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On $\left(\widetilde{T}^{*}-\mathcal{N}\right)^{\eta}$ −Fuzzy Soft Quasi Normal Operators

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ABSTRACT

The concept of fuzzy soft sets is considered more versatile and widely applicable than traditional soft set theory, as the latter struggles to handle the fuzziness of problem parameters effectively. In this study, we introduce and describe the $\left(\widetilde{T}^{*}-\eta\right)$ fuzzy soft quasinormal operator in a more concise manner. This operator is defined on a fuzzy soft Hilbert space, based on the notion of a fuzzy soft vector space, which has been adapted to an FS-inner product space in this work. We present various characterizations and results related to the $\left(\widetilde{T}^{*}-\eta\right)$ fuzzy soft quasinormal operator, demonstrating that it serves as an appropriate example of FS-Hilbert spaces, alongside other related examples. Furthermore, we explore the relationships between these categories. Finally, we provide some fundamental guidelines on this topic.

MSC.

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1. Introduction

Complexity in the real world often arises from ambiguity caused by uncertainty. Consequently, we frequently encounter challenging problems across various fields, including economics, engineering, medicine, environmental science, sociology, and business management. Traditional mathematical methods are insufficient for effectively addressing the uncertainties and assumptions inherent in these scenarios. To overcome uncertainty, In 1965, Zadeh proposed the concept of fuzzy sets as an extension of set theory, where fuzziness is mathematically represented. Fuzzy sets are characterized by a membership function that maps elements from a set X to a range [0,1] [1]. Later, in 1999, Molodtsov introduced soft set theory as an expansion of set theory to address uncertainties and solve complex problems that

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traditional methods cannot handle [2]. This theory found applications in various fields, such as control theory, approximation theory, computer science, numerical analysis, and functional analysis. Soft sets, defined as a specified family of subsets with parameters, provide a mathematical model for handling ambiguity.

Building on this foundation, numerous researchers have developed new concepts based on soft sets, presenting examples and exploring their characteristics. These include soft points [3], soft metric spaces [4], soft normed spaces [5], soft pre-Hilbert spaces [6], and soft Hilbert spaces [7], among others. Additionally, soft set theory applications have been demonstrated in various fields [8],[9].

Despite these advancements, we often still lack complete knowledge of the objects under consideration in real-world problems and circumstances. Maji et al. introduced the idea of a fuzzy soft set, merging fuzzy set and soft set theories to provide more comprehensive results [3]. This innovation extended the applicability of soft sets to fuzzy contexts, which are prevalent in many scenarios. The fuzzy soft Hilbert proposal was introduced by Faried et al. in 2020 [7]. Furthermore, they compared the characteristics of FS-linear operators to those of more traditional forms of operators in the Hilbert space. The notion of fuzzy soft self-adjoint operators was also explained and its characteristics were investigated [10]. In 2021, Salim Dawood presented another concept based on fuzzy soft Hilbert spaces [8],[9], establishing the fuzzy soft pre-Hilbert space based on fuzzy soft vectors and the fuzzy soft Hilbert space.

In this study, we build on these earlier works by demonstrating related theorems and defining the $\left(\widetilde{\mathcal{T}}^{*}-\eta\right)$ fuzzy soft quasinormal operator in fuzzy soft Hilbert spaces. The structure of the article is as follows: Section 2 introduces foundational ideas and descriptions, Section 3 examines the characteristics of the $\left(\widetilde{\mathcal{T}}^{*}-\eta\right)$ fuzzy soft quasinormal operator in fuzzy soft Hilbert spaces, including illustrations and relationships to other classes of fuzzy soft operators, and Section 4 presents the conclusions.

2. Basic Concepts of fuzzy soft bounded linear operators

Definition 2.1. [1]:The set contains the elements of order pairs $\widehat{A}$ is namely a fuzzy set up on a crisp set with denoted as $\mathcal{X}$ is a universal set with the function $\mu_{\widehat{A}}: \mathcal{X} \longrightarrow \mathfrak{T}$, namely membership function where $\mathfrak{T} = [0,1]$ such that denoted by $\hat{A} = \left\{\frac{\mu_{\hat{A}}(x)}{x} : x \in \mathcal{X}\right\}$.

Definitions 2.2. [4]:Assume $\mathcal{X}$ is any characterized set and $\mathcal{P}(\mathcal{X})$ shortly of the family of power set of a crisp set $\mathcal{X}$ with a set E of some parameters such that $\mathcal{D} \subseteq E$, we can give the soft set depend on A, the function $\mathcal{J}: \mathcal{D} \to \mathcal{P}(\mathcal{X})$, where $(\mathcal{J}, \mathcal{D}) = \{\mathcal{J}(a) \in \mathcal{P}(\mathcal{X}): a \in \mathcal{D}\}$ and shortly by the order pair $(\mathcal{J}, \mathcal{D})$.

In [3], P. K. Maji, R. Biswas, and A. R. Roy, introduce the new concept of set theory in mathematics namely fuzzy soft set with summarize ($\mathcal{F}$S-sets), we will recall this sets through the following definition

Definition 2.3. [11]:the fuzzy soft point ($\mathcal{FS}$ −point) of $(\mathcal{J}, \mathcal{D})$ over $\mathcal{X}$, shortly as $\tilde{a}_{\mu_{\mathcal{G}(e)}}$, if $e \in A$ and $a \in \mathcal{X}$, $\tilde{a}_{\mu_{\mathcal{G}(e)}}(x) = \begin{cases} \delta & , if\ a = a_o \in \mathcal{X}\ and\ \ e = e_0 \in A \\ 0 & , if\ a \in \mathcal{X} - \{a_0\}\ or\ \ e \in A - \{e_0\} \end{cases}$, such as $\delta \in (0,1]$,

Definition 2.4. [3]:Let $(\mathcal{J}, \mathcal{D})$ be soft set then which is named fuzzy soft set ($\mathcal{F}$S-sets), up on$\mathcal{X}$, whenever the mapping $\mathcal{J}: \mathcal{D} \to \mathfrak{T}^{\mathcal{X}}$, also $\{\mathcal{J}(a) \in \mathfrak{T}^{\mathcal{X}}: a \in \mathcal{D}\}$, and collection of all $\mathcal{F}$S −set shortly $\mathcal{FSS}(\widetilde{\mathcal{X}})$

Remark 2.5. [12]: A family about any $\mathcal{F}$S −Real numbers shortly as $\mathbb{R}(\hat{A})$, also the family about all $\mathcal{F}$S − complex numbers shortly as $\mathcal{C}(\widehat{A})$,

Proposition 2.6. [5]: A set denoted as $\mathcal{FSV}(\widetilde{\mathcal{X}})$ which is $\mathcal{FS}-$ vector sometime ($\mathcal{FS}-$ linear)space, if satisfy as follow for all $\tilde{x}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{y}_{\mu_{2\mathcal{G}(e_2)}} \in \mathcal{FSV}(\widetilde{\mathcal{X}})$

i) $\tilde{x}_{\mu_{1\mathcal{G}(e_1)}} + \tilde{y}_{\mu_{2\mathcal{G}(e_2)}} = (\widetilde{x+y})_{\left(\mu_{1\mathcal{G}(e_1)} + \mu_{1\mathcal{G}(e_1)}\right)}$

ii) $\tilde{r} \cdot \tilde{x}_{\mu_{\mathcal{G}(e)}} = (\widetilde{r \cdot x})_{\mu_{\mathcal{G}(e)}}$, $\tilde{r} \in \tilde{\mathcal{R}}(A)$

Now, we recall the generalization of norm on fuzzy soft vector, through as follow definition

Definition 2.7.[5]**:** Let $\widetilde{\mathcal{X}}$ be $\mathcal{FS}$ –vector space a mapping denoted by $\widetilde{\|.\|}: \widetilde{\mathcal{X}} \to \mathbb{R}(\hat{A})$ named $\mathcal{FS}$ –norm on $\widetilde{\mathcal{X}}$ if the next statements are realized

i) $\left\|\widetilde{\tilde{a}_{\mu_{\mathcal{G}(e)}}}\right\| \geq \tilde{0}$, for each $\tilde{a}_{\mu_{\mathcal{G}(e)}} \in \widetilde{\mathcal{X}}$, and $\tilde{a}_{\mu_{\mathcal{G}(e)}} = \tilde{\theta}$ if and only if $\left\|\widetilde{\tilde{a}_{\mu_{\mathcal{G}(e)}}}\right\| = \tilde{0}$

ii) $\left\|\widetilde{\tilde{r}\tilde{a}_{\mu_{\mathcal{G}(e)}}}\right\| = |\tilde{r}| \left\|\widetilde{\tilde{a}_{\mu_{\mathcal{G}(e)}}}\right\|$, for each $\tilde{a}_{\mu_{\mathcal{G}(e)}} \in \widetilde{\mathcal{X}}$, and the scalar $\tilde{r} \in \mathcal{C}(\hat{A})$.

iii) $\left\|\tilde{a}_{\mu_{\mathcal{G}(e)}} \widetilde{+ \tilde{b}}_{\mu_{2\mathcal{G}(e_2)}}\right\| \leq \left\|\widetilde{\tilde{a}_{\mu_{\mathcal{G}(e)}}}\right\| + \left\|\widetilde{\tilde{b}_{\mu_{2\mathcal{G}(e_2)}}}\right\|$, for each $\tilde{a}_{\mu_{\mathcal{G}(e_1)}}, \tilde{b}_{\mu_{2\mathcal{G}(e_2)}} \in \widetilde{\mathcal{X}}$. Then is a fuzzy norm on fuzzy $\widetilde{\mathcal{X}}$, and the fuzzy order pair $(\widetilde{\mathcal{X}}, \widetilde{\|\ .\|})$ is namely $\mathcal{FS}$ –normed with shortly $\mathcal{FSN}$

Definition 2.8. [13]: Assume $\widetilde{\mathcal{X}}$ is $\mathcal{FS}$ –vector, the mapping denoted by $\widetilde{\langle .,. \rangle} : \widetilde{\mathcal{X}} \times \widetilde{\mathcal{X}} \to \mathcal{C}(\hat{A})$ namely $\mathcal{FS}-$ inner product on $\widetilde{\mathcal{X}}$ with shortly $\mathcal{FS}$I-space if the follow conditions are realized

i) $\langle \tilde{a}_{\mu_{\mathcal{G}(e)}}\widetilde{,\tilde{a}}_{\mu_{\mathcal{G}(e)}} \rangle \geq \tilde{0}$, for each $\tilde{a}_{\mu_{\mathcal{G}(e)}} \in \widetilde{\mathcal{X}}$ with $\tilde{a}_{\mu_{\mathcal{G}(e)}} = \tilde{\theta}$ if and only if $\langle \tilde{a}_{\mu_{\mathcal{G}(e)}}\widetilde{,\tilde{a}}_{\mu_{\mathcal{G}(e)}} \rangle = \tilde{0}$

ii) $\langle \tilde{a}_{\mu_{1\mathcal{G}(e_1)}}\widetilde{,\tilde{b}}_{\mu_{2\mathcal{G}(e_2)}} \rangle = \overline{\langle \tilde{b}_{\mu_{2\mathcal{G}(e_2)}}\widetilde{,\tilde{a}}_{\mu_{1\mathcal{G}(e_1)}} \rangle}$, for each $\tilde{a}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{b}_{\mu_{2\mathcal{G}(e_2)}} \in \widetilde{\mathcal{X}}$.

iii) $\langle \tilde{\alpha}\tilde{a}_{\mu_{1\mathcal{G}(e_1)}}\widetilde{,\tilde{b}}_{\mu_{2\mathcal{G}(e_2)}} \rangle = \tilde{\alpha} \langle \tilde{a}_{\mu_{1\mathcal{G}(e_1)}}\widetilde{,\tilde{b}}_{\mu_{2\mathcal{G}(e_2)}} \rangle$, for each $\tilde{a}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{b}_{\mu_{2\mathcal{G}(e_2)}} \in \widetilde{\mathcal{X}}$, for all $\tilde{\alpha} \in \mathcal{C}(\hat{A})$.

v) for each $\tilde{a}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{b}_{\mu_{2\mathcal{G}(e_2)}}, \tilde{c}_{\mu_{3\mathcal{G}(e_3)}} \in \widetilde{\mathcal{X}}$, $\langle \tilde{a}_{\mu_{1\mathcal{G}(e_1)}} + \widetilde{\tilde{b}_{\mu_{2\mathcal{G}(e_2)}}, \tilde{c}}_{\mu_{3\mathcal{G}(e_3)}} \rangle = \langle \tilde{a}_{\mu_{1\mathcal{G}(e_1)}}\widetilde{,\tilde{c}}_{\mu_{3\mathcal{G}(e_3)}} \rangle + \langle \tilde{b}_{\mu_{2\mathcal{G}(e_2)}}\widetilde{,\tilde{c}}_{\mu_{3\mathcal{G}(e_3)}} \rangle$ then $(\widetilde{\mathcal{X}}, \widetilde{\langle .,. \rangle})$ say $\mathcal{FS}-$ inner product ($\mathcal{FS}-$ pre-Hilbert) with summarize $\mathcal{FS}$I

Definition 2.9 [13]:A $\mathcal{FS}$ –complete space is a $\mathcal{FS}-$ vector, with property for Every sequence that is FS –Cauchy becomes FS –convergent.

Definition 2.10 [13]:A fuzzy soft Hilbert space is $\mathcal{FS}$ –complete with a addition $\mathcal{FS}$I-space shortly $\mathcal{FSH}$-space.

Definition 2.11 [13]: Assumes that $\widetilde{\mathcal{H}}$ is $\mathcal{FSH}-$ space with $\widetilde{\mathcal{T}}: \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$ is $\mathcal{FS}-$ operator, so have:

i) $\widetilde{\mathcal{T}}$ is namely $\mathcal{FS}$ –linear operator with shortly ($\mathcal{FS}$L –operator) if:

$\widetilde{\mathcal{T}}\left(\tilde{\alpha}\tilde{a}_{\mu_{1\mathcal{G}(e_1)}} + \tilde{\beta}\tilde{b}_{\mu_{2\mathcal{G}(e_2)}}\right) = \tilde{\alpha}\widetilde{\mathcal{T}}\left(\tilde{a}_{\mu_{1\mathcal{G}(e_1)}}\right) + \tilde{\beta}\widetilde{\mathcal{T}}\left(\tilde{b}_{\mu_{2\mathcal{G}(e_2)}}\right)$, where $\tilde{a}_{\mu_{1\mathcal{G}(e_1)}}, \tilde{b}_{\mu_{2\mathcal{G}(e_2)}} \in \widetilde{\mathcal{H}}$ and $\tilde{\alpha}, \tilde{\beta} \in \mathcal{C}(\hat{A})$

ii)$\widetilde{\mathcal{T}}$ which is namely $\mathcal{FS}$ –boubed operator with shortly ($\mathcal{FS}b$ –operator) if:

$\exists\ \tilde{m} \in \mathbb{R}(\hat{A})^{+}$ with property $\left\|\tilde{T}\left(\widetilde{\tilde{a}_{\mu_{1_{\mathcal{G}(e_1)}}}}\right)\right\| \leq \tilde{m}\left\|\widetilde{\tilde{a}_{\mu_{1_{\mathcal{G}(e_1)}}}}\right\|$, for all $\tilde{a}_{\mu_{1_{\mathcal{G}(e_1)}}} \in \tilde{\mathcal{H}}$.

This definition uses a fuzzy soft identity operator as an example. define on $\mathcal{FS}$ −Hilbert space $\tilde{I} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ defined by $\tilde{I}\left(\tilde{a}_{\mu_{1_{\mathcal{G}(e_1)}}}\right) = \tilde{a}_{\mu_{1_{\mathcal{G}(e_1)}}}$, $\forall\ \tilde{a}_{\mu_{1_{\mathcal{G}(e_1)}}} \in \tilde{\mathcal{H}}$,It is obvious that$\mathcal{FS}$ −bounded linear operator on $\tilde{\mathcal{H}}$.

Definition 2.12. [14] Assumes that $\tilde{T}: \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ be $\mathcal{FSB}$ −operator define on $\tilde{\mathcal{H}}$ so, $\mathcal{FS}$ − adjoint of operator symbolize by $\tilde{T}^{*}$ defined as $\langle \widetilde{\tilde{T}\tilde{a}_{\mu_{1_{\mathcal{G}(e_1)}}}, \tilde{b}_{\mu_{2_{\mathcal{G}(e_2)}}}} \rangle = \langle \widetilde{\tilde{a}_{\mu_{1_{\mathcal{G}(e_1)}}}, \tilde{T}^{*}\, \tilde{b}_{\mu_{2_{\mathcal{G}(e_2)}}}} \rangle$, where $\tilde{a}_{\mu_{1_{\mathcal{G}(e_1)}}}, \tilde{b}_{\mu_{2_{\mathcal{G}(e_2)}}} \in \tilde{\mathcal{H}}$.

3. Main Results On $(\tilde{T}^{*} - \eta)$ Quasi Normal Operators

Definition 3.1: Assumes that $\tilde{T} : \tilde{\mathcal{H}} \longrightarrow \tilde{\mathcal{H}}$ be a FSB − operator define on$\mathcal{FSH}$ − *space* $\tilde{\mathcal{H}}$ we say that $\tilde{T}$ is $(\tilde{T}^{*} - \mathcal{N})^{\eta}$ −$\mathcal{FSQ}$ −normal operator and shortly $(\tilde{T}^{*} - \mathcal{N})^{\eta}$ − FSQ − normal operator if satisfy the condition $(\tilde{T}^{*})^{\eta}(\tilde{T}^{*}\tilde{T})^{\eta} = \mathcal{N}(\tilde{T}^{*}\tilde{T})^{\eta}(\tilde{T}^{*})^{\eta}$.

Theorem 3.2: a power of $(\tilde{T}^{*} - \mathcal{N})^{\eta}$ − $\mathcal{FSQ}$ − normal operator is again $(\tilde{T}^{*} - \mathcal{N})^{\eta}$ − $\mathcal{FSQ}$ − normal operator

Proof.

we prove that $\tilde{T}^{m}$ is $(\tilde{T}^{*} - \mathcal{N})^{\eta}$ −$\mathcal{FSQ}$ − normal operator

we proceed by induction

$\tilde{T}$ is $(\tilde{T}^{*} - \mathcal{N})^{\eta}$ − $\mathcal{FSQ}$ −normal operator.

Thus, it holds for $m = 1$

$$(\tilde{T}^{*})^{\eta}\ (\tilde{T}^{*}\tilde{T})^{\eta} = \mathcal{N}(\tilde{T}^{*}\tilde{T})^{\eta}\ (\tilde{T}^{*})^{\eta}$$

Assume it also holds for $m = m_1$

$$\left((\tilde{T}^{*})^{\kappa}\ (\tilde{T}^{*}\tilde{T})^{\eta}\right)^{m_1} = \left(\mathcal{N}\ (\tilde{T}^{*}\tilde{T})^{\eta}\ (\tilde{T}^{*})^{\kappa}\right)^{m_1}$$

We need to show that it holds for $m = m_1 + 1$

That is $\left((\tilde{T}^{*})^{\eta}(\tilde{T}^{*}\tilde{T})^{\eta}\right)^{m_1+1} = \left(\mathcal{N}(\tilde{T}^{*}\tilde{T})^{\eta}(\tilde{T}^{*})^{\eta}\right)^{m_1+1}$

$\left((\tilde{T}^{*})^{\eta}(\tilde{T}^{*}\tilde{T})^{\eta}\right)^{m_1+1} = \left((\tilde{T}^{*})^{\eta}(\tilde{T}^{*}\tilde{T})^{\eta}\right)^{m_1}\left((\tilde{T}^{*})^{\eta}(\tilde{T}^{*}\tilde{T})^{\eta}\right)$

and since $\tilde{T}$ is $(\tilde{T}^{*} - \mathcal{N})^{\eta}$ −$\mathcal{FSQ}$ −normal operator, then

$$\left((\tilde{T}^{*})^{\eta}(\tilde{T}^{*}\tilde{T})^{\eta}\right)^{m_1+1} = \left(\mathcal{N}(\tilde{T}^{*}\tilde{T})^{\eta}(\tilde{T}^{*})^{\eta}\right)^{m_1}\left(\mathcal{N}(\tilde{T}^{*}\tilde{T})^{\eta}(\tilde{T}^{*})^{\eta}\right)$$

$$= \left(\mathcal{N}(\tilde{T}^{*}\tilde{T})^{\eta}(\tilde{T}^{*})^{\eta}\right)^{m_1+1}$$

Which concludes the proof for $m = m_1 + 1$

There for $\widetilde{T}^m$ is also $(\widetilde{T}^* - \mathcal{N})^\eta$ fuzzy soft quasi normal operator for each m where $m \geq 1$

Theorem 3.3: If $\widetilde{T}$ and $\tilde{k}$ be two $(\widetilde{T}^* - \mathcal{N})^\eta - FSQ -$normal operator where satisfy the conditions $\widetilde{T}^* \tilde{k}^* = \widetilde{T}^* \tilde{k} = \widetilde{T}\tilde{k} = \tilde{0}$ then $\widetilde{T} + \tilde{k}$ $(\widetilde{T}^* - \mathcal{N})^\eta$ $-FSQ$ $-$normal operator.

Proof.

$$((\widetilde{T} + \tilde{k})^*)^\eta \left((\widetilde{T} + \tilde{k})^* (\widetilde{T} + \tilde{k})\right)^\eta = (\widetilde{T}^* + \tilde{k}^*)^\eta \left((\widetilde{T}^* + \tilde{k}^*)^\eta (\widetilde{T} + \tilde{k})^\eta\right)$$

$$= \left((\widetilde{T}^*)^\eta + \eta(\widetilde{T}^*)^{\eta-1}\tilde{k}^* + \cdots + (\tilde{k}^*)^\eta\right)\left(\left((\widetilde{T}^*)^\eta + n(\widetilde{T}^*)^{\eta-1}\tilde{k}^* + \cdots + (\tilde{k}^*)^\eta\right)\left((\widetilde{T})^\eta + \eta(\widetilde{T})^{\eta-1}\tilde{k} + \cdots + (\tilde{k})^\eta\right)\right)$$

$= \left((\widetilde{T}^*)^\eta + (\tilde{k}^*)^\eta\right)\left((\widetilde{T}^*)^\eta + (\tilde{k}^*)^\eta\right)\left((\widetilde{T})^\eta + (\tilde{k})^\eta\right)$, this from hypothesis of this theorem

$$= \left((\widetilde{T}^*)^\eta + (\tilde{k}^*)^\eta\right)\left((\widetilde{T}^*)^\eta(\widetilde{T})^\eta + (\widetilde{T}^*)^\eta(\tilde{k})^\eta + (\tilde{k}^*)^\eta(\widetilde{T})^\eta + (\tilde{k}^*)^\eta(\tilde{k})^\eta\right)$$

$$= \left((\widetilde{T}^*)^\eta + (\tilde{k}^*)^\eta\right)\left((\widetilde{T}^*\widetilde{T})^\eta + (\widetilde{T}^*\tilde{k})^\eta + (\tilde{k}^*\widetilde{T})^\eta + (\tilde{k}^*\tilde{k})^\eta\right)$$

$$= \left((\widetilde{T}^*)^\eta + (\tilde{k}^*)^\eta\right)\left((\widetilde{T}^*\widetilde{T})^\eta + (\tilde{k}^*\tilde{k})^\eta\right)$$

$$= (\widetilde{T}^*)^\eta(\widetilde{T}^*)^\eta(\widetilde{T})^\eta + (\widetilde{T}^*)^\eta(\tilde{k}^*)^\eta(\tilde{k})^\eta + (\tilde{k}^*)^\eta(\widetilde{T}^*)^\eta(\widetilde{T})^\eta + (\tilde{k}^*)^\eta(\tilde{k}^*)^\eta(\tilde{k})^\eta,$$

$$= (\widetilde{T}^*)^\eta(\widetilde{T}^*)^\eta(\widetilde{T})^\eta + (\tilde{k}^*)^\eta(\tilde{k}^*)^\eta(\tilde{k})^\eta$$

by assumption $\widetilde{T}$ and $\tilde{k}$ are $(\widetilde{T}^* - \mathcal{N})^\eta$ $-\mathcal{F}SQ$ $-$normal operators. one can have

$$= \mathcal{N}\left((\widetilde{T})^*(\widetilde{T})\right)^\eta(\widetilde{T}^*)^\eta + \mathcal{N}\left((\tilde{k})^*(\tilde{k})\right)^\eta(\tilde{k}^*)^\eta$$

$$= \mathcal{N}\left(\left((\widetilde{T})^*(\widetilde{T})\right)^\eta(\widetilde{T}^*)^\eta + \left((\tilde{k})^*(\tilde{k})\right)^\eta(\tilde{k}^*)^\eta\right)$$

Hence $\widetilde{T} + \tilde{k}$ is $(\widetilde{T}^* - \mathcal{N})^\eta$ fuzzy soft quasi normal operator.

Theorem 3.4: Let $\widetilde{T} : \widetilde{\mathcal{H}} \longrightarrow \widetilde{\mathcal{H}}$ be $(\widetilde{T}^* - \mathcal{N})^\eta$ $-\mathcal{F}SQ$ $-$normal operator. and $\tilde{k} : \widetilde{\mathcal{H}} \longrightarrow \widetilde{\mathcal{H}}$ be $\mathcal{F}S -$ $\widetilde{\mathcal{N}}$ $-$quasi normal, then the product $\widetilde{T}\tilde{k}$ is $(\widetilde{T}^* - \mathcal{N})^\eta$ f$-\mathcal{F}SQ$ $-$normal operator., if the following conditions are had $\widetilde{T}\ \tilde{k} = \tilde{k}\widetilde{T}$ and $\tilde{k}^*\widetilde{T} = \widetilde{T}\ \tilde{k}^*$.

Proof. $((\widetilde{T}\tilde{k})^*)^\eta \left((\widetilde{T}\tilde{k})^*(\widetilde{T}\tilde{k})\right)^\eta = ((\tilde{k}\widetilde{T})^*)^\eta \left((\tilde{k}\widetilde{T})^*(\tilde{k}\widetilde{T})\right)^\eta$

$$= ((\tilde{k}\widetilde{T})^*)^\eta \left(((\tilde{k}\widetilde{T})^*)^\eta(\tilde{k}\widetilde{T})^\eta\right),$$

$$= (\widetilde{T}^*\tilde{k}^*)^\eta \left((\widetilde{T}^*\tilde{k}^*)^\eta(\widetilde{T}^\eta\tilde{k}^\eta)\right),$$

$$= \left((\widetilde{T}^*)^\eta(\tilde{k}^*)^\eta\right)\left(\left((\widetilde{T}^*)^n(\tilde{k}^*)^\eta\right)(\widetilde{T}^\eta\tilde{k}^\eta)\right)$$

$= (\widetilde{T}^{*})^{\eta}\left((\tilde{k}^{*})^{\eta}(\widetilde{T}^{*})^{\eta}\right)\left((\tilde{k}^{*})^{\eta}\widetilde{T}^{\eta}\right)\tilde{k}^{\eta}$

$= (\widetilde{T}^{*})^{\eta}\left((\widetilde{T}^{*})^{\eta}(\tilde{k}^{*})^{\eta}\right)\left(\widetilde{T}^{\eta}(\tilde{k}^{*})^{\eta}\right)\tilde{k}^{\eta}$

$= \left((\widetilde{T}^{*})^{\eta}(\widetilde{T}^{*})^{\eta}\right)\left((\tilde{k}^{*})^{\eta}\widetilde{T}^{\eta}\right)\left((\tilde{k}^{*})^{\eta}\tilde{k}^{\eta}\right)$

$= \left((\widetilde{T}^{*})^{\eta}(\widetilde{T}^{*})^{\eta}\right)\left(\widetilde{T}^{\eta}(\tilde{k}^{*})^{\eta}\right)\left((\tilde{k}^{*})^{\eta}\tilde{k}^{\eta}\right)$

$= \left((\widetilde{T}^{*})^{\eta}(\widetilde{T}^{*})^{\eta}\widetilde{T}^{\eta}\right)\left((\tilde{k}^{*})^{\eta}(\tilde{k}^{*})^{\eta}\tilde{k}^{\eta}\right)$ and sine $\widetilde{T}$ is $(\widetilde{T}^{*}-\mathcal{N})^{\eta}-\mathcal{FSQ}$ −normal operator , that lead to

$= \left(\mathcal{N}(\widetilde{T}^{*})^{\eta}\widetilde{T}^{\eta}(\widetilde{T}^{*})^{\eta}\right)\left((\tilde{k}^{*})^{\eta}\tilde{k}^{\eta}(\tilde{k}^{*})^{\eta}\right)$

$= \mathcal{N}\left(\left((\widetilde{T}^{*})^{\eta}\widetilde{T}^{\eta}\right)\left((\widetilde{T}^{*})^{\eta}(\tilde{k}^{*})^{\eta}\right)\left(\tilde{k}^{\eta}(\tilde{k}^{*})^{\eta}\right)\right)$

$= \mathcal{N}\left(\left((\widetilde{T}^{*})^{\eta}\widetilde{T}^{\eta}\right)\left((\tilde{k}^{*})^{\eta}(\widetilde{T}^{*})^{\eta}\right)\left(\tilde{k}^{\eta}(\tilde{k}^{*})^{\eta}\right)\right)$

$= \mathcal{N}(\widetilde{T}^{*})^{\eta}\left(\widetilde{T}^{\eta}(\tilde{k}^{*})^{\eta}\right)\left((\widetilde{T}^{*})^{\eta}\tilde{k}^{\eta}\right)(\tilde{k}^{*})^{\eta}$

$= \mathcal{N}(\widetilde{T}^{*})^{\eta}\left((\tilde{k}^{*})^{\eta}\widetilde{T}^{\eta}\right)\left(\tilde{k}^{\eta}(\widetilde{T}^{*})^{\eta}\right)(\tilde{k}^{*})^{\eta}$

$= \mathcal{N}\left((\widetilde{T}^{*})^{\eta}(\tilde{k}^{*})^{\eta}\right)\left(\widetilde{T}^{\eta}\tilde{k}^{\eta}\right)\left((\widetilde{T}^{*})^{\eta}(\tilde{k}^{*})^{\eta}\right)$

$= \mathcal{N}(\widetilde{T}^{*}\tilde{k}^{*})^{\eta}(\widetilde{T}\tilde{k})^{\eta}(\widetilde{T}^{*}\tilde{k}^{*})^{\eta}$

$= \mathcal{N}((\tilde{k}\,\widetilde{T})^{*})^{\eta}(\tilde{k}\widetilde{T})^{\eta}((\tilde{k}\,\widetilde{T})^{*})^{\eta}$

$= \mathcal{N}\left((\tilde{k}\,\widetilde{T})^{*}(\tilde{k}\widetilde{T})\right)^{\eta}((\tilde{k}\widetilde{T})^{*})^{\eta}$

$= \mathcal{N}((\tilde{k}\,\widetilde{T})^{*})^{\eta}\left((\tilde{k}\,\widetilde{T})^{*}(\tilde{k}\widetilde{T})\right)^{\eta}$

Therefore; $((\widetilde{T}\tilde{k})^{*})^{\eta}\left((\widetilde{T}\tilde{k})^{*}(\widetilde{T}\tilde{k})\right)^{\eta} = \mathcal{N}\left((\widetilde{T}\tilde{k})^{*}(\widetilde{T}\tilde{k})\right)^{\eta}((\widetilde{T}\tilde{k})^{*})^{\eta}$

Hence the product $\widetilde{T}\tilde{k}$ is $(\widetilde{T}^{*}-\mathcal{N})^{\eta}-\mathcal{FSQ}$ −normal operators

Theorem 3.5*:* Let $\widetilde{T}:\widetilde{\mathcal{H}}\longrightarrow\widetilde{\mathcal{H}}$ is $(\widetilde{T}^{*}-\mathcal{N})^{\eta}-\mathcal{FSQ}$ −normal operators then

$\widetilde{T}/_{\widetilde{\mathcal{M}}}$ is $(\widetilde{T}^{*}-\mathcal{N})^{\eta}-\mathcal{FSQ}$ −normal operator such that $\mathcal{N}=\widetilde{N}/_{\widetilde{\mathcal{M}}}$ where $\widetilde{\mathcal{M}}$ is closed subspace.

Proof

Starting from $\left(\left(\widetilde{T}/_{\widetilde{\mathcal{M}}}\right)^{*}\right)^{\eta}\left(\left(\widetilde{T}/_{\widetilde{\mathcal{M}}}\right)^{*}\left(\widetilde{T}/_{\widetilde{\mathcal{M}}}\right)\right)^{\eta} = \left((\widetilde{T}^{*})^{\eta}/_{\widetilde{\mathcal{M}}}\right)\left((\widetilde{T}^{*}\widetilde{T})^{\eta}/_{\widetilde{\mathcal{M}}}\right),$

$= \left({(\tilde{T}^{*})^{\eta} (\tilde{T}^{*}\tilde{T})^{\eta}} / _{\widetilde{\mathcal{M}}} \right)$ and sine $\tilde{T}$ is $(\tilde{T}^{*} - \mathcal{N})^{\eta} - \mathcal{F}SQ$ −normal operator, that lead to

$= \left(\mathcal{N} (\tilde{T}^{*}\tilde{T})^{\eta} (\tilde{T}^{*})^{\eta} / _{\widetilde{\mathcal{M}}} \right)$,

$= \left(\mathcal{N} / _{\widetilde{\mathcal{M}}} \right) \left((\tilde{T}^{*}\tilde{T})^{\eta} / _{\widetilde{\mathcal{M}}} \right) \left((\tilde{T}^{*})^{\eta} / _{\widetilde{\mathcal{M}}} \right)$

$= \mathcal{N} \left((\tilde{T}^{*}\tilde{T})^{\eta} / _{\widetilde{\mathcal{M}}} \right) \left((\tilde{T}^{*})^{\eta} / _{\widetilde{\mathcal{M}}} \right)$

$= \mathcal{N} \left((\tilde{T}^{*})^{\eta} / _{\widetilde{\mathcal{M}}} \right) \left(\tilde{T}^{\eta} / _{\widetilde{\mathcal{M}}} \right) \left((\tilde{T}^{*})^{\eta} / _{\widetilde{\mathcal{M}}} \right)$

$= \mathcal{N} \left(\left(\tilde{T} / _{\widetilde{\mathcal{M}}} \right)^{*} \left(\tilde{T} / _{\widetilde{\mathcal{M}}} \right) \right)^{\eta} \left(\left(\tilde{T} / _{\widetilde{\mathcal{M}}} \right)^{*} \right)^{\eta}$

Therefore; $\left(\left(\tilde{T} / _{\widetilde{\mathcal{M}}} \right)^{*} \right)^{\eta} \left(\left(\tilde{T} / _{\widetilde{\mathcal{M}}} \right)^{*} \left(\tilde{T} / _{\widetilde{\mathcal{M}}} \right) \right)^{\eta} = \mathcal{N} \left(\left(\tilde{T} / _{\widetilde{\mathcal{M}}} \right)^{*} \left(\tilde{T} / _{\widetilde{\mathcal{M}}} \right) \right)^{\eta} \left(\left(\tilde{T} / _{\widetilde{\mathcal{M}}} \right)^{*} \right)^{\eta}$

Then one have the restriction FS −operator $\widetilde{\tilde{T}} / _{\widetilde{\mathcal{M}}}$ is $(\tilde{T}^{*} - \mathcal{N})^{\eta}$ $-\mathcal{F}SQ$ −normal operator.

4. Conclusions

The primary objective of this study is to introduce and define a new fuzzy soft quasinormal operator within the context of fuzzy Hilbert spaces, which are based on fuzzy vector spaces. This work provides a more explicit and thorough understanding of the subject. Several related concepts emerged from this structure, including FS-normed spaces, the generalization of inner product spaces to FS-inner product spaces, fuzzy soft Hilbert spaces, and fuzzy soft bounded linear operators. Additionally, we introduced another variant of the fuzzy soft quasinormal operator, denoted as the$(\tilde{T}^{*} - \mathcal{N})^{\eta}$ fuzzy soft quasinormal operator, or $(\tilde{T}^{*} - \mathcal{N})^{\eta}$-FS-normal operator. We examined various properties and fundamental operations associated with this concept. Furthermore, the study explores the relationships between this fuzzy soft operator and other types, contributing to a deeper understanding of the field.

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