyset$,h one to one function, there exists $x_1 \in X$; $h^{-1}(y_1) = x_1$, and there exists $x_2 \in X$; $h^{-1}(y_2) = x_2$ and $x_1 \neq x_2$ and ,$x_1 x_2 \in X$, Since X is FT_0_ space there exists U is F_ open set, $[x_1 \in U \wedge x_2 \notin U] \vee [\notin x_1 U \wedge x_2 \in U]$, h^{-1} is F_continuous then h (U) is F_ open $[h(x_1) \in h(U) \wedge h(x_2) \notin h(U)] \vee [h(x_1) \notin h(U) \wedge h(x_2) \in h(U)]$. So Y is FT_0_ space.

Converse: Let Y is FT_0_space .Since $h: (X, \tau) \rightarrow (Y, \tau`)$, there exists h one to one and h onto and h is F_continuous and h^{-1}F_continuous, let $x_1, x_2 \in X$; $x_1 \neq x_2$ then $h^{-1}(x_1)$, $h^{-1}(x_2) \in Y$, h is onto function then $h^{-1}(x_1) \neq \emptyset$, $h^{-1}(x_2) \neq \emptyset$, h is one to one function, there exists $y_1 \in Y$; $h^{-1}(x_1) = y_1$and there exists$y_2 \in Y$; $h^{-1}(x_2) = y_2$;

$y_1 \neq y_2$ and $y_1, y_2 \in Y$, Y is FT_0_space there exists U F_open set. $[y_1 \in U \wedge y_2 \notin U] \vee [y_1 \notin U \wedge y_2 \in U]$, h^{-1}F_continuous; h(U)is F_open set ;$[h(y_1) \in h(U) \wedge h(y_2) \in h(U)] \vee [h(y_1) \notin h(U) \wedge h(y_2) \in h(U)]$So X is FT_0_space.

Theorem(3.1.7) Let (W, τ_w) F_ open subspace of topological space (X, τ) if (X, τ) is FT_0_ space, then (W, τ_w) is FT_0_ space.

Proof: Let $x, y \in W$; $x \neq y$, $x, y \in X$ since $W \subseteq X$,There exists U is F_open in X, $(x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U)$, then $U \cap W$ are F_open in W (by Theorem2.19), $(x \in U \cap W \wedge y \notin U \cap W) \vee (x \notin U W \wedge y \in U \cap W)$, so$(W, \tau_w)$isFT$_0$_space.

Definition(3.1.8) (X, τ)a topological space is defined FT_1_space if and only if for each $a, b \in X$ such that $a \neq b$, there exists U, V is F_open set such that, $[a \in U \wedge b \notin U$ and $b \in V \wedge a \notin V]$.

Example(3.1.9) Let $X = \{a, b, \tau = \{\emptyset, X, \{a\}, \{b\}\}$, The F_ open sets are $\emptyset, X, \{a\}, \{b\}$such that $a \neq b$ there exists $U = \{a\}$ is F_ open set, there exists $V = \{b\}$ is F_open set, $[a \in U \wedge b \notin U$ and $b \in V \wedge a \notin V](X, \tau)$ is FT_1_ space.

Lemma (3.1.10) If X is FT_1_space then X is FT_0_space .

Proof: Let X is FT_1_space and $a, b \in X$ such that $a \neq b$. there exist U, V F_open set, Such that, $[a \in U \wedge b \notin U$ and $b \in V \wedge a \notin V]$, (Since X is$FT_1$_space),There exist U is F_open set Such that $[a \in U \wedge b \notin U] \vee [b \in U \wedge a \notin U]$, so X is FT_0_space.

Example(3.1.11) Let $X = \{a, b\}, \tau = \{\emptyset, X, \{a\}\}$; X is FT_0_space.the F_open set; $\emptyset, X, \{a\}$, Let $a, b \in X$ such that $a \neq b$: there exists U is F_open set such that. $[a \in U \wedge b \notin U] \vee [b \in U \wedge a \notin U]$ but there is not exists V is F_open set such that $[a \in U \wedge b \notin U]$ and $[b \notin V \wedge a \notin V]$, so (X, τ) is not FT_1_space.

Lemma (3.1.12) Every FT_1_space is T_1_space.

Proof: There exists U, V F_ open sets such that $[a \in U \wedge b \notin U] \wedge [b \in V \wedge a \notin V]$, Since every F_open set is open set there exists U, V open set such that$[a \in U \wedge b \notin U]$ and $[b \notin V \wedge a \notin V]$, so X is T_1_ space.

Lemma (3.1.13) Let $\{x\}$ is F_ closed then (X, τ) is FT_1_space for each $x \in X$.

Proof: Let $x, y \in X$ such that $x \neq y$, let $U = X - \{x\}$, ($\{x\}$ F_closed), $V = X - \{y\}$ ($\{y\}$ F_ closed), Such that U, V is F _open set, $[y \in U \wedge x \notin U] \wedge [y \notin V \wedge x \in V]$, So X is FT_1_space.

Theorem(3.1.14) Let (W, τ_w) F_open subspace of topological space (X, τ) if (X, τ) is FT_1_ space, then (W, τ_w) is FT_1_ space.

Proof: Let $x, y \in W$; $x \neq y$, $x, y \in X$(since $W \subseteq X$)Since X is FT_1_space There exists U, V is F_open in X, $[x \in U \wedge y \notin U) \wedge (x \notin V \wedge y \in V]$, then $U \cap W \wedge V \cap W$ are F_ open(by Theorem 2.19) in W.$[x \in U \cap W \wedge y \notin U \cap W) \wedge (x \notin V \cap W \wedge y \in V \cap W]$, So (W, τ_w) is FT_1_ space.

Theorem(3.1.15) h: $(X, \tau) \to (Y, \tau`)$ be F_continuous, onto, one to one then X is FT_1_ space if and only if Y is FT_1 _ space .

Proof: let h: $(X, \tau) \to (Y, \tau`)$ and suppose that X is $FT_1$ _space. Since h: $(X, \tau) \to (Y, \tau`)$ there exists, h is one to one and, onto, F_ continuous, h^{-1} is F_ continuous let $y_1, y_2 \in Y$; $y_1 \neq y_2 \Rightarrow h^{-1}(y_1), h^{-1}(y_2) \in X$, h onto function so $h^{-1}(y_1) \neq \emptyset$, $h^{-1}(y_2) \neq \emptyset$, h one to one function, there exists $x_1 \in X$; $h^{-1}(y_1) = x_1$, and there exists $x_2 \in X$; $h^{-1}(y_2) = x_2$ and $x_1 \neq x_2$ and $x_1 x_2 \in X$, Since X is FT_1_ space there exists U, V is F_ open set , $[x_1 \in U \wedge x_2 \notin U] \wedge [x_1 \notin V \wedge x_2 \in V]$, h^{-1} is F_ continuous then h (U), h(V) is F_ open, $[h(x_1) \in h(U) \wedge h(x_2) \notin h(U)] \wedge [h(x_1) \notin h(V) \wedge h(x_2) \in h(V)]$, So. Y is FT_1_ space.
Converse: Let Y is FT_1_space, Since h: $(X, \tau) \to (Y, \tau`)$ then there exists, h one to one, onto, F_continuous and h^{-1}F_continuous, let $x_1, x_2 \in X$; $x_1 \neq x_2$ then $h^{-1}(x_1), h^{-1}(x_2) \in Y$ h is onto function then $h^{-1}(x_1) \neq \emptyset, h^{-1}(x_2) \neq \emptyset$, h is one to one function, there exists $y_1 \in Y$; $h^{-1}(x_1) = y_1$ and there exists $y_2 \in Y$; $h^{-1}(x_2) = y_2$ and $y_1 \neq y_2$ and $y_1, y_2 \in Y$, Y is FT_1_ space there exists U, V F_open set, $[y_1 \in U \wedge y_2 \notin U] \wedge [y_1 \notin V \wedge y_2 \in V]$, h^{-1}F_continuous, then h (U), h(V)is F_open set; $[h(y_1) \in h(U) \wedge h(y_2) \in h(U)] \wedge [h(y_1) \notin h(V) \wedge h(y_2) \in h(V)]$, So X is FT_1 _space.

Definition(3.1.16) Let (X, τ)topological space is called a FT_2_space if for each pair distinct points $a, b \in X$, the exist F_ open sets U, V and $a \neq b$ such that $[a \in U , b \in V , \text{and } U \cap V = \emptyset]$.

Example(3.1.17) let $X = \{a, b\}$ $\tau = \{\emptyset, X, \{a\}\{b\}\}$ The F_open set in X are $\emptyset, X, \{a\}\{b\}$, for each $a \neq b$ there exists U, V are F_open such that, $[a \in U , b \in V , \text{and } U \cap V = \emptyset]$, So $(X, \tau)$is FT_2_space.

Theorem(3.1.18) Every FT_2_space is FT_1 _space.

Proof: Let X is FT_2 _space and $a \neq b$,There exists U, V is F_open set such that $[a \in U , b \in V , \text{and } U \cap V = \emptyset]$, There exists U, V is F_open sets and $a \neq b$ such that, $[a \in U \wedge b \notin U] \wedge [a \notin V \wedge a \in V]$, So X is FT_1 _space.

Theorem(3.1.19) Let $h: (X, \tau) \to (Y , \tau`)$ be F_ continuous, onto, one to one then Y is FT_2_ space, if and only if X is FT_2 _ space

Proof: Let that Y is FT_2 _ space, there exists $h: (X, \tau) \to (Y , \tau`)$ h is one to one, onto, F_ continuous and h^{-1} is F_continuous let $x_1, x_2 \in X$; $x_1 \neq x \Rightarrow h (x_1) , h (x_2) \in Y$, h onto function then $h (x_1) \neq \emptyset , h (x_2) \neq \emptyset$, h one to one function, there exists $y_1 \in Y$; $h (x_1) = y_1$ and there exists $y_2 \in Y$; $h (x_2) = y_2$ and $y_1 \neq y_2$ and $y_1, y_2 \in Y$ Since Y is FT_2_ space there exists V_1, V_2 are $F-$ open set, $V_1 \cap V_2 = \emptyset$, $[y_1 \in V_1 \wedge y_2 \notin V_2]$, h is F_continuous then $h^{-1} (V_1) = U_1, h^{-1} (V_2) = U_2$ are F_ open, $U_1 \cap U_2 = h^{-1} (V_1) \cap h^{-1} (V_2) = h^{-1}(V_1 \cap V_2) = h^{-1}(\emptyset) = \emptyset, (x_1 \in U_1 \wedge x_2 \in U_2)$, So X is FT_2_space.

Converse let $h: (X, \tau) \to (Y , \tau`)$and suppose that X is $FT_2$_space, Since $h: (X, \tau) \to (Y , \tau`)$there exists h one to one, onto, F_ continuous, h^{-1} is F_ continuous let $y_1, y_2 \in Y$; $y_1 \neq y_2 \Rightarrow h (y_1) , h (y_2) \in X$ h onto function then $h (y_1) \neq \emptyset , h (y_2) \neq \emptyset$, h one to one function, there exists $x_1 \in X$; $h (y_1) = x_1$, and there exists $x_2 \in X$; $h (y_2) = x_2$ and $x_1 \neq x_2$ and $x_1, x_2 \in X$, Since X is FT_2_space there exists V_1, V_2 are F_open set, $V_1 \cap V_2 = \emptyset$, $[x_1 \in V_1 \wedge x_2 \notin V_2]$, h is F_continuous then $h^{-1} (V_1) = U_1, h^{-1} (V_2) = U_2$ are F_open, $U_1 \cap U_2 = h^{-1} (V_1) \cap h^{-1} (V_2) = h^{-1}(V_1 \cap V_2) = h^{-1}(\emptyset) = (y_1 \in U_1 \wedge y_2 \in U_2)$, So Y is FT_2 _ space.

Remark (3.1.20) if (X, τ) is FT_2_space, then not necessary test that space is FT_1_space and every FT_1_space is FT_0_space.

$$FT_0 \nrightarrow FT_1 \nrightarrow FT_2$$

Example (3.1.21) (N, τ_{cof}) is FT_1_space, Let $n, m \in N$ so $n \neq m$ there exists U, V F_open set $U = N /\{m\}, V = N/\{n\}$, Such that $[n \in U \wedge m \notin U] \wedge [n \notin V \wedge m \in V]$,But $(N, \tau_{cof})$is not FT_1 _space. Since $n \neq m$ and $U = N /\{m\}, V = N /\{n\}$, $U \cap V \neq \emptyset$, So $(N, \tau_{cof})$is not FT_2_ space.

Theorem(3.1.22) Let (W, τ_w) F_open subspace of topological space (X, τ) if (X, τ) is FT_2_ space, then (W, τ_w) is FT_2_space.

Proof : Let $x, y \in W$; $x \neq y$. $x, y \in X$(since $W \subseteq X$), Since X is FT_2_ space There exists U, V is F_open in X and ,$U \cap V \neq \emptyset$, $[x \in U \wedge y \in V]$ then $U \cap W \wedge V \cap W$ are F_open in W(by theorem 2.19). $(U \cap W) \cap (U \cap W) = (U \cap V) \cap W = \emptyset \cap W = \emptyset$, and $(x \in U \cap W \wedge y \in V \cap W)$, So (W, τ_w) is FT_2_ space.

References

[1] R. Engelking. General Topology. PWN, Warszawa, 1977.

[2] C. Kuratowski. Topology I. 4th. ed., in French, Hafner, New york, 1958

[3] L. Steen and J. Seebach. GCounterexamples in Topology. Dover Publications, INC, 1995

[4] M. H. Alqahtani "F-open and F-closed Sets in Topological Spaces"European Journal of Pure And Applied Mathematics, Vol. 16, No . 2, 2023, 819-832.

[5] H.M.Darwesh, N.O.Hessean, S_g —open Sets in Topological Spaces, JZS(2015) 17 -1(Part-A).

[6] M. H. Alqahtani "C-open Sets on Topological Spaces ,arXiv:2305.03166(math)on 4 May 2023.