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# On $(K^*\text{-}N)^n$-Quasi Normal Operators

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ABSTRACT

The objective is to present a novel variant of a quasi-normal operator, specifically the $(\boldsymbol{K}^* - \boldsymbol{N})^{\boldsymbol{n}}$ quasi normal operator, alongside the introduction of related theorems, propositions, and illustrative examples elucidating this concept. Additionally, we present the necessary and sufficient conditions for addition and multiplication of this kind of operators.

MSC..

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## 1. Introduction

A.Brow [1] introduced and researched the quasi-normal operator for the first time in 1953. Later Arun B. (1976) [2] provided several properties of the quasi-normal operator. Sh. Lohaj (2010) [3] introduced a novel variation of the quasi-normal operator, namely, N-quasi normal operators, along with presenting some of its fundamental characteristics. Ould A. (2011) submitted the n-power quasi-normal operator along with outlining its characteristics[4]. Valdete R.H. (2013) [5] provided several characteristics of N-quasi normal operators. Saad S. and Laith K (2015) [6] introduced the concept of the quasi-normal operator along with presenting some fundamental properties of this concept. Ahmed M and Salim D. (2017) [7] introduced another type called the $(K - N)^*$ quasi-normal operator, along with presenting some properties.

## 2. Basics

**Definition (2.1)** [8]:-A bounded linear operator $T: \mathcal{K} \to \mathcal{K}$, is n operator satisfying $\|Tx\| \leq k\|x\| \ \forall x \in \mathcal{K}$.

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**Definition (2.2),** [8]**:-**A bounded linear operator $T: \mathcal{K} \to \mathcal{K}$ , is called normal if $TT^* = T^*T$.

**Definition (2.3),** [1]**:-**A bounded linear operator $T: \mathcal{K} \to \mathcal{K}$ is quasi-normal if $T(T^*T) = (T^*T)T$.

**Theorem (2.4),** [2]**:-**Le t$T_1, T_2: \mathcal{K} \to \mathcal{K}$be quasi normal operators, such that $T_2T_1 = T_1T_2 = T_1{}^*T_2 = T_2T_1{}^* = 0$ then $T_1 + T_2$is quasi normal.

**Definition (2.5),** [5]**:-**A bounded linear operator $T: \mathcal{K} \to \mathcal{K}$ be a bounded linear is N-quasi normal if $T(T^*T) = N(T^*T)T$.

**Theorem (2.6),** [5]**:-**Let $T_1: \mathcal{K} \to \mathcal{K}$be N-quasi normal and $T_2: \mathcal{K} \to \mathcal{K}$ be quasi normal then the product $T_1T_2$ is N-quasi normal if the they satisfy $T_1T_2 = T_2T_1$ and$T_2{}^*T_1 = T_1T_2^*$ .

**Definition (2.7),** [4]**:-**A bounded liner operator $T: \mathcal{K} \to \mathcal{K}$is said to be n-power quasi-normal operator if $T^n(T^*T) = (T^*T)T^n$

**Theorem (2.7),** [9]**:-**If $T: \mathcal{K} \to \mathcal{K}$ is n-power quasi-normal operator, then $\lambda T$is also a n-power quasi-normal operator, where$\lambda \in R$.

**Definition (2.8),** [10]**:-** Let $T: \mathcal{K} \to \mathcal{K}$ is a bounded liner operator is said to be (K-N) quasi-normal operator if $T^k(T^*T) = N(T^*T)T^k$.

**Theorem (2.9),** [10]**:-**Let $T_1$ be (K-N) quasi normal operator and $T_2$ is $k$-quasi normal operator, such that,$(T_1T_2) = (T_2T_1)$ and $T_1T_2^* = T_2^*T_1$, then $T_1T_2$is (K-N) quasi normal operator.

**Theorem (2.10),** [10]**:-**Let $T_1, T_2: \mathcal{K} \to \mathcal{K}$ be two (K-N) quasi normal operators such that$T_1^kT_2^* = T_2^kT_1^* = T_1^*T_2 = T_2^*T_1 = 0$ then $T_1 + T_2$is (k-N) quasi normal operator.

**Corollary (2.11),** [10]**:-** Let $T_1, T_2: \mathcal{K} \to \mathcal{K}$ be two (K-N) quasi normal operators, such that$T_1^kT_2^* = T_2^kT_1^* = T_1^*T_2 = T_2^*T_1 = 0$ then $T_1 - T_2$is (K-N) quasi normal operator.

## 3. Main results

**Definition (3,1) -**Let $T: \mathcal{K} \to \mathcal{K}$ be a bounded linear operator then $T$ is said to be$(K^* - N)^{\mathrm{n}}$quasi normal operator ($(K^* - N)^{\mathrm{n}}$-QNO) if $(T^*)^K(T^*T)^n = N(T^*T)^n(T^*)^K$.

**Example (3.2)-**

$T = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ is$(K^* - N)^{\mathrm{n}}$-QNO where $k = 1, n = 1, N = I$

The following result hold for $\lambda \in \mathbb{R}$ but not a complex value of $\lambda$

**Propositions (3.3):** Let $T: \mathcal{K} \to \mathcal{K}$is $(K^* - N)^n$ -QNO then. $\lambda T$ is $(K^* - N)^n$-QNO where $\lambda \in R$.

Proof:

$$((\lambda T)^*)^K\big((\lambda T)^*(\lambda T)\big)^n = (\lambda T^*)^K\big((\lambda T^*)(\lambda T)\big)^n$$

$$= \lambda^K(T^*)^K(\lambda T^*)^n(\lambda T)^n$$

$$= \lambda^K(T^*)^K\lambda^n(T^*)^n\lambda^nT^n$$

$$= \lambda^{K+2n}(T^*)^K(T^*)^nT^n$$

$$= \lambda^{K+2n}N(T^*)^nT^n(T^*)^K$$

$$= N\lambda^n (T^*)^n \lambda^n T^n \lambda^K (T^*)^K$$

$$= N(\lambda T^*)^n (\lambda T)^n (\lambda T^*)^K$$

$$= N\big((\lambda T)^*(\lambda T)\big)^n ((\lambda T)^*)^K$$

$((\lambda T)^*)^K (\lambda T)^* (\lambda T)^n = N(\lambda T)^* (\lambda T)^n ((\lambda T)^*)^K$, hence the $\lambda T$ is $(K^* - N)^n$ – QNO■

**Propositions (3.4):** Let $T: \mathcal{K} \to \mathcal{K}$ is $(K^* - N)^n$ -QNO then. $T/_M$ is $(K^* - N)^n$ -QNO where M is closed sub space.

$$\left(\left(T/_M\right)^*\right)^K \left(\left(T/_M\right)^* \left(T/_M\right)\right)^n = \left(\left(T^*/_M\right)\right)^K \left(\left(T^*/_M\right)\left(T/_M\right)\right)^n$$

$$= \left((T^*)^K/_M\right)\left(T^*T/_M\right)^n$$

$$= \left((T^*)^K/_M\right)\left((T^*T)^n/_M\right)$$

$$= \left(N(T^*T)^n (T^*)^K/_M\right)$$

$$= \left(N(T^*T)^n/_M\right)\left((T^*)^K/_M\right)$$

$$= \left(N\left(\left(T^*/_M\right)\left(T/_M\right)\right)^n\right)\left(\left(T^*/_M\right)\right)^K$$

$$= N\left(\left(T/_M\right)^*\left(T/_M\right)\right)^n \left(\left(T/_M\right)^*\right)^K$$

Then $T/_M$ is $(K^* - N)^n$-QNO ■

**Remark (3.5):** Let $T_1$ and $T_2$ be two $(K^* - N)^n$-QNO then $T_1 + T_2$ is not necessary $(K^* - N)^n$-QNO. To illustrate that consider the following example.

**Example (3.6)-**

$$T_1 = \begin{bmatrix} 2i & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2i \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ i & 0 & 1 \end{bmatrix}$$

$$T_1 + T_2 = \begin{bmatrix} 1+2i & 0 & 2+i \\ 0 & 1 & 0 \\ 2 & 0 & -2i \end{bmatrix}$$

If $n = 1, K = 2$ and $N = I$

$$((T_1 + T_2)^*)^2 \big((T_1 + T_2)^*(T_1 + T_2)\big)^1 = \begin{bmatrix} 32-56i & 0 & -24-52i \\ 0 & 1 & 0 \\ 40-20i & 0 & -40i \end{bmatrix} \ldots (1)$$

$$\big((T_1 + T_2)^*(T_1 + T_2)\big)^1 ((T_1 + T_2)^*)^2 = \begin{bmatrix} -120i & 0 & 40-20i \\ 0 & 1 & 0 \\ 104-52i & 0 & 32+24i \end{bmatrix} \ldots (2)$$

$$(1) \neq (2)$$

Then $T_1 + T_2$ is not $(K^* - N)^n$ -QNO

**Theorem (3.7)** Let $T_1, T_2 : \mathcal{K} \to \mathcal{K}$ be $(K^* - N)^n$-QNO on a Hilbert space s.t. $T_1^* T_2^* = T_1 T_2 = T_1^* T_2 = 0$ then $T_1 + T_2 (K^* - N)^n$ -QNO.

Proof:-

$$\left(((T_1+T_2)^*)^K\left((T_1+T_2)^*(T_1+T_2)\right)^n\right) = \left((T_1^*+T_2^*)^K\left((T_1^*+T_2^*)(T_1+T_2)\right)^n\right)$$

$$= ((T_1^*)^K + K(T_1^*)^{K-1}T_2^* + \cdots + (T_2^*)^K)\left((T_1^* + n(T_1^*)^{n-1}T_2^* + \cdots + (T_2^*)^n)(T_1^n + nT_1^{n-1}T_2 + \cdots + T_2^n)\right)$$
$$= ((T_1^*)^K + (T_2^*)^K)\left(((T_1^*)^n + (T_2^*)^n)(T_1^n + T_2^n)\right)$$

$$= ((T_1^*)^K + (T_2^*)^K)((T_1^*)^n T_1^n + (T_1^*)^n T_2^n + (T_2^*)^n T_1^n + (T_2^*)^n T_2^n)$$

$$= ((T_1^*)^K + (T_2^*)^K)((T_1^*)^n T_1^n + (T_2^*)^n T_2^n)$$

$$= ((T_1^*)^K (T_1^*)^n T_1^n + (T_1^*)^K (T_2^*)^n T_2^n + (T_2^*)^K (T_1^*)^n T_1^n + (T_2^*)^K (T_2^*)^n T_2^n)$$

$$= ((T_1^*)^K T_1^* T_1^n + (T_2^*)^K T_2^* T_2^n)$$

Since $T_1$ and $T_2$ are $(K^* - N)^n$-QNO

$$= N((T_1^*)^n T_1^n (T_1^*)^K) + N((T_2^*)^n T_2^n (T_2^*)^K)$$

$$= N\left(((T_1^*)^n T_1^n (T_1^*)^K) + ((T_2^*)^n T_2^n (T_2^*)^K)\right)$$

Therefore; $T_1 + T_2$ is $(K^* - N)^n$-QNO ■

**Remark (3.8)**

Let $T_1$ and $T_2$ be two $(K^* - N)^n$-QNOs then $T_1 T_2$ is not necessarily a $(K^* - N)^n$ -QNO as shown below.

**Example (3.9)**

$$T_1 = \begin{bmatrix} 2i & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2i \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ i & 0 & 1 \end{bmatrix}$$

$$T_1 T_2 = \begin{bmatrix} 4i & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}$$

If $n = 1, K = 2$ and $N = I$

$$((T_1T_2)^*)^2\left((T_1T_2)^*(T_1T_2)\right)^1 = \begin{bmatrix} -512 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ldots (1)$$

$$\left((T_1T_2)^*(T_1T_2)\right)^1((T_1T_2)^*)^2 = \begin{bmatrix} -512 & 0 & -512i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ldots (2)$$

$$(1) \neq (2)$$

Then $T_1 T_2$ is not $(K^* - N)^n$ -QNO

**Theorem (3.10)**

Let $T_1$ *and* $T_2 : \mathcal{K} \to \mathcal{K}$ be $(K^* - N)^n$ -QNO and $K^*$ quasi normal operator respectively such that, $T_1 T_2 = T_2 T_1$ and $T_2{}^* T_1 = T_1 T_2^*$, then $T_1 T_2$ is $(K^* - N)^n$ -QNOs.

Proof:-

$$((T_1T_2)^*)^K\big((T_1T_2)^*(T_1T_2)\big)^n = ((T_2T_1)^*)^K((T_2^*T_1^*)^n(T_1T_2)^n)$$

$$= \big((T_1^*T_2^*)\big)^K((T_1^*T_2^*)^n(T_1T_2)^n)$$

$$= ((T_1^*)^K(T_2^*)^K)\big(((T_1^*)^n(T_2^*)^n)(T_1^nT_2^n)\big)$$

$$= (T_1^*)^K((T_2^*)^K(T_1^*)^n)((T_2^*)^nT_1^n)T_2^n$$

$$= (T_1^*)^K((T_1^*)^n(T_2^*)^K)(T_1^n(T_2^*)^n)T_2^n$$

$$= ((T_1^*)^K(T_1^*)^n)((T_2^*)^KT_1^n)((T_2^*)^nT_2^n)$$

$$= ((T_1^*)^K(T_1^*)^n)(T_1^n(T_2^*)^K)((T_2^*)^nT_2^n)$$

$$= ((T_1^*)^K(T_1^*)^nT_1^n)((T_2^*)^K(T_2^*)^nT_2^n)$$

$$= N(T_1^*T_1^n(T_1^*)^K)((T_1^*)^nT_2^n(T_2^*)^K)$$

$$= N((T_1^*)^nT_1^n)((T_1^*)^K(T_2^*)^n)(T_2^n(T_2^*)^K)$$

$$= N((T_1^*)^nT_1^n)((T_2^*)^n(T_1^*)^K)(T_2^n(T_2^*)^K)$$

$$= N(T_1^*)^n(T_1^n(T_2^*)^n)((T_1^*)^KT_2^n)(T_2^*)^K$$

$$= N(T_1^*)^n((T_2^*)^nT_1^n)(T_2^n(T_1^*)^K)(T_2^*)^K$$

$$= N(T_1^*T_2^*)^n(T_1^nT_2^n)((T_1^*)^K(T_2^*)^K)$$

$$= N((T_2T_1)^*)^n(T_1T_2)^n\big((T_1^*)(T_2^*)\big)^K$$

$$= N((T_2T_1)^*)^n(T_1T_2)^n((T_2T_1)^*)^K$$

$$= N((T_1T_2)^*)^n(T_1T_2)^n((T_1T_2)^*)^K$$

$$((T_1T_2)^*)^K((T_1T_2)^*)^n(T_1T_2)^n = N((T_1T_2)^*)^n(T_1T_2)^n((T_1T_2)^*)^K$$

Therefore; the product $T_1T_2$ is $(K^* - N)^n$ -QNO

## 4. Conclusion

An introduction to a novel quasi-normal operator is given, specifically the $(K^* - N)^n$ quasi-normal operator. This was accompanied by the presentation of associated theorems, propositions, and illustrative examples to clarify this concept. Furthermore, the necessary and sufficient conditions for both addition and multiplication of operators to be $(K^* - N)^n$ quasi-normal operator is given.

**References**

[1] Arlen Brown, "On a class of operators," *Amer. Math. Soc.*, vol. 4, pp. 723–728, 1953.

[2] ARUN BALA, " A note on quasi normal operators," *INDIAN J. PURE APPL. MATH*, vol. 8, no. 4, pp. 463–468, 1977.

[3] S. Lohaj, "Quasi-normal operators," *Int. Journal of Math. Analysis*, vol. 4, no. 47, pp. 2311–2320, 2010.

[4] O. A. M. S. Ahmed and M. S. Ahmed, "On the class of n-power quasi-normal operators on Hilbert space," *Bull. Math. Anal. Appl*, vol. 3, no. 2, pp. 213–228, 2011.

[5] V. R. Hamiti, "Some Properties of N-Quasinormal Operators," *Gen*, vol. 18, no. 1, pp. 94–98, 2013.

[6] Laith K. Shaakir and Saad S. Marai, "quasi-normal Operator of order n," *Tikrit Journal of Pure Science*, vol. 20, no. 4, pp. 167–169, 2015.

[7] S. D. Muhsin and A. M. Khalaf, "On The Class of (KN)* Quasi-N-Normal Operators on Hilbert Space," *Iraqi Journal of science*, pp. 2172–2176, 2017.

[8] J. N. Sharma and A. R. Vasishtha, *Functional Analysis*, 1st ed. Krishna Prakashan Media, 2013.

[9] T. Veluchamy and K. M. Manikandan, "n-Power quasi normal operators on the hilbert space," *IOSR Journal of Mathematics*, vol. 12, pp. 6–9, 2016.

[10] D. M. Salim and M. K. Ahmed, "On the class of (KN) quasi normal operator on Hilbert Space," *Mathematical theory and modeling*, vol. 5, no. 10, 2015.