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Coefficients Estimates for Certain New Subclasses of Analytic Bi-Univalent Functions

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ABSTRACT

This study introduces two new subclasses $\mathcal{T}_{\Sigma}^{\varrho}(\tau,\lambda,\eta,\gamma;\alpha)$ and $\mathcal{T}_{\Sigma}^{\varrho}(\tau,\lambda,\eta,\gamma;\beta)$, of the function class Σ. These subclasses are defined in the open unit disc and consist of analytic bi-univalent functions. Moreover, in these newly created subclasses, we get approximations for the coefficients $|a_2|$ and $|a_3|$ in the functions. Additional findings have been acquired.

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1. Introduction

Let A, which consists of all normalized analytic functions f in an open unit disk U defined as $U=\{z: z\in\mathbb{C}, |z|<1\}$, and may be expressed in the following form:

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n,\ (z\in U). \tag{1.1}$$

A function f has an inverse f^{-1} is fulfills :

$$f^{-1}(f(z))=z,(z\in U),$$

with

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$$f\left(f^{-1}(w)\right) = w, \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots, (w \in U). \tag{1.2}$$

A function f is said to be bi-univalent in U if both its inverse function f^{-1} and f itself are univalent functions in U. The set of bi-univalent functions defined in the domain U is represented by the symbol Σ [13]. Lewin [13] examined the bi-univalent function class Σ and demonstrated that $|a_2| < 1.51$ for the functions in this class. Following this, Brannan and Clunie [8] hypothesis that $|a_2| < \sqrt{2}$. Netanyahu [14], however, demonstrated that $\max_{f\in\Sigma}|a_2| = \frac{4}{3}$. Additional research is required to address the issue of estimating the coefficient for each value of $|a_n|$ ($n \in \mathrm{N}\backslash\{1,2\}$; $\mathrm{N} \coloneqq \{1,2,\cdots\}$). Brannan and Taha [9] developed several subclasses of the bi-univalent function class Σ. These subclasses can be compared to the well-known subclasses $S^*(\alpha)$ and $K(\alpha)$ of starlike and convex functions with order $\alpha(0 < \alpha \leq 1)$, respectively (see to [1,2,3,4,6,7,22]). Hence, according to the research conducted by Brannan and Taha [9] (also cited in [5,10,17,21]), a function $f \in A$ is classified as a member of the $S^*(\alpha)$ class of extremely bi-starlike functions with order $\alpha(0 < \alpha \leq 1)$ if it satisfies the following criteria:

$$f \in \Sigma \text{ and } \left|arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}$$

and

$$\left|arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2},$$

here, g represents the extension of the inverse function of f to U. Similarly, the function classes $S_\Sigma^*(\alpha)$ and $K_\Sigma(\alpha)$ were defined, along with the classes $S_\Sigma^*(\alpha)$ and K_Σ of bi-starlike functions of order α, respectively.

The first two Taylor-Maclaurin coefficients, $|a_2|$ and $|a_3|$, were calculated for the function classes $S_\Sigma^*(\alpha)$ and $K_\Sigma(\alpha)$ using several sources [9], [16], [18], and [21] . To get further information, kindly refer to the provided references.

Based on the research conducted by Atshan et al. [2,20], Srivastava et al. [19], Frasin and Aouf [12], and Agnes's earlier studies [15], we were motivated to pursue this work.

The primary objective of this study is to define two new subclasses within the function class Σ, $\mathcal{T}_\Sigma^\varrho(\tau,\lambda,\eta,\gamma;\alpha)$ and $\mathcal{T}_\Sigma^\varrho(\tau,\lambda,\eta,\gamma;\beta)$, and we aim to determine the values of the coefficients $|a_2|$ and $|a_3|$ for functions inside these newly defined subclasses, utilizing the methodologies previously utilized by Srivastava et al. [19]. We further expand and enhance the outcomes of Srivastava et al. [19] and Frasin and Aouf [12] that were previously mentioned.

We also mention several new or known particular examples of our findings.

Before deriving our important findings, it is imperative to employ the subsequent lemma.

Lemma 1.1. [7,11] If $p \in \mathcal{P}$, then $|c_j| \leq 2$ for each j, where $\mathcal{P}$ is the family of every functions p analytic in U for which $Rep(z) > 0, p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots$ for $z \in U$.

2. Coefficients Bounds for the Function Class $\mathcal{T}_\Sigma^\varrho(\tau,\lambda,\eta,\gamma;\alpha)$

Definition 2.1. A function f, which is a member of the class Σ and defined by equation (1), is considered to belong to the class $\mathcal{T}_\Sigma^\varrho(\tau,\lambda,\eta,\gamma;\alpha)$ if it fulfills the following conditions:

$$\left|arg\left(1 + \frac{1}{\tau}\left[(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) + \frac{\eta z f''(z)}{(1-\gamma)(f'(z))^\varrho} - 1\right]\right)\right| < \frac{\alpha\pi}{2}, \quad (z \in U) \tag{2.1}$$

and

$$\left|arg\left(1 + \frac{1}{\tau}\left[(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) + \frac{\eta w g''(w)}{(1-\gamma)(g'(w))^\varrho} - 1\right]\right)\right| < \frac{\alpha\pi}{2}, \quad (z \in U), \tag{2.2}$$

where ($\tau \in \mathbb{C} \setminus \{0\};\ \lambda \geq 1; 0 \leq \eta \leq 1;\ 0 \leq \gamma < 1;\ \varrho \geq 0; 0 < \alpha \leq 1$). The function g is defined by equation (1.2).

Theorem 2.1. If the function f is defined by the Taylor-Maclaurin series expansion (1.1) since it is a member of the class $\mathcal{T}_{\Sigma}^{\varrho}(\tau,\lambda,\eta,\gamma;\alpha)$, then

$$|a_2| \leq \frac{2\alpha\tau(1-\gamma)}{\sqrt{\gamma[\tau\alpha A + 4\eta(2\varrho\alpha - 3\alpha) + \alpha(\lambda B + 2\eta(2\lambda+1) + 2 - \gamma) - 2(1+\lambda)^2 + \lambda D + \gamma - 2\eta] + \eta[\alpha C + 4E] + \alpha F + (\lambda+1)^2}}, \quad (2.3)$$

where

$$A = -8\lambda - 4 + 4\gamma\lambda + 2\gamma - 12\eta + 8\eta\varrho; B = 2\lambda + 4 - \gamma\lambda - 2\gamma;$$

$$C = 12\tau - 8\varrho\tau - 4\lambda - 4 - 4\eta;$$

$$D = \gamma\lambda + 2\gamma - 4\eta; E = \lambda + \eta + 1; F = 2\tau(1+2\lambda) - (1+\lambda)^2$$

and

$$|a_3| \leq \frac{4\tau^2\alpha^2(1-\gamma)^2}{\big((1+\lambda)(1-\gamma)+2\eta\big)^2} + \frac{2\tau\alpha(1-\gamma)}{\big((1+2\lambda)(1-\gamma)+6\eta\big)}. \quad (2.4)$$

Proof: Let $f \in \mathcal{T}_{\Sigma}^{\varrho}(\tau,\lambda,\eta,\gamma;\alpha)$ and $g = f^{-1}$, satisfying

$$1 + \frac{1}{\tau}\left[(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) + \frac{\eta z f''(z)}{(1-\gamma)(f'(z))^{\varrho}} - 1\right] = [p(z)]^{\alpha} \quad (2.5)$$

and

$$1 + \frac{1}{\tau}\left[(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) + \frac{\eta w g''(w)}{(1-\gamma)(g'(w))^{\varrho}}\right] - 1 = [q(w)]^{\alpha}, \quad (2.6)$$

then $p(z)$ and $q(w)$ are analytic functions in U, and both $p(0)$ and $q(0)$ are equal to 1,

where $p(z)$ and $q(w)$ be elements of $\mathcal{P}$ and have the following series representations:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \quad (2.7)$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots. \quad (2.8)$$

By equating the coefficients in equation (2.5) and equation (2.6), we obtain

$$\left(\frac{(1-\gamma)(1+\lambda)+2\eta}{\tau(1-\gamma)}\right)a_2 = \alpha p_1, \quad (2.9)$$

$$\left(\frac{(1-\gamma)(1+2\lambda)+6\eta}{\tau(1-\gamma)}\right)a_3 - \frac{4\eta\varrho}{\tau(1-\gamma)}a_2^2 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2, \quad (2.10)$$

$$-\left(\frac{(1+\lambda)(1-\gamma)+2\eta}{\tau(1-\gamma)}\right)a_2 = \alpha q_1, \quad (2.11)$$

and

$$\left(\frac{2(2\lambda+1)(1-\gamma)+4\eta(3-\varrho)}{\tau(1-\gamma)}\right)a_2^2 - \left(\frac{(2\lambda+1)(1-\gamma)+6\eta}{\tau(1-\gamma)}\right)a_3 = \frac{\alpha(\alpha-1)}{2}q_1^2 + \alpha q_2. \quad (2.12)$$

By applying equations (2.9) and (2.11), it can be deduced that

$$p_1 = -q_1, \tag{2.13}$$

and

$$2\left(\frac{2\eta + (1+\lambda)(1-\gamma)}{\tau(1-\gamma)}\right)^2 a_2^2 = \alpha^2(p_1^2 + q_1^2). \tag{2.14}$$

Now adding (2.10) and (2.12) and using (2.14), we obtain

$$\left(\frac{2(1-\gamma)(2\lambda+11) + 4\eta(3-\varrho) - 4\eta\varrho}{\tau(1-\gamma)}\right) a_2^2 = \alpha(p_2 + q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2 + q_1^2)$$

$$= \alpha(p_2 + q_2) + \frac{(\alpha-1)}{\alpha}\left(\frac{(1-\gamma)(1+\lambda) + 2\eta}{\tau(1-\gamma)}\right)^2 a_2^2.$$

$$a_2^2 = \frac{\tau^2(1-\gamma)^2\alpha^2(p_2+q_2)}{\gamma[\tau\alpha A + 4\eta(2\varrho\alpha - 3\alpha) + \alpha(\lambda B + 2\eta(2\lambda+1) + 2 - \gamma) - 2(1+\lambda)^2 + \lambda D + \gamma - 2\eta] + \eta[\alpha C + 4E] + 2\alpha\tau(2\lambda+1)}.$$

By employing lemma 1 to the coefficients p_2 and q_2, we can readily deduce

$$|a_2| \leq \frac{4\alpha\tau(1-\gamma)}{\sqrt{\gamma\left[\tau\alpha A + 4\eta(2\varrho\alpha - 3\alpha) + \alpha\left(\lambda B + 2 - \gamma + 2\eta(2\lambda+1)\right) - 2(\lambda+1)^2 + \lambda D + \gamma - 2\eta\right] + \eta[\alpha C + 4E] + \alpha F + (\lambda+1)^2}},$$

where

$$A = -8\lambda - 4 + 4\gamma\lambda + 2\gamma - 12\eta + 8\eta\varrho;\ B = 2\lambda + 4 - \gamma\lambda - 2\gamma;$$

$$C = 12\tau - 8\varrho\tau - 4\lambda - 4 - 4\eta;\ D = \gamma\lambda + 2\gamma - 4\eta;\ E = \lambda + \eta + 1;$$

$$F = 2\tau(2\lambda+1) - (\lambda+1)^2.$$

Subsequently, to get the upper limit of $|a_3|$, we can obtain the result by subtracting equation (2.12) from equation(2.10).

$$2\left(\frac{(1-\gamma)(2\lambda+1) + 6\eta}{\tau(1-\gamma)}\right) a_3 - \left(\frac{4\eta\varrho + 2(1-\gamma)(2\lambda+1) + 4\eta(3-\varrho)}{\tau(1-\gamma)}\right) a_2^2$$

$$= \alpha(p_2 - q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2 - q_1^2). \tag{2.15}$$

It follows from (2.13), (2.14) and (2.15) that

$$a_3 = \frac{\tau^2\alpha^2(p_1^2 + p_2^2)(1-\gamma)^2}{2\left((\lambda+1)(1-\gamma) + 2\eta\right)^2} + \frac{\tau\alpha(p_2 - q_2)(1-\gamma)}{2\left((2\lambda+1)(1-\gamma) + 6\eta\right)}.$$

By reapplying lemma 1 to the coefficients p_1, p_2, q_1 and q_2, we can achieve this result.

$$|a_3| \leq \frac{4\tau^2\alpha^2(1-\gamma)^2}{\left((\lambda+1)(1-\gamma) + 2\eta\right)^2} + \frac{2\tau\alpha(1-\gamma)}{(2\lambda+1)(1-\gamma) + 6\eta}.$$

The proof of Theorem 2.1 has been completed.

3. Coefficients Bounds for the Function Class $\mathcal{T}_{\Sigma}^{\varrho}(\tau,\lambda,\eta,\gamma;\beta)$

Definition 3.1. A function $f\in\Sigma$ is considered to belong to the class $\mathcal{T}_{\Sigma}^{\varrho}(\tau,\lambda,\eta,\gamma;\beta)$ if it satisfies the following criteria:

$$Re\left(1+\frac{1}{\tau}\left[(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)+\frac{\eta z f''(z)}{(1-\gamma)(f'(z))^{\varrho}}-1\right]\right)>\beta,\quad (z\in U) \tag{3.1}$$

and

$$Re\left(1+\frac{1}{\tau}\left[(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)+\frac{\eta w g''(w)}{(1-\gamma)(g'(w))^{\varrho}}-1\right]\right)>\beta,\quad (w\in U), \tag{3.2}$$

where ($\tau\in\mathbb{C}\setminus\{0\};\ \lambda\geq 1;0\leq\eta\leq 1;\ 0\leq\gamma<1;\ \varrho\geq 0;\ 0\leq\beta<1$). The function g is defined by equation (1.2).

Theorem 3.1. If the function f is defined by equation (1.1) and belongs to the class $\mathcal{T}_{\Sigma}^{\varrho}(\tau,\lambda,\eta,\gamma;\beta)$, then

$$|a_2|\leq\sqrt{\frac{2\tau(1-\beta)(1-\gamma)}{(1-\gamma)(2\lambda+1)+\eta(6-4\varrho)}} \tag{3.3}$$

and

$$|a_3|\leq\frac{4\tau^2(1-\beta)^2(1-\gamma)^2}{\big(2\eta+(\lambda+1)(1-\gamma)\big)^2}+\frac{2\tau(1-\beta)(1-\gamma)}{(1-\gamma)(2\lambda+1)+6\eta}. \tag{3.4}$$

Proof. From equations (3.1) and (3.2), it can be deduced that there exist p and q, which belong to $\mathcal{P}$, such that

$$1+\frac{1}{\tau}\left[(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)+\frac{\eta z f''(z)}{(1-\gamma)(f'(z))^{\varrho}}-1\right]=\beta+(1-\beta)p(z), \tag{3.5}$$

and

$$1+\frac{1}{\tau}\left[(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)+\frac{\eta w g''(w)}{(1-\gamma)(g'(w))^{\varrho}}-1\right]=\beta+(1-\beta)q(w), \tag{3.6}$$

$p(z)$ and $q(w)$ are represented by equations (2.7) and (2.8), respectively. By equating the coefficients in equation (3.5) and equation (3.6), we obtain

$$\left(\frac{(1-\gamma)(1+\lambda)+2\eta}{\tau(1-\gamma)}\right)a_2=(1-\beta)p_1, \tag{3.7}$$

$$\left(\frac{(1-\gamma)(2\lambda+1)+6\eta}{\tau(1-\gamma)}\right)a_3-\frac{4\eta\varrho}{\tau(1-\gamma)}a_2^2=(1-\beta)p_2, \tag{3.8}$$

$$-\left(\frac{(1-\gamma)(1+\lambda)+2\eta}{\tau(1-\gamma)}\right)a_2=(1-\beta)q_1, \tag{3.9}$$

and

$$\left(\frac{2(1-\gamma)(2\lambda+1)+4\eta(3-\varrho)}{\tau(1-\gamma)}\right)a_2^2-\left(\frac{(1-\gamma)(2\lambda+1)+6\eta}{\tau(1-\gamma\gamma)}\right)a_3=(1-\beta)q_2. \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_1=-q_1, \tag{3.11}$$

and

$$2\left(\frac{(1-\gamma)(1+\lambda)+2\eta}{\tau(1-\gamma)}\right)^2 a_2^2=(1-\beta)^2(p_1^2+q_1^2). \tag{3.12}$$

By summing together equations (3.8) and (3.10), we get

$$\left(\frac{2(1-\gamma)(2\lambda+1)+4\eta(3-\varrho)-4\eta\varrho}{\tau(1-\gamma)}\right)a_2^2=(1-\beta)(p_2+q_2).$$

Therefore, we obtain

$$a_2^2=\frac{\tau(1-\beta)(p_2+q_2)(1-\gamma)}{2(1-\gamma)(2\lambda+1)+2\eta(6-4\varrho)}.$$

By utilising lemma (1.1) for the coefficients p_2 and q_2, we obtain

$$|a_2|\le\sqrt{\frac{2\tau(1-\beta)(1-\gamma)}{(1-\gamma)(2\lambda+1)+\eta(6-4\varrho)}}.$$

Now, to find $|a_3|$, by subtracting (3.10) from (3.8), we get

$$2\left(\frac{(1-\gamma)(2\lambda+1)+6\eta}{\tau(1-\gamma)}\right)a_3-\left(\frac{2(1-\gamma)(2\lambda+1)+12\eta}{\tau(1-\gamma)}\right)a_2^2=(1-\beta)(p_2-q_2).$$

Or equivalently

$$a_3=\frac{\tau(1-\beta)(p_2-q_2)(1-\gamma)}{2\big((1-\gamma)(2\lambda+1)+6\eta\big)}+a_2^2.$$

By replacing the value of a_2^2 from equation (3.12), we obtain

$$a_3=\frac{\iota^2(1-\beta)^2(p_1^2+q_1^2)(1-\gamma)^2}{2\big((1-\gamma)(\lambda+1)+2\eta\big)^2}+\frac{\tau(1-\beta)(p_2-q_2)(1-\gamma)}{2\big((1-\gamma)(2\lambda+1)+6\eta\big)}.$$

By using Lemma (1.1) once more to the coefficients p_1, p_2, q_1 *and* q_2, we determine

$$|a_3|\le\frac{4\tau^2(1-\beta)^2(1-\gamma)^2}{\big(2\eta+(\lambda+1)(1-\gamma)\big)^2}+\frac{2\tau(1-\beta)(1-\gamma)}{(2\lambda+1)(1-\gamma)+6\eta}.$$

The proof of Theorem 3.1 has been completed.

4. Corollaries and Consequences

By substituting $\lambda=1$ into Theorem 2.1, we obtain

Corollary 4.1. Let $f(z)$ defined by equation (1.1) belongs to the class $\mathcal{T}_\Sigma^\varrho(\tau,\lambda,\eta,\gamma;\alpha)$. Then

$$|a_2|\le\frac{4\alpha\tau(1-\gamma)}{\sqrt{\gamma[\tau\alpha A+4\eta(2\varrho\alpha-3\alpha)+\alpha(B+6\eta+2-\gamma)-8+D+\gamma-2\eta]+\eta[\alpha C+4E]+6\alpha\tau}},$$

where

$$A=-12+6\gamma-12\eta+8\eta\varrho;\,B=6-3\gamma;\,C=12\tau-8\varrho\tau-8-4\eta;\,D=3\gamma-4\eta;\,E=\eta+2$$

and

$$|a_3| \leq \frac{4\tau^2\alpha^2(1-\gamma)^2}{(2(1-\gamma)+2\eta)^2} + \frac{2\tau\alpha(1-\gamma)}{3(1-\gamma)+6\eta}.$$

By substituting $\lambda = 1$ into Theorem 3.1, we obtain

Corollary 4.2. Let $f(z)$ defined by equation (1.1), which belongs to the class $\mathcal{T}_{\Sigma}^{\varrho}(\tau, \lambda, \eta, \gamma; \beta)$. then

$$|a_2| \leq \sqrt{\frac{2\tau(1-\beta)(1-\gamma)}{3(1-\gamma)+\eta(6-4\varrho)}}$$

and

$$|a_3| \leq \frac{4\tau^2(1-\beta)^2(1-\gamma)^2}{\left(2\eta+2(1-\gamma)\right)^2} + \frac{2\tau(1-\beta)(1-\gamma)}{6\eta+3(1-\gamma)}.$$

Remark . By substituting $\gamma = 0$ and $\varrho = 0$ into theorem 2.1 and theorem 3.1 for symmetric bi-univalent functions, we may get the same findings as those presented by Atshan and Jiben [1]. Furthermore, by substituting $\tau = 1, \eta = 0$ and $\gamma = 0$ into theorem (2.1) and theorem (3.1), we can get the outcomes presented by Frasin and Aouf [12].

References

[1] W. G. Atshan and N. A. Jiben, Coefficients bounds for a general subclasses of m-fold symmetric bi-univalent functions, J. Al-Qadisiyah comput. Sci. Math., 9(2) (2017), 33-39.

[2] S. A. Al-Ammeedee, W. G. Atshan and F. A. Al-Maamori, Coefficients estimates of bi-univalent functions defined by new subclasses functions, J. Phys.: Conf.Ser. 1530(2020), 012105.

[3] W. G. Atshan, E. I. Badawi, Results on coefficients estimates for subclasses of analytic and bi-univalent functions, J. Phys. Conf. Ser., (2019), 1294, 032025, 1-9.

[4] W. G. Atshan, I. A. R. Rahman, A. A. Lupas, Some results of new subclasses for bi-univalent functions using quasi-subordination, Symmetry, (2021), 13(9), 1653, 1-12.

[5] W. G. Atshan, R. A. Al-Sajjad, S. Altinkaya, On the Hankel determinant of m-fold symmetric bi-univalent functions using a new operator, Gazi Univ. J. Sci., 36(1) (2023), 349-360.

[6] E. I. Badiwi, W. G. Atshan, A. N. Alkiffai and A. A. Lupas, Certain results on subclasses of analytic and bi-univalent functions associated with coefficient estimates and quasi-subordination, Symmetry, 15(12) (2023), 2208, 1-12

[7] D. A. Brannan, J. Clunie and W. E. Kirwan, Coefficient estimates for a class of starlike functions, Canad. J. Math., 22 (1970), 476-485.

[8] D. A. Brannan, J. G. Clunie (Eds), Aspects of Contemporary Complex Analysis (Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham; July 1 20, 1979), Academic Press, New York and London, 1980.

[9] D. A. Brannan, T. S. Taha, On some classes of bi-univalent functions, in: S. M. Mazhar, A. Hamoui, N. S. Faour (Eds), Math. Anal. And Appl., Kuwait; February 18-21, 1985, in: KFAS Proceedings Series, vol. 3, Pergamon Press, Elsevier Science Limited, Oxford, 1988, PP. 53-60. See also Studia Univ. Babe, s-Bolyai Math. 31(2), (1986), 70-77.

[10] Darweesh, A. M.; Atshan, W. G.; Battor, A. H.; Mahdi, M. S. On the third Hankel determinant of certain subclass of bi-univalent functions. Math. Model. Eng. Probl., (2023), 10, 1087-1095.

[11] P. L. Duren, Univalent Functions, In: Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Hidelberg and Tokyo, (1983).

[12] B. A. Frasin, M. K. Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett., 24 (2011), 1569-1573.

[13] M. Lewin, On a coefficient problem for bi-univalent function, Proceedings of the American Mathematical Society, Vol. 18 (1967), 63-68.

[14] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Arch. Rational Mech., 32 (1969), 100-112.

[15] ÁO Páll-Szabo, Gl Oros, Coefficient related studies for new classes of bi-univalent functions, mathematics, 2020.

[16] I. A. R. Rahman, W. G. Atshan, G. I. Oros, New concept on fourth Hankel determinant of a certain subclass of analytic functions. Afr. Mat., (2022), 33, 7. 1-15.

[17] P. O. Sabir, H. M. Srivastava, W. G. Atshan, P. O. Mohammed, N. Chorfi and M. V. Cortez, A family of holomorphic and m-fold symmetric bi-univalent functions endowed with coefficient estimate problems, Mathematics, 11(18) (2023), 3970, 1-13.

[18] Shakir, Q.A.; Atshan, W.G. On third Hankel determinant for certain subclass of bi-univalent functions. Symmetry 2024, 16, 239.

[19] H. M. Srivastava, A. K. Mishra, P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett., 23 (2010), 1188-1192.

[20] F. O. Salman, W. G. Atshan, New results on coefficient estimates for subclasses of bi-univalent functions related by a new integral operator,Int. J. Nonlinear Anal. Appl. 14 (2023) 4, 47-54.

[21] T. S. Taha, Topics in univalent function Theory, Ph.D. Thesis, University of London, 1981.

[22] S. Yalcin, W. G. Atshan, H. Z. Hassan, Coefficients assessment for certain subclasses of bi-univalent functions related with quasi-subordination. Publ. L'Institut Math. Nouv. Sér., (2020), 108, 155-162.