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Numerical Solutions of Some Weak Singular Nonlinear Integral Equations of The First Type Using The Spectral Collocation Method

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ABSTRACT

In this paper, a collocation-spectral approximation is proposed for weakly nonlinear and neutral singular Volterra integral-differential equations with rough solutions. We used some appropriate transformations to transform the equation of the equation into a new equation, so that the solution of the new equation has a better order (smoothness) and Jacobi's orthogonal polynomial theory can be easily used. Under appropriate assumptions on the nonlinear part, we were able to perform an acceptable error analysis on the L^{∞} soft and the weighted L^{2} soft. To obtain a numerical approximation, some numerical examples (linear and non-linear) with uneven solutions are considered and numerical results are also presented. Also, a comparison between the proposed method and some existing numerical methods is also provided.

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1. Introduction

Integral equations have many applications in engineering, chemistry and biology. A type of integral equations are weak single integral equations that appear in many problems such as heat transfer, crystal growth and fluid mechanics. Due to the fact that it is not possible to solve many of these equations with analytical methods, numerical methods have been described to solve them. Methods for approximating Abel's integral equations are described in [1], and Professor Jiang's solution in [2] is among those who have investigated integral equation with weakly singular kernel [3]. In [4], a collocation method for Volterra and Fredholm integral equations with a weak single kernel is stated. Volterra's nonlinear integral-differential equations with a weak single kernel are defined as follows:

$$y^{\prime}(x) = f\big(t, y(t)\big) + \int_0^t h_\alpha\left(t, s, y(s)\right) ds, \qquad t \in I := [0, T]$$

Where $0 < \alpha \leq 1, h_\alpha(t,s,y) := p_\alpha(t-s)k(t,s,y)$ and functions $k(t,s,y), f(t,y)$ are smooth functions.

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If the kernel h_α in Volterra's integral equations is dependent on y', then we call Volterra's integral-differential equations neutral, which will be as follows:

$$y'(t) = f(t, y(t)) + (v_\alpha y)(t), t \in I := [0, T]$$

With the initial condition,

$$y(0) = y_0$$

Where v_α is defined as follows:

$$(v_\alpha y)(t) := \int_0^t h_\alpha \left(t, s, y, (s), y'(s)\right) ds$$

And its kernel is defined as follows:

$$h_\alpha(t, s, y, y') = p_\alpha(t - s) j(t, s, y, y'), (0 < \alpha \le 1)$$

The purpose of this research, which is taken from [5], is the numerical solutions of some weak singular nonlinear integral equations of the first type using the spectral collocation method, which will be introduced as follows:

$$f(t) = \lambda \int_D k(x, t) \rho_\alpha (x - t) G(u(x) dx, \quad 0 \le \alpha \le 1, \ D = [0, b]$$

and

$$\rho_\alpha(u) = \begin{cases} u^{-\alpha} & 0 \le \alpha < 1 \\ ln|u| & \alpha = 1 \end{cases}$$

Where $f(t)$ and $k(x, t)$ are known continuous functions in D. It is assumed that $f(0) = 0$ and $G(u(x)) \in C^1(R)$ is an inverse nonlinear function in terms of the unknown function $u(x) : D \to R$. Therefore, according to [6], some important forms of the function $G(u(x))$ will be as follows.

For $u^{(n)}(x)$, it is assumed that $u = u^{(1)} = u^{(2)} = \ldots\ldots = u^{(n-1)} = 0, x = 0$, where $u(n)$ represents the nth derivative function of u with respect to x ; $u^\alpha(x)$, is the α power of $u(x), \alpha \in R$, $sin(u(x))$، $cos(u(x))$، $ln(u(x))$ و $e^u(x)$ or any combination of these functions. Therefore, according to the combined expansion, keeping in mind the above definitions in this research, we are looking for new numerical solutions for some weak singular nonlinear integral equations of the first type.

2. Nonlinear Volterra integral-differential equations with weak singular kernel

Definition 2.1

As we know, the integral equations of the second type with weakly singular kernel are defined as follows [7]:

$$u(x) = g(x) + \int_0^x \frac{\beta}{\sqrt{x - t}} u(t) dt, \ x \in [0, T] \tag{1}$$

They often appear in many fields of physics, mathematics and chemistry, such as heat transfer, electrochemistry, etc. It should be noted that β is a fixed number and $T = 1,2,3$ depends on the model under study. It is also assumed that $g(x)$ is sufficiently smooth, so that the existence of a unique solution for (1) is guaranteed. Integral equation in (1) is placed in the series of single integral equations with kernel $k(x, t) = \frac{1}{\sqrt{x-t}}$ [8].

Recently, it has become a considerable task to obtain the real solution or the approximate solution with high accuracy for these models. Collocation methods have been used to obtain the approximate solution of such equations.

The nonlinear Volterra differential integral equations with weak single kernels are defined as follows:

$$y'(t) = f\big(t, y(t)\big) + \int_0^t h_\alpha(t,s,y(s))ds, \quad t \in I := [0,T], \tag{2}$$

Where $h_\alpha\,(t,s,y) := p_\alpha(t-s)k\,(t,s,y)$ and $0 < \alpha \le 1$ and the functions $f(t,y)$ and $k(\,t,s,y)$ are smooth functions.

If the kernel h_α in Volterra's integral-differential equations depends on y', then we call Volterra's integral-differential equations neutral, which will be as follows:

$$y'(t) = f\big(t, y(t)\big) + (\nu_\alpha y)(t), t \in I := [0,T], \tag{3}$$

With initial condition:

$$y(0) = y_0, \tag{4}$$

Where ν_α is defined as follows:

$$(\nu_\alpha y)(t) := \int_0^t h_\alpha\big(t,s,y(s),(s),y'^{(s)}\big)\,ds, \tag{5}$$

And its Kernel is defined as follows:

$$h_\alpha(t,s,y,y') = p_\alpha(t-s)k(t,s,y,y'), \quad (0<\alpha\le 1),$$

First, we will examine the characteristics of the linear shape, which is defined as follows:

$$f(t,y,y') = a(t)y, \tag{6}$$

$$k(t,s,y,y') = \sum_{\nu=0}^{1} H_\alpha(t,s)y, \tag{7}$$

We consider the given functions a and k which are continuous on I and D respectively. The collocation solution for (4) is calculated in the space of the following smooth piecewise polynomials:

$$S_{m+d}^{(d)}(I_h) \colon \{u_h \in C^d(I) \colon u_h|_{\bar{\sigma}(n} \in \pi_{m+d}(0 \le n \le N-1)\} \tag{8}$$

With $d = 0$ and also the dimension $S_{m+d}^{(d)}(I_h) = N_m + d + 1 = N_m + 1$, the collocation answer u_h is defined as follows:

$$u_h'\,(t) = f\big(t, u_h(t)\big) + (\nu_\alpha)(t), \quad t \in X_n, \tag{9}$$

$$y(0) = y_0,$$

That,

$$X_h := \{t_{n,j} = t_n + c_j h_n \colon 0 \le c_1 < \dots < c_m \le 1 (0 \le n \le N-1)\}$$

$$y_n := u_h(t_n), \qquad Y_{n.j} \colon u_h'\big(t_{n.j}\big)$$

And,

$$u_h(t_n + \upsilon h_n) = y_n + h_n \sum_{j=1}^{m} \beta_j(\upsilon) Y_{n.j}\,, \quad \upsilon \in [0,1] \tag{10}$$

That,

$$B_j(\upsilon) := \int_0^\upsilon L_j\,(s)ds. \tag{11}$$

By placing the collocation Lagrange representation in (11), with $t = t_{n,j}\big(i.e.\, \upsilon = c_j, j = 1, \dots m\big)$ the system of algebraic equations for $Y_n \in R_m$ is obtained. From the answer u_h of this device, the values of the collocation solution are obtained from (10) and its derivative values from (9) [9].

3. Numerical approximation and convergence analysis

We consider the following nonlinear weakly integro-differential equation of Volterra:

$$\begin{cases} y'(t) = f\big(t, y(t)\big) + \int_0^t (t-\sigma)^{-\mu} K\big(t, \sigma, y(\sigma), y'(\sigma)\big) d\sigma, \quad t \in [0, T] \\ y(0) = y_0, \end{cases} \tag{12}$$

Where $y(t), 0 < \mu < 1$ is an unknown function and the given continuous functions $f(t, y)$ and $k(t, \sigma, y\,, y',)$ respectively on $[0, T] \times R$ and $(D \coloneqq \{(t, \sigma) 0 \le \sigma \le t \le T\}) D \times \mathbb{R}^2$ and the relation to y and y' are nonlinear. We assume that (12) has a unique solution and the functions $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 k}{\partial y \partial t}$ respectively on $[0, T] \times R$ and $D \times \mathbb{R}^2$ are continuous. The purpose of this chapter is to present a collocation-spectral method for the above equation and analyze the error of the numerical method [10].

As we mentioned in the previous chapter, for every integer m, the mth derivative of the solution $y(t)$ of (12), when $t \longrightarrow 0$ behaves like $y^{(m)}(t) \sim t^{2-u-m}$ has that in $y \,/\!\in C^m\,(I), m \ge 2$, to overcome the unevenness problem, we consider the solution of the following transformations:

$$t = \gamma(s) = s^q, \qquad \sigma = \gamma(\eta) = \eta^q$$

Using this transformation, (12) is written as follows:

$$\begin{cases} v'(s) = \bar{f}\big(s, v(s)\big) + \int_0^s \big(\gamma(s) - \gamma(\eta)\big)^{-\mu} \bar{K}\big(s, \eta, v(\eta), v'(\eta)\big) d\eta \\ v\,(s) = y_0 \end{cases} \tag{13}$$

Where q is a positive integer and:

$$v\,(s) = y\big(\gamma(s)\big), \quad \bar{f}\big(s, v(s)\big) = \gamma'(s) f\big(\gamma(s), v(s)\big),$$

$$\bar{K}\big(s, \eta, v(\eta), v'(\eta)\big) = \gamma'(s)\gamma'(\eta) k\left(\gamma(s), \gamma(\eta), v\,(\eta) \frac{1}{\gamma'(\eta)} v'(\eta)\right)$$

We rewrite the solution of the first equation of (13) as follows:

$$v\,(s) = \bar{f}\big(s, v(s)\big) + \int_0^s (s-\eta)^{-\mu}\, \hat{K}\big(s, \eta, v'(\eta)\big) d\eta, \qquad s \in \big[0, \sqrt[q]{T}\big] \tag{14}$$

$$\hat{k}\big(s, \eta, v(\eta), v'^{(\eta)}\big) = \begin{cases} \left(\frac{\gamma(s) - \gamma(\eta)}{s - \eta}\right) \hat{K}\big(s, \eta, v(\eta), v'^{(\eta)}\big), & s \ne \eta, \\ \big(\gamma'(s)\big)^{-u} \bar{K}\big(s, \eta, v(\eta), v'^{(\eta)}\big), \end{cases}$$

And $\hat{k}\big(s, \eta, v(\eta), v'(\eta)\big)$ is smooth, if $\bar{K}\big(s, \eta, v(\eta), v'(\eta)\big)$ is smooth. Therefore, the solution of (14) with the initial condition $v(0) = y_0$ applies in the following relationship [11]:

$$v^m(s) \sim s^{q(2-\mu)-m}, \qquad s \to 0,$$

Therefore, the solution of the transformed equation can be smoothed with the appropriate choice of q, and therefore we can easily use the collocation-spectral method to solve it. When $v(s)$ is known, using inverse transformations:

$$y'(t) = \frac{v'\big(\gamma^{-1}(t)\big)}{\gamma'\big(\gamma^{-1}(t)\big)}, \qquad y(t) = v\big(\gamma^{-1}(t)\big).$$

$$y'(t) = \frac{v'\big(\gamma^{-1}(t)\big)}{\gamma'\big(\gamma^{-1}(t)\big)}, \qquad y(t) = v\big(\gamma^{-1}(t)\big).$$

The value of the main function $y(t)$ and its derivative $y'(t)$ are obtained.

3.1. Numerical approximation with Jacobi collocation method

In this section, we use the collocation-spectral Jacobi method to numerically solve the transformed equation. We assume that $w^{\alpha,\beta}(x) = (1-x)^{\alpha}(1-x)^{\beta}$ is a weight function where $\alpha, \beta > -1$ for a given positive integer N Gauss points Jacobi corresponding to the weight function $w^{\alpha,\beta}(x)$ is denoted by $\{x_i\}_{i=0}^{N}$ and its corresponding weights are denoted by $\{w_i\}_{i=0}^{N}$. Suppose that P_N represents the space of all maximal polynomials of degree N. For any continuous function ν in the Lagrange polynomial, we show it as $\mathfrak{T}_N^{\alpha,\beta}\nu \in P_N$, which applies to the following relationship [12]:

$$\mathfrak{T}_N^{\alpha,\beta}\nu(x_i) = \nu(x_i), \qquad 0 \le i \le N,$$

The Lagrange interpolator polynomial can be written as follows:

$$\mathfrak{T}_N^{\alpha,\beta}\nu(x_i) = \sum_{i=0}^{N} \nu(x_i)F_i(x),$$

Where $F_i(x)$ is the basis function of the Lagrange interpolator depending on the points $\{x_i\}_{i=0}^{N}$. To apply spectral methods on the standard interval $[-1,1]$ using variable transformations:

$$s = \frac{\sqrt[q]{T}}{2}(1+x), \qquad \eta = \frac{\sqrt[q]{T}}{2}(1+\xi)$$

We write (13) with its initial condition as follows:

$$u'(x) = g(x,u(x)) + \int_{-1}^{x} (x-\xi)^{-u}\tilde{K}\big(x,\xi,u(\xi),u'(\xi)\big)d\xi, \quad [-1,1] \tag{15}$$

$$u(-1) = u_{-1} = y_0, \tag{16}$$

In which:

$$u(x) = \nu\left(\frac{\sqrt[q]{T}}{2}(1+x)\right), \qquad g(x,u(x)) = \frac{\sqrt[q]{T}}{2}\bar{f}\left(\frac{\sqrt[q]{T}}{2}(1+x), u(x)\right),$$

$$\tilde{k}(x,\xi,u(\xi),u'(\xi)) = \left(\frac{\sqrt[q]{T}}{2}\right)^{2-u} \tilde{K}\left(\frac{\sqrt[q]{T}}{2}(1+x), \frac{\sqrt[q]{T}}{2}(1+\xi), u(\xi), \frac{2}{\sqrt[q]{T}}u'(\xi)\right),$$

To facilitate the analysis of equations, we rewrite (15) as follows:

$$u(x) = u_{-1} + \int_{-1}^{x} u'(\xi)d\xi$$

First, we assume that (15) and (16) hold at the collocation points $\{x_i\}_{i=0}^{N} \in (-1,1)$, or equivalently:

$$u'(x_i) = g(x_i,u(x_i)) + \int_{-1}^{x_i} u'(x_i-\xi)^{-\mu}\tilde{k}(x_i,\xi,u(\xi),u'(\xi))d\xi \tag{17}$$

$$u(x_i) = u_{-1} + \int_{-1}^{x_i} u'^{(\xi)}\, d\xi \tag{18}$$

To calculate the integral expressions in the above equations by Gauss integration rule, we move the integration interval $[-1, x_i]$ to the interval [-1,1]. Therefore, based on this idea, using variable change:

$$\xi = \xi(x_i,\theta) = \frac{1+x_i}{2}\theta + \frac{x_i-1}{2}, \qquad -1 \le \theta \le 1, \tag{19}$$

The integral expressions in (17) and (18) are written as follows:

$$\int_{-1}^{x_i} u'(x_i-\xi)^{-\mu}\tilde{k}(x_i,\xi,u(\xi),u'^{(\xi)})d\xi = \int_{-1}^{-1} (1-\theta)^{-\mu}k_1(x_i,\xi(x_i,\theta),u(\xi(x_i,\theta)),u'^{(\xi(x_i,\theta))})d\theta \quad (20)$$

$k_1\left(x_i,\xi(x_i,\theta),u(\xi(x_i,\theta)),u'(\xi(x_i,\theta))\right)d\theta,$

$$\int_{-1}^{x_i} u'(\xi)d\xi = \frac{1+x_i}{2}\int_{-1}^{1} u'\xi(x_i,\theta)d\theta \quad (21)$$

In which,

$$\left(x_i,\xi(x_i,\theta),u(\xi(x_i,\theta)),u'(\xi(x_i,\theta))\right) = \left(\frac{1+x_i}{2}\right)^{1-u}\tilde{k}\left(x_i,\xi(x_i,\theta),u(\xi(x_i,\theta)),u'(\xi(x_i,\theta))\right)$$

Using Gauss integral formulas, we approximate the integral expressions, in which case the above equations are written as follows:

$$\int_{-1}^{-1} (1-\theta)^{-\mu}k_1\left(x_i,\xi(x_i,\theta),u(\xi(x_i,\theta)),u'(\xi(x_i,\theta))\right)d\theta$$

$$\approx \sum_{k=0}^{N} k_1\left(\xi(x_i,\bar{\theta}_k),u\left(\xi(x_i,\bar{\theta}_k)\right),u'^{\left(\xi(x_i,\bar{\theta}_k)\right)}\right)\overline{w}_k \quad (22)$$

$$\int_{-1}^{-1} u'^{(\xi(x_i,\theta))}d\theta \approx \sum_{k=0}^{N} u'^{\left(\xi(x_i,\hat{\theta}_k)\right)\widehat{w}_k} \quad (23)$$

Where $\{\bar{\theta}_k\}_{k=0}^{N}$ and $\{\hat{\theta}_k\}_{k=0}^{N}$ respectively represent Gauss-Jacobi points corresponding to weights $w^{-\mu,0}(\theta)$ and $w^{0,0}(\theta)$ [13].

Now we assume that u_i' and u_i are the approximate values of the functions $u(x_i)$ and $u'(x_i)$, respectively. We expand the functions $u'(x)$ and $u(x)$ using Lagrange interpolator polynomials as follows:

$$u'(x) \approx u_N'(x) = \sum_{j=0}^{N} u_j' F_j(x), \qquad u(x) \approx u_N(x) = \sum_{j=0}^{N} u_j F_j(x) \quad (24)$$

In this case, Jacobi's collocation method is used to search for the values $\{u_i'\}_{i=0}^{N}$ and $\{u_i\}_{i=0}^{N}$, which apply to the following collocation equations:

$$u_i' = g(x_i,u_i) + \sum_{k=0}^{N} k_1\left(x_i,\xi(x_i,\bar{\theta}_k),\sum_{j=0}^{N} u_j F_j\left(\xi(x_i,\bar{\theta}_k)\right),\sum_{j=0}^{N} u_j' F_j\left(\xi(x_i,\bar{\theta}_k)\right),\overline{w}_k\right) \quad (25)$$

$$u_i = u_{-1} + \frac{1+x_i}{2}\sum_{j=0}^{N} u_j'\left(\sum_{j=0}^{N} F_j\left(\xi(x_i,\hat{\theta}_k)\widehat{w}_k\right)\right) \quad (26)$$

The above equations form a $2N+2$ equation device with $2N+2$ unknowns. After solving this device and finding the unknowns u_i' and u_i, we can obtain the approximate solutions of $u_N'(x)$ and $u_N(x)$ by placing these values in (24) [14].

Note 3.1

Since the degree of polynomials $F_{j,}\ j = 0,1,\dots,N$ does not exceed N, therefore we have:

$$\int_{-1}^{x_i}\sum_{j=0}^{N} u_j' F_j(\xi)d\xi = \frac{1+x_i}{2}\int_{-1}^{1}\sum_{j=0}^{N} u_j'F_j\left(\xi(x_i,\theta)\right)d\theta = \frac{1+x_i}{2}\sum_{j=0}^{N} u_j'\left(\sum_{j=0}^{N} F_j\left(\xi(x_i,\hat{\theta}_k)\widehat{w}_k\right)\right)$$

4. Numerical examples

In this part, in order to show the efficiency and accuracy of Jacobi's collocation approximation for the solution of non-linear weakly Volterra integro-differential equations, some examples of these equations are solved with this approximation method and we compare the obtained numerical results with the numerical results of the proposed methods in other references. To analyze the behavior of the approximate solutions $u_N(x)$ and $u'_N(x)$, we define the softs L^2 and L^∞ for them as follows:

$$\|e_u\|_{L^2_{w^{\alpha,\beta}(-1,1)}} = \left(\int_{-1}^{1} |e_u|^2 w^{\alpha,\beta}(x)dx\right)^{\frac{1}{2}},$$

$$\|e_u\|_{L^2_{w^{\alpha,\beta}(-1,1)}} = \left(\int_{-1}^{1} |e_{u'}|^2 w^{\alpha,\beta}(x)dx\right)^{\frac{1}{2}},$$

$$\|e_u\|_{L^\infty}(-1,1) = \max_{0\le i\le N},$$

$$\|e_{u'}\|_{L^\infty}(-1,1) = \max_{0\le i\le N} |u'(x_i) - u'_N(x_i)|,$$

Where $w^{\alpha,\beta}(x) = (1-x)^\alpha (1-x)^\beta, \alpha,\beta > -1$, it should be noted that all numerical calculations using It was done by Mathematica software.

Example 4.1

We consider the following nonlinear weakly integro-differential equation of Volterra:

$$\begin{cases} y'(t) = f\big(t, y(t)\big) + \int_0^t (t-\sigma)^{-\frac{1}{3}} K(t,\sigma,y(\sigma),y'(\sigma))d\sigma, \quad t \in [0,8], \\ y(0) = e^{-1}, \end{cases} \tag{27}$$

In which,

$$K\big(t,\sigma,y(\sigma),y'(\sigma)\big) = \frac{1}{9} t^{-\frac{2}{3}} \sigma^{-\frac{2}{3}} \left(t^{\frac{2}{3}} + t^{\frac{1}{3}}\sigma^{\frac{1}{3}} + \sigma^{\frac{2}{3}}\right)\left(t^{\frac{1}{3}} - 1\right)^2 \left(e^{-2\left(\sigma^{\frac{1}{3}}-1\right)} y^2(\sigma) + log\left(3\sigma^{\frac{2}{3}} y'(\sigma)\right)\right),$$

$$f(t,y) = \frac{y}{3} t^{\frac{2}{3}} - \frac{1}{162} log^2(y) t^{\frac{1}{3}} \left(27 + 2\sqrt{3\pi}\left(2 + \frac{2^{\frac{2}{3}}\sqrt{\pi}}{\Gamma\left[\frac{7}{6}\right]}\right)\right)$$

The exact solution of this equation is equal to. $y(t) = e^{\sqrt[3]{t}-1} = e^{t-3}$ in order to overcome the unevenness of the derivatives, we use the following variable change for the solution of this equation near $t = 0$:

$t = \gamma(s) = s^3, \qquad \sigma = \gamma(\eta) = \eta^3$

By using this variable change, the solution of the modified equation is equal to $v(s) = y(\gamma(s)) = e^{s-1}$, in order to numerically approximate the solution of this equation and also the derivative of this solution, first this transferred the equation to the interval $-1,1$ using variable change,

$s = 1 + x, \qquad \eta = 1 + \xi$

And then we approximate the solution of the modified equation using the collocation method.

$u(x) = v(1 + x).$

Table 1 shows the approximate solution error $u_n(x)$ and its derivative $u'_n(x)$ for different values of N.

In Fig. 1, we have drawn the corresponding numerical error in the L^2 and L^∞ software. These numerical results show that with the increase of N, the error decreases rapidly, so the speed of the spectral method can be seen from these results.

Table 1 - Error of $u_N(x)$ and $u'_N(x)$ for $\alpha = -\frac{13}{50}$ and $\beta = -\frac{7}{25}$ values in Example 4.1.

N	$\|u - u_N\|_{w^{\alpha,\beta}}$	$\|u - u_N\|_\infty$	$\|u' - u'_N\|_{w^{\alpha,\beta}}$	$\|u' - u'_N\|_\infty$
2	$3/66 \times 10^{-2}$	$1/3 \times 10^{-2}$	$3/25 \times 10^{-2}$	$1/87 \times 10^{-2}$
3	$1/14 \times 10^{-2}$	$1/54 \times 10^{-2}$	$1/76 \times 10^{-2}$	$3/29 \times 10^{-2}$
4	$7/41 \times 10^{-4}$	$8/91 \times 10^{-4}$	$8/19 \times 10^{-4}$	$1/65 \times 10^{-3}$
5	$5/10 \times 10^{-5}$	$5/60 \times 10^{-5}$	$6/72 \times 10^{-5}$	$1/54 \times 10^{-4}$
6	$2/86 \times 10^{-6}$	$2/1 \times 10^{-6}$	$3/31 \times 10^{-6}$	$2/67 \times 10^{-6}$
7	$1/69 \times 10^{-8}$	$5/98 \times 10^{-7}$	$1/85 \times 10^{-7}$	$2/74 \times 10^{-7}$
8	$9/23 \times 10^{-9}$	$2/3 \times 10^{-9}$	$1/01 \times 10^{-8}$	$8/58 \times 10^{-9}$

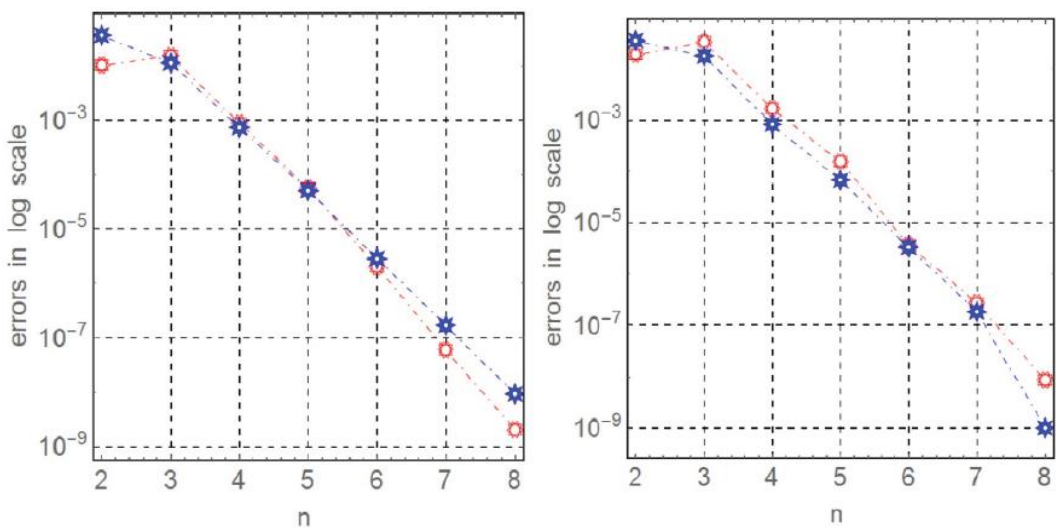

Fig. 1 - $L^\infty LL^2_{w\alpha,\beta}$ errors for the function $u_N(x)$ (left) and its derivative $u'_N(x)$ (right) in Example 4.1.

In Fig. 2, we have drawn the exact solution of $u(x)$ and its derivative $u'(x)$, as well as the approximations of these functions, i.e. $u_N(x)$ and $u'_N(x)$. As can be seen, for a small value of $N = 3$, the approximate and original solutions are completely matched with each other, which indicates the high accuracy of the approximation.

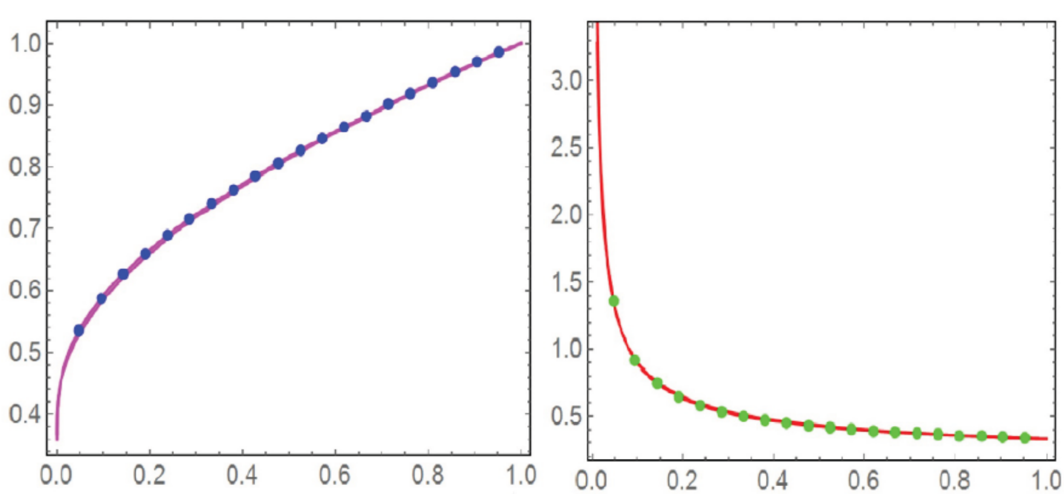

Fig. 2 - Comparison of y and its approximation y_N (left) and comparison of y' and its approximation y'_N (right) in Example 4.1.

Example 4.2

We consider the following weakly nonlinear integro-differential equation of Volterra:

$$\begin{cases} y'(t) = f\big(t, y(t)\big) + \int_0^t (t-\sigma)^{-\frac{1}{3}} K(t,\sigma,y(\sigma),y'(\sigma))d\sigma, & t \in [0,1] \\ \quad y(0) = 0, \end{cases} \tag{28}$$

In which:

$$k(t,\sigma,y(\sigma),y'(\sigma)) = y^3(\sigma)$$

$$f\big(t,y(t)\big) = \frac{1}{3}t^{-\frac{2}{3}} - \frac{9}{10}t^{\frac{5}{3}}$$

The exact answer of this equation is equal to $y(t) = \sqrt[3]{t}$. As in the previous example, to overcome the unevenness of the derivatives of the solution of this equation near $t = 0$, we use the following variable change.

By using this variable change, the solution of the changed equation is equal to $v(s) = y\big(\gamma(s)\big) = s$, in order to numerically approximate the solution of this equation as well as the derivative of this solution, first this equation by using the variable change to the interval $[-1,1]$ transfer data.

$$s = \frac{1+x}{2}, \qquad \eta = \frac{1+\xi}{2},$$

And then we approximate the solution of the modified equation using the collocation method.

$$u(x) = v\left(\frac{1+x}{2}\right).$$

Table 2 shows the approximate answer error $u_N(x)$ and its derivative $u'_N(x)$ for different values of $N = 3$.

Table 2 - Error of $u_N(x)$ and $u'_N(x)$ for $\alpha = -\frac{13}{50}$ and $\alpha = -\frac{13}{50}$ values in Example 4.2.

N	$\|u - u_N\|_{w^{\alpha,\beta}}$	$\|u - u_N\|_\infty$	$\|u' - u'_N\|_{w^{\alpha,\beta}}$	$\|u' - u'_N\|_\infty$
2	$4/9 \times 10^{-5}$	$7/1 \times 10^{-5}$	$1/41 \times 10^{-4}$	$2/42 \times 10^{-4}$
3	$3/85 \times 10^{-7}$	$9/16 \times 10^{-10}$	$1/81 \times 10^{-6}$	$4/18 \times 10^{-6}$
4	$5/17 \times 10^{-9}$	$1/51 \times 10^{-8}$	$2/65 \times 10^{-8}$	$7/46 \times 10^{-8}$
5	$2/76 \times 10^{-10}$	$8/88 \times 10^{-10}$	$1/40 \times 10^{-9}$	$4/45 \times 10^{-10}$
6	$2/51 \times 10^{-12}$	$8/61 \times 10^{-12}$	$1/27 \times 10^{-11}$	$4/35 \times 10^{-11}$
7	$2/72 \times 10^{-13}$	$9/71 \times 10^{-13}$	$1/38 \times 10^{-12}$	$4/91 \times 10^{-12}$
8	$4/57 \times 10^{-15}$	$1/20 \times 10^{-14}$	$2/64 \times 10^{-14}$	$9/73 \times 10^{-14}$

In Fig. 3, we have drawn the corresponding numerical error in the L^2 and L^∞ softs. These numerical results show that with the increase of N, the error decreases rapidly, and therefore the speed of the spectral method can be seen from these results.

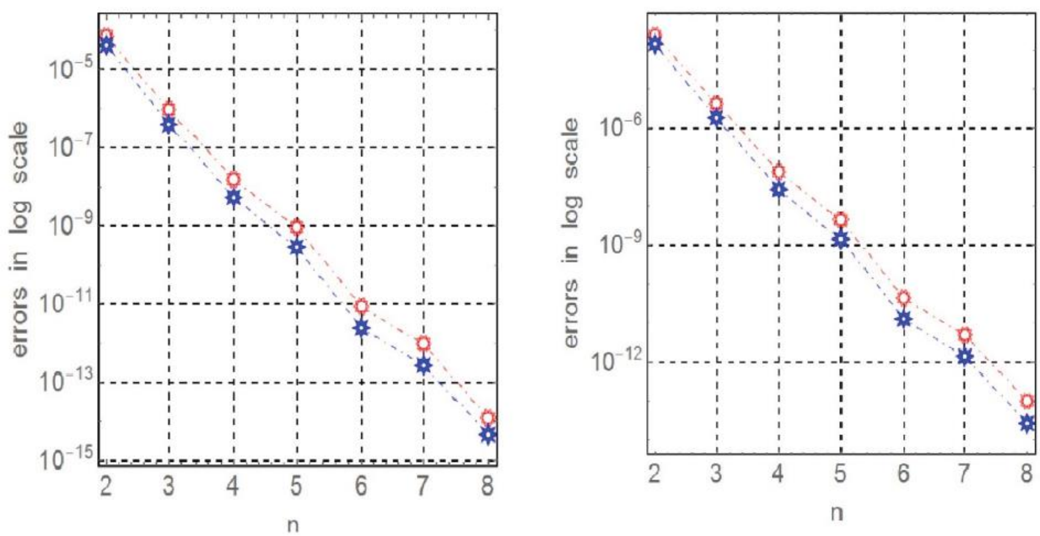

Fig. 3 - L^∞ and $L^2_{w^{\alpha,\beta}}$ errors for the function $u_N(x)$ (left) and its derivative $u'_N(x)$ (right) in Example 4.2.

5. Conclusion

In this paper, the collocation method for obtaining an approximate solution and an approximate derivative of the solution of weak nonlinear neutral integral-differential equations with smooth solutions is explained in full detail. This method is used by changing some variables in order to change the original equation to a new equation that has a better order, and Jacobi's orthogonal polynomial theory can be easily used. With reasonable assumptions about non-linearity, we proved the convergence of the method and obtained the error in the soft L^∞ and the weighted soft L^2. Numerical examples confirm the theoretical results and show the significant improvement of the proposed methods over some other methods.

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