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On e*-Singular-Hollow-Lifting Modules

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ABSTRACT

This research introduces the innovative notions in module X over a ring R. The first is called $F_{e*S}I$-lifting module, which is an inference of e*S-lifting. The second concept is e*S-hollow-lifting, which is a generalization of the e*S-lifting module. We will illustrate a few of these concept attributes.

MSC.

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1. Introduction

In this paper X will be a unitary left R-module, and R is any ring with identity. Notationally, asubmodule T of an R-module X is considered small, which is well known. $T \ll X$, if for all submodule of X, T + L = X, then L = X, [1],[2]. A new submodule type was created by Baanoon in [3] and it is a generalization of essential submodule called e∗-essential as follows. For any non-zero cosingular submodule B of X, if $A \cap B \neq 0$, we say that A is an e*-essential submodule in X. Denoted by $A \leq_{e*} X$. This is the definition of the singular submodule: Z(X) = {m in X: ann(m) $\leq_e$R}[4]. We generalized Z(X) to Z_{e*}(X), by applying e*_essential submodules. Let X be a module define Z_{e*}(X) = {w in X: ann(w) $\leq_{e*}$ R}, X is called e*_singular module if Z_{e*}(X) = X, and X is called e*_nonsingular module if Z_{e*}(X) = 0 [5]. The generalization of small submodule known as e*S-small submodule is introduced in [5], by A. Kabban and W. Khalid. A submodule T of X is called e*S-small submodule of X (signified by $T \ll_{e*S} X$) if whenever X = T + H, with $Z_{e*}(\frac{X}{H}) = \frac{X}{H}$ implies that X = H. A non-zero module X is called e*S-hollow if each proper submodule of X is e*S-small [6]. Let $H \subseteq W \subseteq X$, if $\frac{W}{H} \ll \frac{X}{H}$, then H is called acoessential submodule of W in X [7],[8]. A generalization of the coessential submodule, we present the following as the e*S-coessential submodules in [6]. Let an R–module X and

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W, H ⊆ X. Such that W ⊆ H ⊆ X, then W is called e*S-coessential submodule of H in X (denoted by W $\subseteq_{e*S_{ce}}$ H in X) if $\frac{H}{W} \ll_{e*S} \frac{X}{W}$. We say that H is called e*S-coclosed submodule of X (denoted by H $\subseteq_{e*S_{cc}}$ X) if whenever T $\subseteq_{e*S_{ce}}$ H, (i.e. $\frac{H}{T} \ll_{e*S} \frac{X}{T}$) implies that T = H [6]. Let V and H be submodules of the R-module X. If X = V + H and V ∩ H $\ll_{e*S}$ H (V ∩ B $\ll_{e*S}$ X), then V is called e*S-supplement of H in X. If each a submodule of X has e*S-supplement, then X is called e*S-supplemented module [9]. An R-module X is called ⊕e*S–supplemented if each submodule of X has an e*S–supplement which is a direct summand of X [9]. Defined T β_{e*S}V if $\frac{T+V}{T} \ll_{e*S} \frac{X}{T}$ and $\frac{T+V}{V} \ll_{e*S} \frac{X}{V}$. X is $\underset{e*S}{H}$– Supplemented module if for each submodule T of X there is adirect summand V of X such that T β_{e*S} V, [9]. Any R-module X is called e*S-lifting if for each submodule L of X there is asubmodule V of L such that X = V ⊕ D, where D ⊆ X and L ∩ D $\ll_{e*S}$ D. A submodule L of X is a fully invariant if *h* (L) ⊆ L for every *h* ∈ End (X). Any R-module X is called duo if each submodule of X is fuly invariant [10]. The notion of hollow-lifting modules was first proposed by Orhan, Keskin, and Tribak. An R-module is considered hollow-lifting if for any submodule L of X with $\frac{X}{L}$ is hollow, there is a direct summand V of X such that V isa coessential submodule of L in X, [11]. In this research we will present e*S-hollow-lifting as a generalization of this last concept. We will give properties and examples of it with proofs.

2. Fully invariant e*_Singular-lifting module.

This section presents the notion of fully invariant e*_Singular-lifting modules, including examples and fundamental characteristics.

Remember that amodule X is called FI-lifting if for each a fully invariant submodule L of X, there is adecomposition X = T ⊕ V, such that T ⊆ L and L ∩ V ≪ V. See [12].

Definition 2.1: An R-module X is called **Fully invariant e*_Singular-lifting module** (shortly $\underset{e*S}{FI}$-lifting module) if for each fully invariant submodule L of X, there exists asubmodule T of L such that X = T ⊕ V, where V ⊆ X and L ∩ V $\ll_{e*S}$ V.

The following proposition gives characterization of $\underset{e*S}{FI}$-lifting modules.

Remark 2.2: Let X be any R-module. Then X is $\underset{e*S}{FI}$-lifting module if and only if for each fully invariant submodules L of X, there is asubmodule T of L such that X = T ⊕ V, where V ⊆ X and L ∩ V $\ll_{e*S}$ X.

Proof: Clear by Proposition 13 [5].

Examples and Remarks 2.3:

1) Z_6 as Z-module is $\underset{e*S}{FI}$-lifting module.

2) Q as Z-module is $\underset{e*S}{FI}$-lifting module.

3) Z as Z-module isn't $\underset{e*S}{FI}$-lifting. Since the only direct summand contained in the fully invariant submodule 2Z is {0}, and 2Z is not e*S-small submodule in Z. See examples and remarks 2 [5].

4) It is clear that every e*S-lifting is $\underset{e*S}{FI}$-lifting module. But the converse need not be accurate in general. For example, let X = $Z_8 \oplus Z_2$ as Z-module. Since Z_8 and Z_2 are $\underset{e*S}{FI}$-lifting modules, then X is $\underset{e*S}{FI}$-lifting, see Theorem 2.11, but not e*S-lifting.

5) Let X be a duo module. Then X is e*S-lifting if and only if X is $\underset{e*S}{FI}$-lifting.

6) $\underset{e*S}{FI}$-lifting modules are closed under isomorphisms.

Theorem 2.4: The statements that follow are equivalent if X is an R-module.

1) X is $\underset{e*S}{\text{F I}}$-lifting module.

2) Each fully invariant submodule N of X can be written as N = K ⊕ S, where K is a direct summand of X and S $\ll_{e*S}$ X.

3) Each fully invariant submodule N of X can be written as N = K + S, where K is a direct summand of X and S $\ll_{e*S}$ X.

4) For each fully invariant submodule N of X, there is adirect summand K of X such that K ⊆ N and K $\subseteq_{e*S_{ce}}$N in X.

Proof: 1⇒2) Assume that X is a$\underset{e*S}{\text{F I}}$-lifting module and let N be afully invariant submodule of X, then there is asubmodule K of N such that X = K ⊕ Ḱ, Ḱ ⊆ M and N ∩ Ḱ $\ll_{e*S}$ X, by Remark (2.2). Now, N = N ∩ X = N ∩ (K ⊕ Ḱ) = K ⊕ (N ∩ Ḱ), by Modular law. Hence, we get the result.

2⇒3) Obvious.

3⇒4) Let N be afully invariant submodule of X. By (3), N = K + S, where K is adirect summand of X and S $\ll_{e*S}$ X. So, X = K ⊕ Ḱ, Ḱ ⊆ X. Since Ḱ is e*S-supplement of K in X and S $\ll_{e*S}$X, then Ḱ is an e*S-supplement of K + S = N in X, by Proposition (2.10) [9]. To show that K $\subseteq_{e*S_{ce}}$N in X, let φ: Ḱ → $\frac{X}{K}$ be a map defined by $\varphi(x)$ = x + K, for every $x \in$ Ḱ. Clearly φ is an isomorphism. Since N ∩ Ḱ $\ll_{e*S}$ Ḱ, then φ (N ∩ Ḱ) = $\frac{N}{K} \ll_{e*S} \frac{X}{K}$. Thus, K $\subseteq_{e*S_{ce}}$N in X.

4⇒1) Let N be a fully invariant submodule of X. By (4) there exists a direct summand K of X such that K ⊆ N and $\frac{N}{K} \ll_{e*S} \frac{X}{K}$. We want to prove that N ∩ Ḱ $\ll_{e*S}$ Ḱ. Let Ḱ = (N ∩ Ḱ) + B with $Z_{e*}(\frac{Ḱ}{B}) = \frac{Ḱ}{B}$, where B ⊆ Ḱ. Since X = K + Ḱ = K + (N ∩ Ḱ) + B, then $\frac{X}{K} = \frac{K + (N \cap Ḱ) + B}{K} = \frac{K+(N \cap Ḱ)}{K} + \frac{B+K}{K}$. Since K ⊆ K + (N ∩ Ḱ) ⊆ N and K $\subseteq_{e*S_{ce}}$ N in X. Then K $\subseteq_{e*S_{ce}}$ K + (N ∩ Ḱ) in X, by Proposition (3.5) [6], and $\frac{X}{K+B} = \frac{K+Ḱ}{K+B} = \frac{Ḱ+(K+B)}{K+B} \cong \frac{Ḱ}{Ḱ \cap (K+B)} = \frac{Ḱ}{B}$, by (the Second Isomorphism Theorem and Modular law). Since $Z_{e*}(\frac{Ḱ}{B}) = \frac{Ḱ}{B}$, then $Z_{e*}(\frac{X}{K+B}) = \frac{X}{K+B}$, since $\frac{K+(N \cap Ḱ)}{K} \ll_{e*S} \frac{X}{K}$, and hence $\frac{X}{K} = \frac{B+K}{K}$, implies that X = B + K. Since B ⊆ Ḱ and K ∩ Ḱ = {0}, then B ∩ K = {0} and hence X = K ⊕ B, that is Ḱ = B. Thus, X is $\underset{e*S}{\text{F I}}$-lifting module.

Theorem 2.5: The statements that follow are equivalent if X is an R-module.

1) X is $\underset{e*S}{\text{F I}}$-lifting module.

2) Each fully invariant submodule V of X has an e*S-supplement T in X, where T ⊆ X such that T ∩ V is adirect summand of V.

Proof: 1⇒2) Let X be an $\underset{e*S}{\text{F I}}$-lifting module and V is fully invariant submodule of X. By Theorem 2.4, there exists adirect summand A of X such that A ⊆ V, X = A ⊕ T and A $\subseteq_{e*S_{ce}}$V in X and V ∩ T $\ll_{e*S}$ T. Now V = V ∩ X = V ∩ (A ⊕ T) = A ⊕ (T ∩ V), by Modular law. Since A ⊆ V, then X = V + T and V ∩ T $\ll_{e*S}$ T. Hence T is e*S-supplement of Vin X and T ∩ V is a direct summand of V.

2⇒1) Let V be a fully invariant submodule of X. By (2) V has e*S-supplement T in X such that T ∩ V is a direct summand of V. Then X = T + V, T ∩ V $\ll_{e*S}$ T, and V = (T ∩ V) ⊕ Y, where Y ⊆ V. Since X = T + V = T +

(T ∩ V) + Y = T + V and {0} = T ∩ V ∩ Y = T ∩ Y. Then X = T ⊕ Y and T ∩ V $\ll_{e*S}$ T. Therefore, X is $\underset{e*S}{\text{F I}}$-lifting module.

Proposition 2.6: Let X be any R-module. Then X is $\underset{e*S}{\text{F I}}$-lifting module if and only if for each fully invariant submodule L of X, there is an idempotent φ ∈ End(G) such that φ(G) ⊆ L and (I – φ) (L) $\ll_{e*S}$ (I – φ) (X).

Proof: ⇒) Suppos that X is $\underset{e*S}{\text{F I}}$-lifting module and let L be a fully invariant asubmodule of X. By Theorem 2.5, L has e*S-supplement T in X such that L ∩ T is a direct summand of L, then X = L + T, T ∩ L $\ll_{e*S}$ T and L = (L ∩ T) ⊕ Y, where Y ⊆ L. Now X = L + T = (L ∩ T) + Y + T = Y + T, and L ∩ T ∩ Y = T ∩ Y = {0}, implies that X = T ⊕ Y. Consider the projection map φ: X ⟶ Y it is clear that φ is an idempotent and φ(X) ⊆ Y ⊆ L. It is sufficient to show that (I – φ) (L) $\ll_{e*S}$ (I – φ) (X). One can easily show that (I – φ) (L) = L ∩ (I – φ) (X) = L ∩ T $\ll_{e*S}$ T = (I – φ) (X).

⇐) Let L be a fully invariant submodules of X. By our assumption there is an idempotent φ ∈ End (X) such that φ(X) ⊆ L and (I – φ) (L) $\ll_{e*S}$ (I – φ) (X), clearly that X = φ(X) ⊕ (I – φ)(X) and L ∩ (I – φ) (X) = (I – φ) (L) $\ll_{e*S}$ (I – φ) (X). Therefore, X is $\underset{e*S}{\text{F I}}$-lifting module.

The following Proposition gives another characterization of $\underset{e*S}{\text{F I}}$-lifting module.

Proposition 2.7: The statements that follow are equivalent if X is an R-module.

1) X is $\underset{e*S}{\text{F I}}$-lifting module.

2) Each fully invariant submodule of X has a direct summand e*S-supplement.

3) For every fully invariant submodule N of X, there is an e*S-coclosed submodule T of X and a direct summand e*S-supplement L of T such that T $\subseteq_{e*S_{ce}}$N in X and each homomorphism f: X ⟶ $\frac{X}{T \cap L}$ can be lifted to an endomorphism g: X⟶X such that g (x) + (T ∩ L) = f (x) for all x ∈ X.

Proof: 1⇔2) Suppose that X is $\underset{e*S}{\text{F I}}$-lifting and let N be a fully invariant submodule of X, then there is a direct summand T of X such thatT ⊆ N, X = T ⊕ L, where L ⊆ X and N ∩ L $\ll_{e*S}$ L. Clearly that L is an e*S-supplement of N. Conversely, let N be a fully invariant submodule of X. By our assumptions, there is a direct summand T of X such that X = T ⊕ L and T is an e*S-supplement of N in X. It is enough to show that L ⊆ N. Let the projection map P: M ⟶ L. Since N is fully invariant submodule of X, P(N) = (N + T) ∩ L = X ∩ L = L ⊆ N. Thus, X is $\underset{e*S}{\text{F I}}$-lifting module.

1⇒3) Let N be a fully invariant submodule of X. Since X is $\underset{e*S}{\text{F I}}$-lifting module, there exists a decomposition X = T ⊕ L where T ⊆ N and T $\subseteq_{e*S_{ce}}$N in X, then T is e*S-coclosed submodule of X and it is clear that L is a direct summand e*S-supplement of T in X. Since T ∩ L = {0}, then the result is obtained.

3⇒1) Let N be a fully invariant submodule of X. By (3) there is an e*S-coclosed submodule T of X and a direct summand e*S-supplement L of T such that T ⊆ N and T $\subseteq_{e*S_{ce}}$N in X. It follows from ([13], lemma 2.2), that T is a direct summand of X. Thus, X is $\underset{e*S}{\text{F I}}$-lifting module.

Proposition 2.8: Let X be an R-module. Consider the following statement.

1) X is e*S-lifting module.

2) X is ⊕e*S-supplemented module.

3) X is $\underset{e*S}{\text{F I}}$-lifting module. Then (1)⇒(2)⇒(3). If X is a duo module, then (3)⇒(1).

Proof: 1⇒2) Clear.

2⇒3) Assume that X is ⊕e*S-supplemented and let L be a fully invariant submodule of X, then L has an e*S-supplement which is a direct summand, hence X is $\underset{e*S}{\mathrm{F\,I}}$-lifting module, by Proposition 2.7.

3⇒1) Clear.

Proposition 2.9: Let X be an $\underset{e*S}{\mathrm{H}}$–Supplemented module such that every direct summand of X is e*_Singular, then X is $\underset{e*S}{\mathrm{F\,I}}$-lifting module.

Proof: Let X be an $\underset{e*S}{\mathrm{H}}$–Supplemented and let N be a fully invariant submodule of X, there is a direct summand V of X such that X = V ⊕ K, where K ⊆ X, and N β_{e*S}V. Since X = V + K and $\frac{X}{K} \cong V$ is e*_Singular. Let $\frac{X}{N} = \frac{V+K}{N} = \frac{N+V}{N} + \frac{N+K}{N}$ with $Z_{e*}\left(\frac{X}{N+K}\right) = \frac{X}{N+K}$ and $\frac{N+V}{N} \ll_{e*S} \frac{X}{N}$, then X = N + K. And N = N ∩ X = N ∩ (V ⊕ K) = (N ∩ V) ⊕ (N ∩ K), so $\frac{N+V}{V} \cong \frac{N}{N\cap V} \cong N \cap K$. Since $\frac{N+V}{V} \ll_{e*S} \frac{X}{V}$, thus $N \cap K \ll_{e*S} X$. Then K is an e*S-supplement of N in X. Thus, by Proposition 2.7, X is $\underset{e*S}{\mathrm{F\,I}}$-lifting module.

Proposition 2.10: Let X be a $\underset{e*S}{\mathrm{F\,I}}$-lifting module and let V be a fully invariant direct summand of X, then V is $\underset{e*S}{\mathrm{F\,I}}$-lifting module.

Proof: Let X = V ⊕ L be a $\underset{e*S}{\mathrm{F\,I}}$-lifting and let V be a fully invariant submodule of X. To show that V is a$\underset{e*S}{\mathrm{F\,I}}$-lifting, let Y be a fully invariant submodule of V, then Y is a fully invariant submodule of X, by (Lemma 1.1 [14]), and hence Y = K ⊕ S, where K is a direct summand of X and $S \ll_{e*S} X$, implies that K is a direct summand of V and $S \ll_{e*S} V$, by Proposition 13 [5]. Thus, Vis $\underset{e*S}{\mathrm{F\,I}}$-lifting module.

The following Theorem shows that a finite direct sum of $\underset{e*S}{\mathrm{F\,I}}$-lifting modules is $\underset{e*S}{\mathrm{F\,I}}$-lifting.

Theorem 2.11: Let $X = \oplus_{i=1}^{n} X_i$ be a direct sum of $\underset{e*S}{\mathrm{F\,I}}$-lifting modules. Then X is $\underset{e*S}{\mathrm{F\,I}}$-lifting.

Proof: Let L be a fully invariant submodule of X, then $L = \oplus_{i=1}^{n} (L \cap X_i)$ and $L \cap X_i$ is a fully invariant submodule of X_i, ∀ i = 1, …, n, by (Lemma 1.1 [14]). Since each of X_i is $\underset{e*S}{\mathrm{F\,I}}$-lifting, then $L \cap X_i = K_i \oplus T_i$, where K_i is a direct summand of X_i and $T_i \ll_{e*S} X_i$, ∀ i = 1, …, n. Let $K = \oplus_{i=1}^{n} K_i$ and $T = \oplus_{i=1}^{n} T_i$. It is clear that K is a direct summand of X and $T \ll_{e*S} X$. Thus, X is $\underset{e*S}{\mathrm{F\,I}}$-lifting module.

Proposition 2.12: Let $X = X_1 \oplus X_2$. Then X_2 is$\underset{e*S}{\mathrm{F\,I}}$-lifting module if and only if for each a fully invariant submodule $\frac{N}{X_1}$ of $\frac{X}{X_1}$, there exists a direct summand K of X such that $K \subseteq X_2$, X = K + N and $N \cap K \ll_{e*S} K$.

Proof: ⇒) Assume that X_2 is $\underset{e*S}{\mathrm{F\,I}}$-lifting and let $\frac{N}{X_1}$ be a fully invariant submodule of $\frac{X}{X_1}$. Then $N \cap X_2$ is a fully invariant submodule of X_2, see [15]. Since X_2 is $\underset{e*S}{\mathrm{F\,I}}$-lifting, then there exists a direct summand K of X_2 such that $K \subseteq N \cap X_2$, $X_2 = K \oplus \acute{K}$, $X_2 = (N \cap X_2) + \acute{K}$ and $N \cap \acute{K} \ll_{e*S} \acute{K}$. Clearly that $X = \acute{K} + N$.

⇐) To show thatX_2 is $\underset{e*S}{\mathrm{F\,I}}$-lifting, let N be a fully invariant submodule of X_2. Then $\frac{N \oplus X_1}{X_1}$ is a fully invariant submodule of $\frac{X}{X_1}$, see [15]. By hypothesis, then there exist a direct summand K of X such that K $\subseteq X_2$, $X = K + N + X_1$ and $K \cap (N + X_1) \ll_{e*S} K$. Now, $X_2 = X_2 \cap X = X_2 \cap (K + N + X_1) = K + N$. Itis easy to show that K is an e*S-supplement of N in X_2. Therefore, X_2 is $\underset{e*S}{\mathrm{F\,I}}$-lifting, by Proposition 2.7.

Proposition 2.13: Let X be an $\underset{e*S}{F}I$-lifting module. Then $\frac{X}{N}$ is $\underset{e*S}{F}I$-lifting for every fully invariant submodule N of X.

Proof: Let $\frac{L}{N}$ be a fully invariant submodule of $\frac{X}{N}$. Then by Lemma 1.1 [14], L is fully invariant submodule of X. Since X is $\underset{e*S}{F}I$-lifting module there is B ⊆ L such that X = B ⊕ A, for some submodule A of X, and $\frac{L}{B} \ll_{e*S} \frac{X}{B}$. By Lemma 5.4 [11], we have $\frac{X}{N} = \frac{B+N}{N} \oplus \frac{A+N}{N}$ with $\frac{B+N}{N} \subseteq \frac{L}{N}$. So, $\frac{\frac{L}{N}}{\frac{B+N}{N}} \cong \frac{L}{B+N}$ and $\frac{\frac{X}{N}}{\frac{B+N}{N}} \cong \frac{X}{B+N}$, since $\frac{L}{B} \ll_{e*S} \frac{X}{B}$, by proposition 12 [5], $\frac{L}{B+N} \ll_{e*S} \frac{X}{B+N}$. Therefore, $\frac{X}{N}$ is $\underset{e*S}{F}I$-lifting module.

3. e*_Singular-hollow-lifting Module.

Here we define some of the basic characteristics of e*S-hollow-lifting modules. Moreover, we will show some new results.

Definition 3.1: Let X be any R-module. Then X is called **e*_Singular-hollow-lifting** module (denoted by e*S-hollow-lifting), if for each submodule L of X with $\frac{X}{L}$ is e*S-hollow, there is a direct summand Y of X such that X = Y ⊕ V, for some V ⊆ X and $Y \subseteq_{e*S_{ce}}$ L in X.

We then provide some e*S-hollow-lifting module characterization.

Theorem 3.2: Any R-module X is e*S-hollow-lifting if and only if for each submodule L of X with $\frac{X}{L}$ e*S-hollow, there exists a direct summand Y of L such that X = Y ⊕ V, for some V ⊆ X and $L \cap V \ll_{e*S} V$.

Proof:⇒) Let L be a submodule of X with $\frac{X}{L}$ e*S-hollow. Since X is e*S-hollow-lifting, then there exists a direct summand Y of X such that $Y \subseteq_{e*S_{ce}}$ L in X and X = Y ⊕ V, where V ⊆ X, L = L ∩ X = L ∩ (Y ⊕ V) = Y ⊕ (L ∩ V), by Modular law. We want to show that $L \cap V \ll_{e*S} V$. Where U ⊆ V, Let (L ∩ V) + U = V, with $Z_{e*}(\frac{V}{U}) = \frac{V}{U}$. Then X = L + U. Now $\frac{X}{Y} = \frac{L+U}{Y} = \frac{L}{Y} + \frac{U+Y}{Y}$, since $\frac{X}{U+Y} = \frac{Y+V}{U+Y} = \frac{(U+Y)+V}{U+Y} \cong \frac{V}{V \cap (U+Y)} = \frac{V}{U+(Y \cap V)} = \frac{V}{U}$, by (Second Isomorphism and Modular law). Since $Z_{e*}(\frac{V}{U}) = \frac{V}{U}$, then $Z_{e*}(\frac{X}{U+Y}) = \frac{X}{U+Y}$ and $\frac{L}{Y} \ll_{e*S} \frac{X}{Y}$, therefore $\frac{X}{Y} = \frac{U+Y}{Y}$, so X = U + Y. Since X = Y ⊕ V, and U ⊆ V, then V = U. Thus $L \cap V \ll_{e*S} V$.

⇐) Let L be a submodule of X with $\frac{X}{L}$ is e*S-hollow, then by our assumption, there exists adirect summand Y of L such that X = Y ⊕ V, where V ⊆ X and $L \cap V \ll_{e*S} V$. Let $\frac{L}{Y} + \frac{U}{Y} = \frac{X}{Y}$, with $Z_{e*}(\frac{X}{U}) = \frac{X}{U}$ and U is submodule of X containing Y. Thus X = L + U. By Modular law L = L ∩ X = L ∩ (Y ⊕ V) = Y ⊕ (L ∩ V), and hence X = L + U = Y + (L ∩ V) + U = (L ∩ V) + U. Now, since $L \cap V \ll_{e*S} V$, by Proposition 12 [5], $L \cap V \ll_{e*S}$ X and $Z_{e*}(\frac{X}{U}) = \frac{X}{U}$. So, X = U and $\frac{X}{Y} = \frac{U}{Y}$. Then $\frac{L}{Y} \ll_{e*S} \frac{X}{Y}$, therefore $Y \subseteq_{e*S_{ce}}$ L in X. Thus, X is e*S-hollow-lifting.

Remark 3.3: Any R-module X is e*S-hollow-lifting if and only if foreach submodule L of X with $\frac{X}{L}$ e*S-hollow, there is a direct summand V of L such that X = V ⊕ Y, where Y ⊆ X and $L \cap Y \ll_{e*S}$ X.

Proof: Clear by Proposition 12 [5].

Examples and Remarks 3.4:

1) Z_4 as Z-module is e*S-hollow-lifting module.

2) Q as Z-module is not e*S-hollow-lifting module.

3) The Z-module Z is not e*S-hollow-lifting. For the submodule 4Z, since $\frac{Z}{4Z} \cong Z_4$ is e*S-hollow, and the only direct summand contains in 4Z is {0}. So, $\frac{4Z}{\{0\}} \cong$ 4Z is not e*S-small submodule in $\frac{Z}{\{0\}} \cong$ Z.

4) Let X = $Z_2 \oplus Z_4$ module, obviously Z_2 and Z_4 as Z-module are e*S-hollow modules. Since X = $Z_2 \oplus Z_4$ is an e*S-lifting then itis e*S-hollow-lifting. See by Proposition 3.5.

5) Every simple module is e*S-hollow-lifting. And every e*S-hollow module is e*S-hollow-lifting.

6) Every module with no e*S-hollow factor module is e*S-hollow-lifting.

7) Every e*S-lifting module is e*S-hollow-lifting. But the converses need not be accurate in general. For example, let X be a nonzero indecomposable module with no e*S-hollow factor. Hence X is e*S-hollow-lifting by (6). Claim that X is not e*S-lifting. If not, we have that X is an indecomposable e*S-lifting module. By Proposition 1.6 [6], X is e*S-hollow, and by Corollary 2.7 [6], $\frac{X}{B}$ is e*S-hollow for any proper submodule B of X, which is a contradiction.

Proposition 3.5: Let X_1 and X_2 be e*S-hollow modules. The statements that follow are equivalent if for the module X = $X_1 \oplus X_2$.

1) X is e*S-hollow-lifting module.

2) X is e*S-lifting module.

Proof: 1⇒2) Let L be a submodule of X. Deem the two natural projections homomorphism π_1:X ⟶ X_1 and π_2:X ⟶ X_2. We have two cases:

Case I: If $\pi_1(L) \neq X_1$ and $\pi_2(L) \neq X_2$. Then by our assumption $\pi_1(L) \ll_{e*S} X_1$ and $\pi_2(L) \ll_{e*S} X_2$. So, by Proposition 12 [5], we get $\pi_1(L) \oplus \pi_2(L) \ll_{e*S} X_1 \oplus X_2$. Now, claim that $L \subseteq \pi_1(L) \oplus \pi_2(L)$, to see that, let $l \in L$ then $l \in X = X_1 \oplus X_2$ and hence $l = (x_1, x_2)$, where $x_1 \in X_1, x_2 \in X_2$. Now, $\pi_1(l) = \pi_1(x_1, x_2) = x_1$ and $\pi_2(l) = \pi_2(x_1, x_2) = x_2$. This implies that $l = (\pi_1(l), \pi_2(l))$ and we get $L \subseteq \pi_1(L) \oplus \pi_2(L)$, hence $L \ll_{e*S} X$. Thus, X is e*S-lifting module.

Case II: Now, if $\pi_1(L) = X_1$, then $\pi_1(L) = \pi_1(X)$. So, itis easy to see that X = L + X_2. By (Second Isomorphism Theorem), $\frac{X}{L} = \frac{L+X_2}{L} \cong \frac{X_2}{L \cap X_2}$. Since X_2 is e*S-hollow, then $\frac{X_2}{L \cap X_2}$ is e*S-hollow and thus $\frac{X}{L}$ is e*S-hollow. But X is e*S-hollow-lifting, therefore there is an e*S-coessential submodule of Lin X which is adirect summand of X. Hence, X is e*S-lifting.

2⇒1) Clear.

The following proposition gives acondition to make afactor of e*S-hollow-lifting is an e*S-hollow-lifting module.

Proposition 3.6: Let X be any R-module. If X is e*S-hollow-lifting module, then $\frac{X}{L}$ is e*S-hollow-lifting for each fully invariant submodule L of X.

Proof: Let $\frac{Y}{L}$ be a submodule of $\frac{X}{L}$ such that $\frac{X/L}{Y/L} \cong \frac{X}{Y}$ is e*S-hollow, by (Third Isomorphism Theorem). Since X is e*S-hollow-lifting module, then there is a submodule T of X such that $T \subseteq_{e*S_{ce}} Y$ in X and X = T ⊕ H, for some H ⊆ X. Now, obviously T + L ⊆ Y and thus $\frac{T+L}{L} \subseteq \frac{Y}{L}$. Let $f: \frac{X}{T} \longrightarrow \frac{X}{T+L}$ be a mapping defined by $f(x + T) = x + (T + L)$, for all $x \in X$. It is easy to verify that f is an epimorphism. Since $T \subseteq_{e*S_{ce}} Y$ in X, then by Proposition 12 [5], $f(\frac{Y}{T}) \ll_{e*S} \frac{X}{T+L}$ and thus $f(\frac{Y}{T}) = \frac{Y}{T+L} \ll_{e*S} \frac{X}{T+L}$. So, $T + L \subseteq_{e*S_{ce}} Y$ in X. By (Third

Isomorphism Theorem), we get $\frac{T+L}{L} \subseteq_{e*S_{ce}} \frac{Y}{L}$ in $\frac{X}{L}$. Now, since L is a fully invariant submodule of X, then by Lemma 5.4 [11], $\frac{X}{L} = \frac{T+L}{L} \oplus \frac{H+L}{L}$. Hence $\frac{T+L}{L}$ is a direct summand of $\frac{X}{L}$. Thus, $\frac{X}{L}$ is e*S-hollow-lifting.

A condition under which a direct summand of an e*S-hollow-lifting module is e*S-hollow-lifting is provided by the following Corollary.

Corollary 3.7: Let X be a duo e*S-hollow-lifting module. Then each a direct summand of X is an e*S-hollow-lifting.

Proof: Obvious by Proposition 3.6.

Theorem 3.8: Any R-module X is e*S-hollow-lifting, if and only if for each submodule L of X with $\frac{X}{L}$ e*S-hollow, has e*S-supplement V in X such that V ∩ L is a direct summand of L.

Proof: ⇒) Assume that X is e*S-hollow-lifting and let L ⊆ X with $\frac{X}{L}$ e*S-hollow. Then there is a submodule V of L such that $V \subseteq_{e*S_{ce}}$ L in X and X = V ⊕ C, for some C ⊆ X. By Modular law, L = L ∩ X = L ∩ (V ⊕ C) = V ⊕ (L ∩ C). Then (L ∩ C) is a direct summand of L and X = L + C. Using the same Theorem 3.2 argument, we have $L \cap C \ll_{e*S} C$. Thus, C is e*S-supplement of L in X.

⇐) Let L be a submodule of X with $\frac{X}{L}$ e*S-hollow, thus based on our assumption, there is X = L + V, L ∩ V $\ll_{e*S}$ V, and L = (L ∩ V) ⊕ K, where K ⊆ L. Now, X = L + V = (L ∩ V) + K + V = K + V. Itis clear that K ∩ V = {0}, so X = K ⊕ V. Let $\frac{L}{K} + \frac{Y}{K} = \frac{X}{K}$, with $Z_{e*}(\frac{X}{Y}) = \frac{X}{Y}$, where Y ⊆ X containing K. Then X = L + Y. So, X = (L ∩ V) ⊕ K + Y = (L ∩ V) + Y. Now, since $L \cap V \ll_{e*S} V$, and by Proposition 12 [5], $L \cap V \ll_{e*S} X$, and $Z_{e*}(\frac{X}{Y}) = \frac{X}{Y}$. Then X = Y, and $\frac{Y}{K} = \frac{X}{K}$, thus $\frac{L}{K} \ll_{e*S} \frac{X}{K}$, therefore $K \subseteq_{e*S_{ce}}$ L in X. Then X is e*S-hollow-lifting.

Theorem 3.9: Let X be any R-module. The statements that follow are equivalent.

1) X is e*S-hollow-lifting.

2) Each submodule L of X with $\frac{X}{L}$ e*S-hollow, can be written as L = V ⊕ H, with V is a direct summand of X and $H \ll_{e*S} X$.

3) Each submodule L of X with $\frac{X}{L}$ e*S-hollow, can be written as L = V + H, with V is a direct summand of X and $H \ll_{e*S} X$.

Proof: 1⇒2) Let L be asubmodule of X, with $\frac{X}{L}$ e*S-hollow. Since X is e*S-hollow-lifting, then there is a submodule V of X such that $V \subseteq_{e*S_{ce}}$ L in X and X = V ⊕ Y, where Y ⊆ X. By (Modular law) L = L ∩ X = L ∩ (V ⊕ Y) = V ⊕ (L ∩ Y). By the same argument of Theorem 3.2, we have $L \cap Y \ll_{e*S} Y$, by proposition 12 [5], $L \cap Y \ll_{e*S} X$. Let H = L ∩ Y, so L = V ⊕ H, where V is a direct summand of X and $H \ll_{e*S} X$.

2⇒3) Obvious.

3⇒1) Let L be a submodule of X with $\frac{X}{L}$ e*S-hollow. By (3) L can be written as L = V + H, with V is a direct summand of X and $H \ll_{e*S} X$. We want to show that $V \subseteq_{e*S_{ce}}$ L in X. Let V ⊆ C and $\frac{L}{V} + \frac{C}{V} = \frac{X}{V}$, with $Z_{e*}(\frac{X}{C}) = \frac{X}{X}$. Then X = L + C = V + H + C = H + C. Since $H \ll_{e*S} X$, and $Z_{e*}(\frac{X}{C}) = \frac{X}{C}$, then X = C and $\frac{C}{V} = \frac{X}{V}$. Thus, $\frac{L}{V} \ll_{e*S} \frac{X}{V}$, therefore $V \subseteq_{e*S_{ce}}$ L in X, and X is e*S-hollow-lifting.

Proposition 3.10: Let X be an e*S-hollow-lifting. If X = V + L, where L is a direct summand of X and $\frac{X}{V \cap L}$ is e*S-hollow, then L contains an e*S-supplement of V in X.

Proof: Since X is e*S-hollow-lifting and $\frac{X}{V \cap L}$ is an e*S-hollow, then by Theorem 3.9, V ∩ L = Y ⊕ C, where Y is a direct summand of X and C $\ll_{e*S}$ X. But L is a direct summand of X and C ⊆ L, thus by Proposition 13 [5], C $\ll_{e*S}$ L. Let X = Y ⊕ H, where H ⊆ X. By (Modular law) L = L ∩ X = L ∩ (Y ⊕ H) = Y ⊕ (L ∩ H). Let D = L ∩ H, so X = V + Y + D = V + D. Also, V ∩ L = V ∩ (Y ⊕ D) = Y ⊕ (V ∩ D). Let π: Y ⊕ D ⟶ D be the natural projection map. So, we have V ∩ D = π(Y ⊕ (V ∩ D)) = π(V ∩ L) = π(Y ⊕ C) = π(C). Since C $\ll_{e*S}$ L = Y ⊕ D, then by Proposition 12 [5], π(C) $\ll_{e*S}$ D, and hence V ∩ D $\ll_{e*S}$ D. Thus, D is an e*S-supplement of V in X and D is contained in L.

Proposition 3.11: Let X = $X_1 \oplus X_2$ be a duo module. Then X is e*S-hollow-lifting if and only if X_1 and X_2 are e*S-hollow-lifting.

Proof: ⇒) Obvious by Corollary 3.7.

⇐) Let L be a submodule of X with $\frac{X}{L}$ e*S-hollow. By Lemma 5.4 [11], $\frac{X}{L} = \frac{L + X_1}{L} \oplus \frac{L + X_2}{L}$, since $\frac{X}{L}$ is e*S-hollow, we can assume that $\frac{L + X_1}{L} = \frac{X}{L}$, then $X_2 \subseteq L$. Since $\frac{L + X_1}{L} \cong \frac{X_1}{L \cap X_1}$, by (Second Isomorphism Theorem), and X_1 is e*S-hollow-lifting. Then there is a direct summand V of X_1 such that $\frac{L \cap X_1}{V} \ll_{e*S} \frac{X_1}{V}$, since L = L ∩ X = L ∩ ($X_1 \oplus X_2$), then L = (L ∩ X_1) ⊕ (L ∩ X_2), we get $\frac{L}{V \oplus X_2} \ll_{e*S} \frac{X}{V \oplus X_2}$. Furthermore, it is obvious V ⊕ X_2 is a direct summand of X. Hence, X is e*S-hollow-lifting.

Proposition 3.12: Let X be any R-module. Then X is e*S-hollow-lifting module if and only if for each submodule L of X with $\frac{X}{L}$ e*S-hollow, there is an idempotent Q ∈ End (X) with Q(X) ⊆ L and (I − Q) (L) $\ll_{e*S}$ (I − Q) (X).

Proof: ⇒) Let L be a submodule of X with $\frac{X}{L}$ e*S-hollow. Since X is e*S-hollow-lifting, then by Theorem 3.8, L has an e*S-supplement V in X such that L ∩ V is a direct summand of V, then X = L + V, L ∩ V $\ll_{e*S}$ V and L = (L ∩ V) ⊕ Y, where Y ⊆ L. Then X = L + V = (L ∩ V) + Y + V = Y + V and L ∩ V ∩ Y = V ∩ Y = {0}, and hence X = V ⊕ Y. Define the map that follows now. Q: X→ Y be the natural projection map. It is easy to show that Q is an idempotent and Q (X) ⊆ Y. Since Y ⊆ L, then Q (X) ⊆ L. Now, (I − Q) (X) = {(I − Q) (x), x ∈ X} = {(I − Q) ($c + w$), where c ∈ Y, w ∈ V} = {(I − Q) ($c + w$) = $c + w - c = w$} = V. We aim to prove that (I − Q) (L) = L ∩ (I − Q) (X). Let n ∈ (I − Q) (L), then there is l ∈ L, such that n = (I − Q) (l) = $l - f(l)$. Thus n ∈ L and n ∈ (I − Q) (X). So, n ∈ L ∩ (I − Q) (X). Hence, (I − Q) (L) ⊆ L ∩ (I − Q) (X). Let d ∈ L ∩ (I − Q) (X), then d ∈ L and d ∈ (I − Q) (X). There is y ∈ X such that d = (I − Q) (y) = $y - Q$ (y). Hence, $d + Q$ (y) = y ∈ L, then d ∈ (I − Q) (L). So, (I − Q) (L) = L ∩ (I − Q) (X) = L ∩ V $\ll_{e*S}$ V. Hence (I − Q) (L) $\ll_{e*S}$ (I − Q) (X).

⇐) Let L be a submodule of X with $\frac{X}{L}$ e*S-hollow. By our assumption, there is an idempotent Q ∈ End (X) with Q (X) ⊆ L and (I − Q) (L) $\ll_{e*S}$ (I − Q) (X). Claim that X = Q (X) ⊕ (I − Q) (X). To show that, let x ∈ X, then $x = x + Q(x) - Q(x) = Q(x) + x - Q(x) = Q(x)$ + (I − Q) (x). Thus X = Q (x) + (I − Q) (x). Now, let w ∈ Q (X) ∩ (I − Q) (X), then $w = Q(x_1)$ and w = (I − Q) (x_2), for some x_1, x_2 ∈ X. So, $Q(w) = Q(x_1) = Q$ ((I − Q) (x_2)) = $Q(x_2) - Q(x_2)$ = {0}, then Q (Q (x_1) = Q (x_1) = {0}, thus w = {0}. Thus X = Q (X) ⊕ (I − Q) (X). Obviously, L ∩ (I − Q) (X) = (I − Q) (L). Since (I − Q) (L) $\ll_{e*S}$ (I − Q) (X), then L ∩ (I − Q) (X) $\ll_{e*S}$(I − Q) (X). Thus, X is e*S-hollow-lifting.

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