Coconvex Polynomial Approximation

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Abstract *Let* $f \in L_p(I)$ *change its convexity finitely many times in the interval* I *, say* s *times at* Y_s *, We estimate the degree of approximation of* f *by polynomials which change convexity exactly at the points where* f *does.*

1. Introduction and Main Results

We consider the space $L_p(I)$, consisting of all measurable functions f on I, for which

$$\|f\|_p^p := \int_I |f(x)|^p \, dx < \infty .$$

Recall that $\|f\|_p \le 2^{\frac{1}{p}-1} \|f\|_1$, that is $L_1 \subset L_p$. the L_p norm is not actually a norm for $p < 1$. Nevertheless, it is not hard to see that $L_p(I)$ is a complete metric space.

Now, we will settle the question of which $L_p, p < 1$ embed into L_q for $q \ge 1$. Or which subspaces of $L_p(I)$ on which all of the various $L_p(I)$ quasi-norms for $0 < p < q$ are equivalent. The key in this article is due to Kadec and Pelzynyski from [10]:

For $0 < \varepsilon < 1$ and $0 < p < \infty$, consider the following subset of $L_p(I)$

$$M(p,\varepsilon) := \{ f \in L_p(I) : \text{meas } \{x : |f(x)| \ge \varepsilon \|f\|_p \} \ge \varepsilon \},$$

where by "meas" , we mean the measure of a set.
Notice that if $\varepsilon_1 < \varepsilon_2$, then $M(p,\varepsilon_2) \subset M(p,\varepsilon_1)$. Also $\bigcup_{\varepsilon > 0} M(p,\varepsilon) = L_p(I)$, since for any nonzero $f \in L_p(I)$ we have meas $\{|f| \ge \varepsilon\} \to \{f \ne 0\}$ as $\varepsilon \to 0$. In fact, any finite subset of

$L_p(I)$is contained in an $M(p,\varepsilon)$ for same $\varepsilon>0$. Finally note that meas $\{|f|\geq\|f\|_p\}\geq 1$ implies $|f|=\|f\|_p$ almost every where.

The following theorem puts this observation to good use

Theorem 1.1. [3] For a subset S of $L_p(I)$, the following are equivalent

(i) $S\subset M(p,\varepsilon)$ for some $\varepsilon>0$. (ii) For each $0<p<q$, there exists a constant $c(q)<\infty$ such that $\|f\|_q\leq\|f\|_p\leq c(q)\|f\|_q$ for all $f\in S$. (iii) For some $0<p<q$, there exists a constant $c(q)<\infty$ such that $\|f\|_q\leq\|f\|_p\leq c(q)\|f\|_q$, for all $f\in S$.

Theorem 1.2 . [5] If U is any neighborhood of zero in $L_p(I)$, then

$$L_p(I)=\operatorname{conv}(U)$$

In particular

$$L_p(I)=\operatorname{conv}\left\{f:\|f\|_p^p<1\right\},$$

where $\operatorname{conv}(U)$ is a smallest convex neighborhood of zero contains U.

In our work we will use moduli of smoothness which are connected with difference of higher orders.

The r^{th} *symmetric difference* of f is given by:

$$\Delta_h^r(f,x,J):=\Delta_h^r(f,x):=\begin{cases}\sum_{i=0}^{r}(-1)^i\binom{r}{i}f\left(x+\frac{r}{2}h-ih\right) & \left(x\pm\frac{r}{2}h\right)\in J\\ 0 & o.w\end{cases}.$$

The *forward* r^{th} differences of f are defined respectively by:

$$\vec{\Delta}_h^r(f,x,J):=\vec{\Delta}_h^r(f,x):=\begin{cases}\sum_{i=0}^{r}(-1)^i\binom{r}{i}f(x+(r-i)h) & x,(x+rh)\in J\\ 0 & o.w\end{cases},$$

Then the r^{th} *usual (ordinary) modulus of smoothness* defined by:

$$\omega_r(f,t,J)_p:=\sup_{0<h\leq t}\left\|\Delta_h^r(f,.)\right\|_{L_p(J)},\ t\geq 0.$$

For the forward differences, $\overline{\Delta}_h^r$, the ordinary modulus of smoothness is defined by:

$$\overline{\omega}_r(f,t,J)_p:=\sup_{0<h\leq t}\left\|\overline{\Delta}_h^r(f,.)\right\|_{L_p(J)},\ t\geq 0.$$

A new way of measuring smoothness was moduli of smoothness with weighted, a modification of the moduli of smoothness based on differences Δ_u^r in which the step is of the form $u:=\varphi(x)h$ and is therefore allowed to depend up on the position of x in the interval $J:=[a,b]$. The motivation for this lies in the properties of algebraic polynomial approximation. The requirements on the smoothness of f can be relaxed if x is close to a and b, without impairing the error of approximation. Several authors, among them Ivanov [9], have introduced moduli of this type, but the most useful proved to be the last version, which is introduced by Ditzian and Totik [8].

Thus the need for this new concept arises to solve some basic problems, such as characterizing the behavior of best polynomial approximation in $L_p(J)$.

The *Ditzian – Totik modulus of smoothness* which is defined for $f \in L_p(J)$ as follows:

$$\omega_r^{\varphi}(f,t,J)_p := \operatorname{Sup}_{0<h\le t} \left\|\Delta^r_{h\varphi(.)}(f,.)\right\|_{L_p(J)}, \ t \ge 0 \text{ and } \varphi(x) := \sqrt{\left(1-x^2\right)} \quad .$$

If the interval $I := [-1,1]$ is used in any of the above notations, it will be omitted for the sake of simplicity, for example

$$\omega_r(f,t)_p := \omega_r(f,t,I)_p \text{ and } \omega_r^{\varphi}(f,t)_p := \omega_r^{\varphi}(f,t,I)_p .$$

We always have that $\omega_r^{\varphi}(f,t,J)_p \le \omega_r(f,t,J)_p$, $0 < p \le \infty$. But the converse is not true in general. However in [2] E. Bhaya, there has been proved that the moduli ω_r^{φ} and ω_r for a function f defined on $J := [a, b] \subseteq I$ are equivalent, if $|J| \approx \Delta_n(a)$, where $\Delta_n(a) := \frac{1}{n}\sqrt{\left(1-a^2\right)} + \frac{1}{n^2}$, namely

$$\omega_r(f,\Delta_n(a),J)_p \sim \omega_r^{\varphi}(f,n^{-1},J)_p .$$

The conservation of certain geometric properties of the data by the designed mathematical might be the main point of view in many applications. These properties include positivity, monotonicity, convexity and in general, k-convexity. This is the topic that so called *shape preserving approximation or constrained approximation* is concerned with. Whenever constraints emerge, the situation becomes more difficult to get direct estimates, but the researches concentrating on such point have been widely acquired attention in recent times for the theory of non-constrained problems is still useful. We intend to refer to those modifications which are essential in making a break-through approach. We do this for coconvex approximation by algebraic polynomials. The main objective of this thesis is to provide the answer of the question that whether the constraint cost anything or can we achieve the same degree of approximation as in the non-constrained case?

In coconvex polynomial approximation, we are given a function f changes its convexity finitely many times in the interval I. We are interested in estimating the degree of approximation of f by polynomials which are coconvex with it, namely, polynomials that change their convexity exactly at the points where f does. Question of this nature first appeared in the work of D. J. Newman and et al (see [18], [19] and [20]).

To be specific, Let $s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and let $Y_s = \{y_i\}_{i=1}^{s}$ be the set of points such that $-1 := y_0 < y_1 < y_2 < \ldots < y_s < y_{s+1} := 1$, where $s = 0$, $Y_0 := \phi$. For Y_s we set

$$\pi(x) := \pi(x, Y_s) := \prod_{i=1}^{s}(x - y_i) \text{ and } \quad \delta(x) := \operatorname{sgn}(\pi(x)) ,$$

where the empty product is equal to 1.

Let $\Delta^2(Y_s)$, be the set of all functions f that change convexity at the points $y_i \in Y_s$, and are convex near 1. In particular, if $s=0$, then f is convex on I, and write $f \in \Delta^2$, that is (the *divided differences* $[x_0, x_1, x_2; f]$ are nonnegative for all choices of three distinct points x_0, x_1 and x_2), where the divided difference of a function f at the points $x_0, x_1, \ldots, x_n$ are defined by(see [2])

$$[x_0, x_1, \ldots, x_n; f] := \sum_{j=0}^{n} \frac{f(x_i)}{\prod_{\substack{i=0\\ i\neq j}}^{n} (x_j - x_i)}.$$

Moreover, if f is twice differentiable function on I (i.e., $f \in C^2(I)$), then

$f \in \Delta^2(Y_s)$, if and only if $f''(x)\pi(x) \geq 0, \quad \forall x \in I$,

(or equivalently, if and only if $f''(x)\delta(x) \geq 0, \ \forall x \in I$).

In our work, the approximation will be carried out by polynomials $P_n \in \Pi_n$. Now, for $f \in L_p(I) \cap \Delta^2(Y_s)$, we denote by

$$E_n^{(2)}(f, Y_s)_p := \inf_{P_n \in \Pi_n \cap \Delta^2(Y_s)} \| f - P_n \|_p ,$$

the *degree of coconvex polynomial approximation*.

First of all in 1981, Beatson and Leviatan gave a remark in [1] it might be possible to obtain a Jackson – type estimate of coconvex approximation of a function with only one regular convexity – turning point, and Yu [24] obtained a Jackson – type estimate of coconvex approximation of functions with one regular convexity – turning point also quoted her result of functions $f \in C^k(I)$ and $k \geq 3$ (the space of all function such that $f^{(k-1)}$ are absolutely continuous in I and $f^{(k)} \in C(I)$), with some extra conditions on convexity turning points.

In 1993, Wu and Zhou [23] and Zhou [25], they proved that for $0 < p \leq \infty$, it is impossible to get a Jackson – type estimate of coconvex approximation involving $\omega_4(f,1)_p$ with constants independent of n and f.

Afterwards, in 1995 Kopotun [11] obtained the following result for twice differentiable functions.

Theorem 1.3 [11] For a function $f \in C^2(I) \cap \Delta^2(Y_s)$ with $1 \leq s < \infty$, there is a polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$ such that

$$\| f - P_n \| \leq C(s) \frac{1}{n^2} \omega^{\varphi}\left(f'', \frac{1}{n}\right),$$

$$\| f' - P_n' \| \leq C(s) \frac{1}{n} \omega^{\varphi}\left(f'', \frac{1}{n}\right),$$

and

$$\| f'' - P_n'' \| \leq C(s) \omega^{\varphi}\left(f'', \frac{1}{n}\right),$$

for all $n \geq N := N(Y_s)$ is a constant depending on the location of points of Y_s in I, and $C(s)$ is a constant depend only on s - the number of convexity change.

Then in 2003 E. Bhaya [2] improved Kopotun's result for functions $f \in L_p^1(I) := \{ f\ ;\ f, f' \in L_p(I) \}$ with $1 \le p < \infty$. Also, in 1999 Kopotun, Leviatan and Shevchuk [13] improved Kopotun's result for the uniform norm space, but not simultaneously. Namely, they proved

Theorem 1.4. If $f \in C(I) \cap \Delta^2(Y_s)$ with $1 \le s < \infty$, then there is a polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$ such that

$$\|f - P_n\| \le C(s,p)\,\omega_3^{\varphi}\left(f, \frac{1}{n}\right) \tag{1.5}$$

for all $n \ge N := N(Y_s)$.

Thereafter, several other results have been achieved for coconvex polynomial approximation throughout a number of researches by Leviatan and Shevchuk [16], [17] and by Kopotun, Leviatan and Shevchuk [14] [15].

Our achievement in this area is to emphasize that the estimate (1.5) is exact in the quasi-norm spaces L_p with $0 < p < \infty$. Namely, we prove

Theorem 1.6. Let $f \in L_p(I)$ with $0 < p < \infty$ have s changes of convexity at $Y_s := \{y_i\}_{i=1}^s$, and denote $d(Y_s) := \min\ \{1 + y_1, y_2 - y_1, \dots, y_s - y_{s-1}, 1 - y_s\}$. Then there exists a constant $A(s)$ such that for $n > N := N(Y_s) := \dfrac{A(s)}{d(Y_s)}$, there is a polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$, such that

$$\|f - P_n\|_p \le C(s,p)\,\omega_3^{\varphi}\left(f, \frac{1}{n}\right)_p , \tag{1.7}$$

hence

$$\|f - P_n\|_p \le C(s,p)\,\omega_3\left(f, \frac{1}{n}\right)_p . \tag{1.8}$$

The estimate (1.5) is best possible in that one can not replace $\omega_3^{\varphi}(f, 1/n)_p$ by any higher modulus of smoothness, even not with the larger ordinary modulus of smoothness. This is due to the work of Shvedov [22] in case $s = 0$, and to Wu and Zhou [23], Zhou [25] in case $s > 0$, (as we mentioned above).

As an immediate consequence of (1.6), we have the following corollary

Corollary 1.9. Let $f \in W_p^r(I)$ with $r = 1,2,3$ have s changes of convexity at $Y_s := \{y_i\}_{i=1}^s$, and denote $d(Y_s) := \min\ \{1 + y_1, y_2 - y_1, \dots, y_s - y_{s-1}, 1 - y_s\}$. Then there exists a constant $A(s)$ such that for $n > N := N(Y_s) := \dfrac{A(s)}{d(Y_s)}$, there is a polynomial $P_n \in \Pi_n \cap \Delta^2(Y_s)$, such that

$$\left\| f - P_n \right\|_p \le C(s,p) n^{-r} \left\| f^{(r)} \right\|_p .$$

This corollary is obtained directly from (1.7). However, we can not obtain it from (1.8), in the case $p < 1$, since the inequality $\omega_{r+k}(f,t)_p \le c(p) t^k \omega_r(f^{(k)},t)_p$ is not satisfied in that case.

2. Proof of Theorem 1.6:

Throughout this paper we use the following notations, given $n \in N$, we set $x_{-1} = 1$, $x_n = -1$ and

$x_j := x_{j,n} := \cos\left(\frac{j\pi}{n}\right)$, the Chebyshev partition of interval I,

we denote $I_j := I_{j,n} := [x_j, x_{j-1}]$, $h_j := |I_j| := x_{j-1} - x_j$,

and $\psi_j := \psi_{j,n} := \dfrac{h_j}{|x - x_j| + h_j}$, $j = 0,1,\ldots,n$.

Also we need the following theorem

Theorem 2.1. [21]

Suppose that $f : I \to \Re$ is convex function, then f satisfies Lipshchitz condition on any closed subinterval $[a,b]$ of I° (interior of I), f is absolutely continuous on $[a,b]$ and in particular it's continuous on I°, f has left and right nondecreasing derivatives, $f'_-(x)$ and $f'_+(x)$ on I. Furthermore, the set E where f' fail to exist is countable, and f' is continuous on $I \setminus E$.

We use the mathematical induction on s - the number of convexity changes of f and the idea of flipping technique of f, which originally introduced by Beatson and Leviatan in [1].
For $s = 0$ (i.e., f is convex in I), then the theorem is valid and it was proved by DeVore, Leviatan and Hu [6]. Thus we will assume that $s \ge 1$, and it is clear that f is either concave or convex in the interval $[-1, y_1]$, and each case where will need a separate through similar construction. We will detail the construction for the case where f is concave in $[-1, y_1]$. For the sake of simplicity in written we write $\alpha = y_1$.

Now, we may assume that $\alpha \in [x_{j_0}, x_{j_0-1})$. Then, if $n > N_\alpha := \max\left\{\frac{50}{y_2 - \alpha}, \frac{50}{1+\alpha}\right\}$, we are assured that $x_{j_0+3} \ge -1$ and that $x_{j_0-4} \le y_2$. Set $h := c\Delta_n(\alpha) < \frac{1}{6} h_{j_0}$, where c is chosen sufficiently small to guarantee the right inequality. We observing that this implies

$$x_{j_0+1} < \alpha + 2h < \alpha - 2h < x_{j_0-2} .$$

We are going to replace f on the interval $[\alpha-h,\alpha+h]$ in a way that will keep us near the original function (see g below) will be smoother at α. As we said above, the case $s=0$ is known and serves as the beginning of the induction process; it has been proved by DeVore, Leviatan and Hu [6], and in this case (1.7) holds, for all $n\geq 2$. Thus, we proceed by induction.

To this end, we observe that either $\Delta_h^2(f,\alpha)\geq 0$ or $\Delta_h^2(f,\alpha)<0$.

In the first case, let $\ell_1(x)$ be denote the linear function interpolating f at $\alpha-h$ and α. Then the function $\bar{f}:=f-\ell_1$ satisfies

$$\bar{f}(\alpha-h)=\bar{f}(\alpha)=0\,,\bar{f}(\alpha+h)\geq 0, \text{ and } \bar{f}(x)\leq 0,\ -1\leq x<\alpha-h\ .$$

Hence, for $J:=[x_{j_0+1},x_{j_0-2}]$, we have,

$$\begin{aligned}0\leq \bar{f}(\alpha+h)&\leq \bar{f}(\alpha+h)-\bar{f}(\alpha-2h)\\ &=\bar{f}(\alpha+h)-3\bar{f}(\alpha)+3\bar{f}(\alpha+h)-\bar{f}(\alpha-2h)\\ &=\vec{\Delta}_h^3(\bar{f},\alpha).\end{aligned}$$

Now, since $\bar{f}$ is concave in $[-1,\alpha]$ and it's convex in $[\alpha,y_2]$, then from Theorem 2.1, we have $\bar{f}$ is continuous on $(-1,\alpha)$ and (α,y_2), and since $\bar{f}(\alpha)=0$ which is finite, so $\bar{f}$ is bounded on $(-1,y_2)$, then $\Delta_h^3(\bar{f},\alpha)$ is finite on J. On the other hand, we have from the definition of ordinary modulus of smoothness that

$$\left\|\vec{\Delta}_h^3(\bar{f},\alpha)\right\|_{L_p(J)}\leq\vec{\omega}_3(\bar{f},h,J)_p\leq\vec{\omega}_3(f,h,J)_p\leq C(p)\omega_3(f,h,J)_p\,,$$

then

$$\left|\Delta_h^3(\bar{f},\alpha)\right|\leq|J|^{-\frac{1}{p}}\omega_3(f,h,J)_p\,.$$

Thus

$$0\leq\bar{f}(\alpha+h)\leq\left|\Delta_h^3(\bar{f},\alpha)\right|\leq|J|^{-\frac{1}{p}}\omega_3(f,h,J)_p\,.$$

Similarly, in the latter case, let $\ell_1(x)$ be denote the linear function interpolating f at α and $\alpha+h$. Then the function $\bar{f}:=f-\ell_1$ satisfies

$$\bar{f}(\alpha)=\bar{f}(\alpha+h)=0\,,\bar{f}(\alpha-h)<0, \text{ and } \bar{f}(x)\geq 0,\ -1\leq x<\alpha-h\ .$$

Hence, for $J:=[x_{j_0+1},x_{j_0-2}]$, we have,

$$\begin{aligned}0\leq -\bar{f}(\alpha-h)&\leq\bar{f}(\alpha+2h)+\bar{f}(\alpha-h)\\ &=\bar{f}(\alpha+2h)-3\bar{f}(\alpha+h)+3\bar{f}(\alpha)-\bar{f}(\alpha-h)=\overleftarrow{\Delta}_h^3(\bar{f},\alpha)\end{aligned}$$

Also

$$0\leq\left|\bar{f}(\alpha-h)\right|\leq\left|\overleftarrow{\Delta}_h^3(\bar{f},\alpha)\right|\leq|J|^{-\frac{1}{p}}\omega_3(f,h,J)_p\,.$$

Thus in both cases we have,

$$\max\left\{\left|\bar{f}(\alpha-h)\right|,\left|\bar{f}(\alpha)\right|,\left|\bar{f}(\alpha+h)\right|\right\}\leq|J|^{-\frac{1}{p}}\omega_3(f,h,J)_p\,,$$

which in turn implies that the quadratic polynomial ℓ_2

interpolating f at $\alpha-h,\alpha$ and $\alpha+h$, is bounded by the same quantity on $[\alpha-h,\alpha+h]$. This means that

$$|\ell_2(x)| \le |J|^{-\frac{1}{p}}\omega_3(f,h,J)_p$$
$$\le c(2h)^{-\frac{1}{p}}\omega_3(f,h,J)_p, \quad \forall\ x\in[\alpha-h,\alpha+h],$$

so

$$\|\ell_2\|_{L_p[\alpha-h,\alpha+h]} \le c\,\omega_3(f,h,J)_p,$$

then by applying the following lemma from [12], we obtain

Lemma 2.2. [12]

Let J_1 and J_2 be subintervals such that $J_1\subset J_2$. If $q_k\in\Pi_k$, then for $0<p\le\infty$

$$\|q_k\|_{L_p(J_2)} \le c(k,p)\left(\frac{|J_2|}{|J_1|}\right)^{k+\frac{1}{p}}\|q_k\|_{L_p(J_1)}.$$

$$\|\ell_2\|_{L_p(J)} \le C(p)\omega_3(f,h,J)_p \tag{2.3}$$

At the same time applying Whitney's theorem we conclude that

$$\|\bar{f}-\ell_2\|_{L_p(J)} \le C(p)\omega_3(f,h,J)_p. \tag{2.4}$$

Hence

$$\|\bar{f}\|_{L_p(J)} \le C(p)\omega_3(f,h,J)_p. \tag{2.5}$$

Now, since ℓ_2 is bounded on I, then

$$\|\ell_2\|_{L_p(I_j)} \sim \|\ell_2\|_{L_p(J)},\ \forall\ j=1,2,\dots,n. \tag{2.6}$$

By the theorem 2.1 about the property of convex function we have.:

$$\omega_3(f,h,J)_p \sim \omega_3(f,h_j,I_j)_p,\ \forall\ j=1,2,\dots,n \tag{2.7}$$

Then by using (2.3), (2.6) and (2.7), we conclude that for each $j=1,2,\dots,n$

$$\|\ell_2\|_{L_p(I_j)} \le C(p)\|\ell_2\|_{L_p(J)} \le C(p)\omega_3(f,h,J)_p \le C(p)\omega_3(f,h_j,I_j)_p.$$

So by applying the following lemma from [6], we obtain:

Lemma 2.8. [6]

For a function $f\in L_p(I)$ with $0<p<\infty$ and $k\in N$, the following inequality holds

$$\left(\sum_{j=1}^{n}\omega_r(f,h_j,\Im_j)_p^p\right)^{\frac{1}{p}} \le C(B_0,r,p)\omega_r^{\varphi}(f,n^{-1})_p,$$

where, for every j, $I_j\subseteq\Im_j$ is such that $|\Im_j|\le B_0|I_j|$.

$$\|\ell_2\|_p \le C(p)\omega_3^{\varphi}(f,n^{-1})_p.$$

Analogously

$$\|\bar{f}-\ell_2\|_p \le C(p)\omega_3^{\varphi}(f,n^{-1})_p,$$

hence

$$\|\bar{f}\|_p \le C(p)\omega_3^{\varphi}(f,n^{-1})_p. \tag{2.9}$$

Now, let

$$\hat{f}(x) := \begin{cases} -\bar{f}(x) & -1 \le x \le \alpha \\ \bar{f}(x) & o.w\ , \end{cases}$$

and

$$g(x) := \begin{cases} \hat{f}(x) & x \notin [\alpha - h, \alpha + h] \\ \max\ \{\hat{f}(x), 0\} & x \in [\alpha - h, \alpha + h]. \end{cases}$$

by virtue of (2.9) we have

$$\left\| \hat{f} - g \right\|_p \le C(p)\,\omega_3^{\varphi}(f, n^{-1})_p \tag{2.10}$$

Thus, by using (2.9), the inequality $\omega_r^{\varphi}(f,t)_p \le c(r,p)\|f\|_p$,[7]

and $\frac{1}{10}\left(\left|x - x_j\right| + h_j\right) < \left|x - x_j\right| + \Delta_n(x) < 2\left(\left|x - x_j\right| + h_j\right)$.[7]

we obtain

$$\begin{aligned} \omega_3^{\varphi}(g, n^{-1})_p &\le C(p)\left(\omega_3^{\varphi}(\hat{f} - g, n^{-1})_p + \omega_3^{\varphi}(\hat{f}, n^{-1})_p\right) \\ &\le C(p)\left(\left\| \hat{f} - g \right\|_p + \left\| \hat{f} \right\|_p\right) \\ &\le C(p)\,\omega_3^{\varphi}(f, n^{-1})_p \end{aligned} \tag{2.11}$$

It is readily that $g \in L_p(I)$, that it is convex in $[-1, y_2]$ and that it changes convexity at $\hat{Y}_{s-1} := Y_s \setminus \{y_1\}$. If, on the other hand, f was convex in $[-1, \alpha]$, then g would be concave in $[-1, y_2]$ and change convexity at $\hat{Y}_{s-1}$. Thus in any case g had fewer convexity changes, so by induction, we may assume that for $n > \frac{A(s)}{d(s)}$, there exists an n^{th} degree polynomial q_n which is coconvex with g, and satisfies the analogues of (1.5). Namely (by (2.11))

$$\|g - q_n\|_p \le C(s-1,p)\ \omega_3^{\varphi}\left(g, \frac{1}{n}\right)_p \le C(s,p)\,\omega_3^{\varphi}\left(f, \frac{1}{n}\right)_p . \tag{2.12}$$

Note that, since $g(\alpha) = 0$, we may assume that $q_n(x) = 0$.

We fix $n > \max\left\{\frac{A(s-1)}{d(s-1)}, N_\alpha\right\}$ readily leads to the definition of $A(s)$. Kopotun [4] has constructed, for α, q_n sufficiently large $\mu \ge 2$, and for each n like above, two polynomials V_n and W_n of degree $\le C(s)n$ such that the polynomial

$$P_n(x) := \int_\alpha^x \left[\left(q_n'(u) - q_n'(\alpha)\right)V_n(u) + q_n'(\alpha)W_n(u)\right]du\ ,$$

is coconvex with f , and the following inequalities are satisfied $x \in I$,

$$V_n(x)\,\mathrm{sgn}\ (x - \alpha) \ge 0\ ,$$

$$V_n'(x)\,q_n''(x)\left(q_n'(x) - q_n'(\alpha)\right)\mathrm{sgn}\ (x - \alpha) \ge 0\ ,$$

$$\left|V_n(x) - \mathrm{sgn}\ (x - \alpha)\right| \le C(s)\,\psi_{j_0}^{\mu}\ , \tag{2.13}$$

$$\left|W_n(x) - \mathrm{sgn}\ (x - \alpha)\right| \le C(s)\,\psi_{j_0}^{\mu}\ , \tag{2.14}$$

and

$$\left|\mathcal{V}_n'(x)\right| \le C(s) h_{j_0}^{-1} \psi_{j_0}^{\mu}. \tag{2.15}$$

Observe that $P_n(x) := P_n(x) + \ell_1(x)$ is of same degree of P_n and it too is coconvex with f, so we conclude the induction step by proving (2.3) for P_n, and to this end, we begin with

$$\begin{aligned}
\| f - P_n \|_p &= \| \bar{f} - P_n \|_p = \| \hat{f}(x) \operatorname{sgn}(x-\alpha) - P_n \|_p \\
&\le C(p) \left(\| \hat{f} - g \|_p + \| g(x) \operatorname{sgn}(x-\alpha) - P_n \|_p \right) \\
&\le C(p) \left(\omega_3^{\varphi}\left(f, \frac{1}{n}\right)_p + \| g(x) \operatorname{sgn}(x-\alpha) - P_n \|_p \right) \\
&\le C(p) \left(\omega_3^{\varphi}\left(f, \frac{1}{n}\right)_p + \| g - q_n \|_p + \Big\| q_n(x) \operatorname{sgn}(x-\alpha) \right. \\
&\qquad \left. - \int_{\alpha}^{x} q_n'(u) \mathcal{V}_n(x)\,du \Big\|_p + \Big\| \left| q_n'(\alpha) \right| \int_{\alpha}^{x} \left(\mathcal{V}_n(u) - W_n(u) \right) du \Big\|_p \right) \\
&\le C(s,p) \left(\omega_3^{\varphi}\left(f, \frac{1}{n}\right)_p + \Big\| q_n(x) \operatorname{sgn}(x-\alpha) - \int_{\alpha}^{x} q_n'(u) \mathcal{V}_n(x)\,du \Big\|_p \right. \\
&\qquad \left. + \Big\| \left| q_n'(\alpha) \right| \int_{\alpha}^{x} \left(\mathcal{V}_n(u) - W_n(u) \right) du \Big\|_p \right) \\
&=: C(s,p) \left(E_1 + E_2 + E_3 \right),
\end{aligned}$$

where we applied (2.12) and (2.10) in the first and last inequality respectively. Recalling that $q_n(\alpha) = 0$, integration by parts, (2.13) and (2.15) yield

$$\begin{aligned}
E_2 &= \Big\| q_n(x) \operatorname{sgn}(x-\alpha) - \int_{\alpha}^{x} q_n'(u) \mathcal{V}_n(x)\,du \Big\|_p \\
&\le \Big\| \left| q_n(x) \right| \left| \operatorname{sgn}(x-\alpha) - \mathcal{V}_n(x) \right| - \int_{\alpha}^{x} \left| q_n(u) \right| \left| \mathcal{V}_n'(x) \right| du \Big\|_p \\
&\le C(s) \Big\| \left| q_n(x) \right| \psi_{j_0}^{\mu} - \int_{\alpha}^{x} \left| q_n(u) \right| h_{j_0}^{-1} \psi_{j_0}^{\mu}(u)\,du \Big\|_p \\
&\le C(s,p) \left(\| q_n \|_p + \Big\| \int_{\alpha}^{x} \left| q_n(u) \right| h_{j_0}^{-1} \psi_{j_0}^{\mu}(u)\,du \Big\|_p \right) \\
&=: C(s,p) \left(E_{2,1} + E_{2,2} \right).
\end{aligned}$$

To estimate E_2, we need estimate $E_{2,1}$ and $E_{2,2}$.

By virtue of (2.9), (2.10) and (2.12),

$$E_{2,1} = \|q_n\|_p \le C(p)\left(\|g - q_n\|_p + \|\hat{f} - g\|_p + \|\hat{f}\|_p \right)$$
$$\le C(s,p)\,\omega_3^{\varphi}\left(f,\frac{1}{n}\right)_p , \tag{2.16}$$

Now, to estimate $E_{2,2}$, we separate the cases $p \ge 1$, from the cases $0 < p < 1$, and we reall Jensen's inequality from[26], which is

$$\phi\left\{ \frac{\int_a^b f(x)p(x)dx}{\int_a^b p(x)dx} \right\} \le \frac{\int_a^b \phi(f(x))p(x)dx}{\int_a^b p(x)dx}, \tag{2.17}$$

where ϕ is convex in interval $d \le f(x) \le e$, that $d \le f(x) \le e$ in $a \le x \le b$, the $p(x)$ is nonnegative and $\ne 0$, and all the integrals in the inequality exists. First: For $1 \le p < \infty$, since we have $\|.\|_p$ is convex, and all integrals exist, so by applying the Jensen's inequality (2.17) and $\int_{-1}^{1}\psi_j^{\alpha}(x)dx \le ch_j$, $\alpha \ge 2$ we obtain

$$E_{2,2} := \left\| \int_{\alpha}^{x} |q_n(u)|\, h_{j_0}^{-1}\psi_{j_0}^{\mu}(u)du \right\|_p \le \left\| \int_{-1}^{1} |q_n(u)|\, h_{j_0}^{-1}\psi_{j_0}^{\mu}(u)du \right\|_p$$
$$\le \int_{-1}^{1} \|q_n\|_p\, h_{j_0}^{-1}\psi_{j_0}^{\mu}(u)du \le C\,\|q_n\|_p .$$

Second: For the other case, fix $0 < p < 1$, since $q_n h_{j_0}^{-1}\psi_{j_0}^{\mu} \in M(p,\varepsilon)$, for some $\varepsilon > o$, then by choosing (for example $q = 2$) from Theorem 1.1, it follows that

$$\left\| \int_{\alpha}^{x} q_n(u) h_{j_0}^{-1}\psi_{j_0}^{\mu}(u)du \right\|_p \le c \left\| \int_{\alpha}^{x} q_n(u) h_{j_0}^{-1}\psi_{j_0}^{\mu}(u)du \right\|_2 ,$$

then we use the Jensen's inequality (2.17), to obtain

$$\left\| \int_{\alpha}^{x} |q_n(u)|\, h_{j_0}^{-1}\psi_{j_0}^{\mu}(u)du \right\|_2 \le \int_{-1}^{1} \|q_n\|_2\, h_{j_0}^{-1}\psi_{j_0}^{\mu}(u)du \le C\,\|q_n\|_2 .$$

Hence by using Theorem 1.2, in new, we obtain

$$E_{2,2} := \left\| \int_{\alpha}^{x} |q_n(u)|\, h_{j_0}^{-1}\psi_{j_0}^{\mu}(u)du \right\|_p \le c\|q_n\|_2 \le C(p)\|q_n\|_p .$$

Thus, in each case, we have

$$E_{2,2} \le C(p)\|q_n\|_p := C(p)E_{2,1} .$$

So by virtue (2.16)

$$E_2 \le C(s,p)\,\omega_3^{\varphi}\left(f,\frac{1}{n}\right)_p \tag{218}$$

Finally, it remains only to estimate E_3 , to do so, we notice that q_n is convex in $[-1, y_2]$, then q_n' is monotone increasing there. If $q_n'(\alpha) \ge 0$, then by mean value theorem, for some $\beta \in (\alpha, \alpha + h_{j_0})$,

$$0 \le q'_n(\alpha) \le q'_n(\beta) = \frac{q_n(\alpha + h_{j_0}) - q_n(\alpha)}{h_{j_0}} = h_{j_0}^{-1} q_n(\alpha + h_{j_0}).$$

Then, by the inequality $\sum_{j=1}^{n} \psi_j^{\alpha}(x) \le c,\ \alpha \ge 2$.

, (2.13), (3.3.14) and (2.15) we have

$$E_3 = \left\| |q'_n(\alpha)| \int_{\alpha}^{x} (\nu_n(u) - W_n(u))du \right\|_p$$

$$\le \left(\int_{-1}^{1} \left(\left| h_{j_0}^{-1} q'_n(\alpha + h_{j_0}) \right| \int_{-1}^{1} C(s) \psi_{j_0}^{\mu}(u) du \right)^p dx \right)^{\frac{1}{p}}$$

$$\le C(s,p) \left(\int_{-1}^{1} \left| q'_n(\alpha + h_{j_0}) \right|^p dx \right)^{\frac{1}{p}} = C(s,p) \|q_n\|_p$$

$$\le C(s,p)\, \omega_3^{\varphi}(f, n^{-1})_p .$$

Similarly, if $q'_n(\alpha) < 0$, then, for some $\beta \in (\alpha - h_{j_0}, \alpha)$,

$$0 \le -q'_n(\alpha) \le -q'_n(\beta) = \frac{q_n(\alpha - h_{j_0}) - q_n(\alpha)}{h_{j_0}} = h_{j_0}^{-1} q_n(\alpha - h_{j_0}),$$

then

$$E_3 \le C(s,p)\, \omega_3^{\varphi}\left(f, \frac{1}{n}\right)_p .$$

This implies our assertion ◊

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