The Completion of $\oplus$-measure

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1- Abstract

The theory of measure is an important subject in mathematics; in Ash [4,5] discusses many details about measure and proves some important results in measure theory.

In 1986, Dimiev [7] defined the operation addition and multiplication by real numbers on a set $E = (-\infty,1) \subset R$, he defined the operation multiplication on the set E and prove that E is a vector space over R and for any a>1 E_a is field, also he defined the fuzzifying functions on arbitrary set X.

In 1989, Dimiev [6] discussed the field E_a as in [7] and defined the operations addition, multiplication and multiplication by real number on a set of all fuzzifying functions defined on arbitrary set X, and also defined $\oplus$-measure on a measurable space and proved some results about it.

we mention the definition of the field E_a, and the fuzzifying functions on the arbitrary set X also we mention the definition of the operations.

Definition (1.1.1) [7]:

Let (R, +,.) be a field of real numbers with usual order and E= (-∞,1)⊆R, we introduce the operations addition $\oplus$ and scalar multiplication $\odot$ on the set E as follows:

For any $x, y \in E$ and $\lambda \in$ R we have

$x \oplus y = x + y - xy$, $\quad \lambda \odot x = 1-(1-x)^{\lambda}$.

Proposition (1.2) [7]:

The set E with the operations $\oplus$, $\odot$ and the relation order, represent ordered linear space.

Definition (1.3) [6]:

Let $a > 1$, we introduce an operation multiplication on the set E as follows

For any $x, y \in E$ we have $x \circ y = 1 - a^{-\log_a(1-x)\log_a(1-y)}$.

Proposition (1.4) [6]:

The set E with the operations $\oplus, \circ$ is a field which is denoted by E_a .

Remark (1.5):

Let $x, y \in E_a$, we denote $x \Theta y = x \oplus (-w) \circ y$ and $\Theta x = (-w) \circ x$ where $w = 1 - a^{-1}$ the unit element in the field E_a.

Definition (1.6)[6]:

Let X be arbitrary set, the map $f : X \rightarrow E_a$ is said to be E_a-valued fuzzifying function.

2- $\oplus$ - Measure:

In this section we mention the definition of $\oplus-$measure on a measurable space and proved some results about it, also we defined $\oplus-$outer measure and proved some results about it.

Definition (2.1)[5]:

A collection $\mathcal{F}$ of subsets of a set Ω is said to be:

a) σ-ring if

1- $\varphi \in \mathcal{F}$, where φ is empty set.

2- if A,B$\in \mathcal{F}$ then $A|B \in \mathcal{F}$.

3- if $\{A_n\}$ is a sequence of sets in $\mathcal{F}$ then $\overset{\infty}{\underset{n=1}{\cup}} A_n \in \mathcal{F}$.

b) σ-field (or σ-algebra) if

1- $\Omega \in \mathcal{F}$.

2- if A$\in \mathcal{F}$ then $A^c \in \mathcal{F}$.

3- if $\{A_n\}$ is a sequence of sets in $\mathcal{F}$ then $\overset{\infty}{\underset{n=1}{\cup}} A_n \in \mathcal{F}$. A measurable space is a pair $(\Omega, \mathcal{F})$ where Ω is a set and $\mathcal{F}$ is σ-ring or σ-field and a measurable set is a subset A of Ω such that A$\in \mathcal{F}$.

Definition (2.2) [6]:

Let $(\Omega, \mathcal{F})$ be a measurable space, a fuzzifying function $\mu : \mathcal{F} \rightarrow E_a$ is said to be:

1- $\oplus$-additive if $\mu(A \cup B) = \mu(A) \oplus \mu(B)$ for every disjoint sets A, B in $\mathcal{F}$.

2- Accountability $\oplus$-additive if $\mu(\bigcup_{n=1}^{\infty} A_n) = \oplus_{n=1}^{\infty} \mu(A_n)$ for every disjoint sequence $\{A_n\}$ of sets of $\mathcal{F}$.

3- $\oplus$- measure, if μ is accountability $\oplus$- additive and non-negative

The triple $(\Omega, \mathcal{F}, \mu)$ is called a space with $\oplus$-measure.

Theorem (2.3):

Let $(\Omega, \mathcal{F}, \mu)$ be a space with $\oplus$- measure and A, B $\in \mathcal{F}$ then:

1- $\mu(\varphi) = 0$.

2- $\mu(A) = \mu(A \cap B) \oplus \mu(A \cap B^C)$.

3- $\mu(A \cup B) \oplus \mu(A \cap B) = \mu(A) \oplus \mu(B)$.

4- if $A \subseteq B$ then:

(a) $\mu(B|A) = \mu(B) \oplus (-w) \circ \mu(A)$.

(b) $\mu(A) \leq \mu(B)$.

Proof:

1- Since A = $A\cup\varphi$ and $A\cap\varphi=\varphi$.

$$\mu(A)=\mu(A\cup\varphi)=\mu(A)\oplus\mu(\varphi).$$

Since E_a is a field $\Rightarrow \mu(\varphi)=0$.

2- Since $A=(A\cap B)\cup(A\cap B^c)$).

and ($A\cap B$)$\cap(A\cap B^c)=\varphi$.

$$\Rightarrow \mu(A)=\mu((A\cap B)\cup(A\cap B^c)).$$

$$=\mu(A\cap B\ )\oplus\mu(A\cap B^c).$$

3- Since $A\cup B=(A\cap B^c)\ \cup B$ and $(A\cap B^c)\cap B=\varphi$.

$$\Rightarrow \mu(A\cup B)=\mu(A\cap B^c))\cup B)$$

$$=\mu(A\cap B^c)\ \oplus\mu(B).$$

$$\mu(A\cup B)\oplus\mu(A\cap B)=(\mu(A\cap B^c)\ \oplus\mu(B))\oplus\mu(A\cap B).$$

$$=(\mu(A\cap B^c)\oplus\mu(A\cap B))\oplus\mu(B).$$

$$=\mu(A)\oplus\mu(B).$$

4- (a) Since $A\subseteq B\Rightarrow B=A\cup(B|A)$ and $A\cap(B|A)=\varphi$.

$$\mu(B)=\mu(A\cup(B|A)).$$

$$=\mu(A)\oplus\mu(B|A).$$

Since E_a is a field $\Rightarrow \mu(B|A)=\mu(B)\oplus(-w)\circ\mu(A)$.

(b) Since $\mu(B|A)\geq\circ$ from (a) we get that $\mu(A)\leq\mu(B)$.

Definition (2.4):

Let $(\Omega,\mathcal{F})$ be a measurable space and let the fuzzifying $\mu:\mathcal{F}\to E_a$ be a $\oplus-$additive, we say that μ is :

1. $\oplus-$continuous from below at A$\in\mathcal{F}$ if $\mu(A_n)\to\mu(A)$.

   For every non – decreasing sequence $\{A_n\}$ of sets in $\mathcal{F}$ which converge to A (i.e $A_n\uparrow A$).

2. $\oplus-$continuous from below at A$\in\mathcal{F}$ if $\mu(A_n)\to\mu(A)$.

   For every non- increasing sequence $\{A_n\}$ of sets in $\mathcal{F}$ converge to A (i.e $A_n\uparrow A$).

3. $\oplus-$continuous at A$\in\mathcal{F}$ if it is continuous at A from above and from below.

Theorem (2.5):

Let μ be $\oplus-$additive fuzzifying function on measurable space $(\Omega,\mathcal{F})$, then the following are valid.

1- If μ is countable $\oplus-$additive, then μ is $\oplus-$continuous at A for all A$\in\mathcal{F}$.

2- If μ is $\oplus-$continuous from below at every A$\in\mathcal{F}$, then μ is countable $\oplus-$additive.

3- If μ is continuous from above at φ then μ is countable $\oplus-$additive.

Proof:

1- Let $\{A_n\}$ be an increasing sequence of sets in $\mathcal{F}$ which converge to A, i.e $A_n \uparrow A$.

(a) Let $B_1 = A_1$, $B_n = A_n | A_{n-1}$ $\quad \forall n \geq 2$.

$\Rightarrow \quad B_n \cap B_m = \varphi, \forall n \neq m$ and $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n = A$.

$\mu(A) = \mu(\bigcup_{n=1}^{\infty} B_n) = \mu(A_1) \oplus (\bigoplus_{N=2}^{\infty} \mu(B_n))$.

$= \mu(A_1) \oplus (\bigoplus_{n=2}^{\infty} \mu(A_n | A_{n-1})$.

$\mu(A) = \mu(A_1) \oplus \lim_{K\to\infty} \bigoplus_{n=2}^{K} (\mu(A_n | A_{n-1}) = \lim_{K\to\infty} \mu(A_K)$.

$\Rightarrow \mu$ is $\oplus -$continuous from below at $A \in \mathcal{F}$.

(b) Suppose that $A_n \downarrow A \rightarrow A_1 | A_n \uparrow A_1 | A$.

$\Rightarrow \mu(A_1 | A_n) \rightarrow \mu(A_1 | A) \Rightarrow \mu(A_n) \rightarrow \mu(A)$.

So μ is $\oplus -$continuous from above at $A \in \mathcal{F}$.

From (a) and (b) we get that μ is $\oplus -$continuous at $A \in \mathcal{F}$.

2-Let $\{A_n\}$be a disjoint sequence of sets in $\mathcal{F}$, and $A = \bigcup_{n=1}^{\infty} A_n$.

Put $\quad B_n = \bigcup_{\varsigma=1}^{n} A_i \Rightarrow B_n \in F \Rightarrow B_n \uparrow A$.

Since μ is $\oplus -$continuous from below at $A \in \mathcal{F}$.

$\Rightarrow \mu(B_n) \rightarrow \mu(A)$.

Since μ is $\oplus -$additive $\Rightarrow \mu(B_n) = \mu(\bigcup_{i=1}^{n} A_i) = \bigoplus_{i=1}^{n} \mu(A_i)$.

$\Rightarrow \bigoplus_{i=1}^{n} \mu(A_i) \rightarrow \mu(\bigcup_{n=1}^{\infty} A_n) \Rightarrow \mu(\bigcup_{n=1}^{\infty} A_n) = \bigoplus_{n=1}^{\infty} \mu(A_n)$.

So μ is countable $\oplus -$additive.

3-In the notation of (2) put $C_n = A | B_n \quad \Rightarrow C_n \in \mathcal{F}, n = 1,2,.......$

$\Rightarrow C_n \downarrow \varphi$.

$\Rightarrow \mu(C_n) \rightarrow \mu(\varphi) = 0 \Rightarrow \mu(A | B_n) \rightarrow 0$.

$\mu(A) = \bigoplus_{i=1}^{n} \mu(A_i) \oplus \mu(C_n)$.

So that $\mu(A) = \bigoplus_{i=1}^{\infty} \mu(A_i)$.

3- The completion of ⊕-measure

In this section we construct the completion of $\oplus-$measure.

Definition (3.1)

Let $(\Omega,\mathcal{F})$ be a measurable space with $\mathcal{F}$ a σ-ring and μ is ⊕-measure on $\mathcal{F}$, $E\in\mathcal{F}$ is said to be μ-null set if $\mu(E)=0$.The ⊕-measure μ is said to be complete on $\mathcal{F}$ if $\mathcal{F}$ contains the subsets of every μ-null sets.

Theorem (3.2):

Let $(\Omega,\mathcal{F},\mu)$ be a space with ⊕-measure where $\mathcal{F}$ is σ-ring and $N_\mu=\{E:E\subset A\in\mathcal{F}\quad and\quad \mu(A)=0\}$ then N_μ is a σ-ring.

Proof:

1- Clearly $\varphi\in N_\mu$.

2- Let $E_1,E_2\in N_\mu\Rightarrow$ there exists $A_1,A_2\in\mathcal{F}$ such that $E_1\subseteq A_1,E_2\subset A_2$ and $\mu(A_1)=0,\mu(A_2)=0$.

$E_1|E_2\subset E_1\subset A_1\in\mathcal{F}$ So $E_1|E_2\in N_\mu$.

3- Let $\{E_i\}$ be a sequence of sets in N_μ $i=1,2,....\Rightarrow$ there exist a sequence $\{A_i\}$ $i=1,2,....$ of sets in $\mathcal{F}$ such that $E_i\subset A_i$ and $\mu(A_i)=0$.

$\bigcup_{i=1}^{\infty}E_i\subset\bigcup_{i=1}^{\infty}A_i$ Since $\mathcal{F}$ is σ- ring $\Rightarrow\bigcup_{i=1}^{\infty}A_i\in\mathcal{F}$.

$\mu(\bigcup_{i=1}^{\infty}A_i)\le\bigoplus_{i=1}^{\infty}\mu(A_i)=0\Rightarrow\mu(\bigcup_{i=1}^{\infty}A_i)=0$.

So $\bigcup_{i=1}^{\infty}E_i\in N_\mu$ therefore N_μ is σ-ring.

Theorem (3.3):

Let $(\Omega,\mathcal{F},\mu)$ be a space with ⊕-measure where $\mathcal{F}$ is a σ-ring, define $\overline{\mathcal{F}}=\{(E\cup E_1)-E_2:E\in\mathcal{F},E_1,E_2\in N_\mu\}$ then $A\in\overline{\mathcal{F}}$ iff there exist sets $M,N\in\mathcal{F}$ such that $M\subset A\subset N$ and $\mu(N-M)=0$.

Proof:

Let $M,N\in\mathcal{F}$ and $M\subset A\subset N$ such that $\mu(N-M)=0$, so $A=(N\cup\varphi)-(N-A)$.

Since $N-A\subset N-M\in\mathcal{F}$ and $\mu(N-M)=0$.

$\Rightarrow N-A\in N_\mu$.

Therefore $A\in\overline{\mathcal{F}}$.

Suppose that $A\in\overline{\mathcal{F}}$.

Then $A=(E\cup E_1)-E_2$, $E\in\mathcal{F}, E_1,E_2\in N_\mu$.

Therefore there exist $A_1,A_2\in\mathcal{F}$ such that $\mu(A_i)=0$ and $E_i\subset A_i$, $E-A_2\subset A\subset E\cup A_1$

$E\cup A_1,E-A_2\in\mathcal{F}$ and

$\mu((E\cup A_1)-(E-A_2))=\mu((A_1-E)\cup(A_2\cap E))$.

$=\mu((A_1-E))\oplus\mu(A_2\cap E)$.

Since $A_1-E\subset A_1$ and $A_2\cap E\subset A_2$.

$\Rightarrow\mu(A_1-E)=0\quad\wedge\mu(A_2\cap E)=0$.

So $\mu((E\cup A_1)-(E-A_2))=0$.

Corollary (3.4):

Let $(\Omega,\mathcal{F},\mu)$ be a space with ⊕-measure where $\mathcal{F}$ is σ-ring then $A\in\overline{\mathcal{F}}$ iff $A=E\cup M$, $E\in \mathcal{F}$ and $M\in N_{\mu}$.

Proof:

Suppose that $A\in\overline{\mathcal{F}}$.

By theorem (1.3.3) there exist $M,N\in\mathcal{F}$ such that $N\subset A\subset M$ and $\mu(M-N)=0$

$A=N\cup(A-N)$, $N\in\mathcal{F}$.

Since $A-N\subset M-N\in\mathcal{F}$ and $\mu(M-N)=0$.

$\Rightarrow A-N\in N_{\mu}$.

Conversely suppose $A=E\cup M$, $E\in\mathcal{F}\wedge M\in N_{\mu}$.

$$A=(E\cup M)-\varphi \quad \varphi\in N_{\mu}.$$

$$\Rightarrow A\in\overline{\mathcal{F}}.$$

Corollary (3.5):

Let $(\Omega,\mathcal{F},\mu)$ be a space with ⊕ -measure where $\mathcal{F}$ is σ-ring then $A\in\overline{\mathcal{F}}$ iff $A=E-D$ with $E\in\mathcal{F}$ and $D\in N_{\mu}$.

Proof:

Suppose that $A\in\overline{\mathcal{F}}$.

⇒There exist $M,N\in\mathcal{F}$ such that $M\subset A\subset N$.

and $\mu(N-M)=0$.

$A=N-(N-A)$, $N\in\mathcal{F}$.

Since $N-A\subset N-M\in F$ and $\mu(N-M)=0$.

So $N-A\in N_{\mu}$.

Conversely suppose that $A=E-D$ where $E\in\mathcal{F}$ Λ $D\in N_{\mu}$.

$$\Rightarrow A=(E\cup\varphi)-D \quad D,\ \varphi\in N_{\mu}.$$

$$\Rightarrow A\in\overline{F}.$$

Theorem (3.6):

Let $(\Omega,\mathcal{F},\mu)$ be a space with ⊕ -measure where $\mathcal{F}$ is a $\sigma-$ring then $\overline{\mathcal{F}}$ is σ-ring.

Proof:

1-clearly $\varphi\in\overline{\mathcal{F}}$.

2-Let $\{A_i\}$ i =1,2,… be a sequence of sets such that $A_i\in\overline{\mathcal{F}}\Rightarrow A_i=M_i\cup N_i$ where $M_i\in\mathcal{F}$ and $N_i\in N_{\mu}$.

$\bigcup_{i=1}^{\infty}A_i=\bigcup_{i=1}^{\infty}(M_i\cup N_i)$.

$=(\bigcup_{i=1}^{\infty}M_i)\cup(\bigcup_{i=1}^{\infty}N_i)$.

Since $\mathcal{F}$ and N_{μ} are σ-ring.

$$\Rightarrow\bigcup_{i=1}^{\infty}M_i\in F_1$$

$$\bigcup_{i=1}^{\infty}N_i\in N_{\mu}$$

So $\bigcup_{i=1}^{\infty} A_i \in \overline{\mathcal{F}}$.

3- Let $A, B \in \overline{\mathcal{F}}$ from Corollary (1.3.4) we obtain $A = M_1 \cup N_1 \quad B = M_2 \cup N_2$.

$A - B = (M_1 \cup N_1) - (M_2 \cup N_2)$.

$= ((M_1 - M_2) - N_2) \cup ((N_1 - M_2) - N_2)$.

$= [(M_1 - M_2) - E_2) \cup (E_2 - N_2) \cap (M_1 - M_2))] \cup ((N_1 - M_2) - N_2)$

$N_2 \subset E_2 \in \mathcal{F}, \quad \mu(E_2) = 0$

$A - B \in \overline{\mathcal{F}}$.

Therefore $\overline{\mathcal{F}}$ is σ-ring.

Theorem (3.7):

Let $(\Omega, \mathcal{F}, \mu)$ be a space with $\oplus$ -measure and $\overline{\mu} : \overline{\mathcal{F}} \to E_a$ defined as follows $\overline{\mu}(A) = \mu(M)$ where $A = (M \cup N)$, $M \in \mathcal{F}$ and $N \in N_\mu$.

Then $\overline{\mu}$ is complete $\oplus$-measure on $\overline{\mathcal{F}}$, where is restriction to $\mathcal{F}$ is μ.

Proof:

1- $\overline{\mu}(\varphi) = \mu(\varphi) = 0$.

2-Let $\{A_i\}$ be a sequence of sets in $\overline{\mathcal{F}} \quad i = 1,2,...$

$\Rightarrow$There exist a sequence of sets $\{E_i\}$ in $\mathcal{F}$ and a sequence of sets $\{N_i\}$ in N_μ such that $A_i = E_i \cup N_i$.

$\overline{\mu}(\bigcup_{i=1}^{\infty} A_i) = \overline{\mu}(\bigcup_{i=1}^{\infty} (E_i \cup N_i))$.

$= \overline{\mu}((\bigcup_{i=1}^{\infty} E_i) \cup (\bigcup_{i=1}^{\infty} N_i))$

$= \mu(\bigcup_{i=1}^{\infty} E_i) = \bigoplus_{i=1}^{\infty} \mu(E_i) = \bigoplus_{i=1}^{\infty} \overline{\mu}(A_i)$

So $\overline{\mu}$ is $\oplus$-measure on $\overline{\mathcal{F}}$.

3-Let $A \in \mathcal{F} \quad , A = A \cup \varphi, \varphi \in N_\mu$.

$\overline{\mu}(A) = \overline{\mu}(A \cup \varphi) = \mu(A)$.

μ is $\oplus$-restriction of $\overline{\mu}$ to $\mathcal{F}$.

4- Let $E \in \overline{\mathcal{F}}$ and $\overline{\mu}(E) = 0$, $A \subset E$.

$E = M \cup N \quad , M \in \mathcal{F}, N \in N_{\mu}$.

$\overline{\mu}(E) = \mu(M) \quad \Rightarrow \mu(M) = 0$.

Since $N \in N_{\mu} \Rightarrow$ There exists $E_1 \in \mathcal{F}$ such that $N \subset E_1$ and $\mu(E_1) = 0$, since $\mu(E_1) = \mu(M) = 0 \Rightarrow M, E \in N_{\mu}$.
$A \subset E = M \cup N \subset M \cup E_1 \Rightarrow A \subset M \cup E_1 \in \mathcal{F}, \mu(M \cup E_1) = \mu(M) \oplus \mu(E_1) = 0 \Rightarrow A \in N_{\mu}$
$A = (M \cup E_1) - ((M \cup E_1) - A)$, $M \cup E_1 \in \mathcal{F}, (M \cup E_1) - A \in N_{\mu} \Rightarrow A \in \overline{\mathcal{F}} \Rightarrow \overline{\mu}$ is complete on $\overline{\mathcal{F}}$.

5- To show that the definition of $\overline{\mu}$ is well defined.
Let $A \in \overline{\mathcal{F}} \Rightarrow A = M \cup N \quad , M \in \mathcal{F}$ and $N \in N_{\mu}$.
$\Rightarrow \exists E \in \mathcal{F} \quad N \subset E$ and $\mu(E) = 0$.
The relations $M \cup N = (M - E) \cup (E \cap (M \cup N))$.

and $M \Delta N = (M - E) \cup (E \cap (M \Delta N))$ show that

the class $\overline{\mathcal{F}}$ may also be decried as there class of the form $M \Delta N, M \in \mathcal{F}$ and $N \in N_{\mu}, \overline{\mu}(M \Delta N) = \overline{\mu}(M \cup N) = \mu(M)$.

Let $F_1 \Delta N_1 = F_2 \Delta N_2$.

$F_i \in \mathcal{F} \quad , N_i \subseteq E_i \in \mathcal{F} \quad , \mu(E_i) = 0$ i=1,2.

Then $F_1 \Delta F_2 = N_1 \Delta N_2$.

Therefore $\mu(F_1 \Delta F_2) = 0 \Rightarrow \mu(F_1) = \mu(F_2) \Rightarrow \overline{\mu}(F_1 \Delta N_1) = \overline{\mu}(F_2 \Delta N_2)$.

So the definition of $\overline{\mu}$ is well defined .

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