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Review In Fourier series

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ABSTRACT

It is known that Fourier series are one of linear transformations and have many uses in mathematics, physics, engineering...etc. It is used to influence on periodic functions that repeats in a certain period to transform it into a sum of sine and cosine functions also used for non-periodic functions, as we will notice, and it has many other uses. In this article, I study, with examples, the important uses of Fourier series in physics, engineering, and mathematics, with proofs and examples where it necessary. One of results that we have reached in this article is that use Fourier series to facilitated a solution of differential equations in particular partial differential equations by the sinusoidal part of it, also obtaining the best approximation element for a particular function by means of its mathematical mean which called Feger operators. Moreover, there are more than one auxiliary conclusion for, example if a function has two different periods, then the sum of these periods, its subtraction, and its multiplying by constant, will be a new period for this function. Here, I refer to another conclusion that, Fourier series itself do not give the best approximation element for a particular function.

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1.Introduction

Here I divide introduction into two parts:

1.1 Writer Introduction

Fourier series [1],[2],[3] are an important tool for converting periodic functions (including complex periodic functions or those with irregular frequency or wavelength) into a simple series in terms of sines and cosines functions the reason for choosing the sines and cosines functions in a new series is our certain knowledge of the algebraic and differential properties, of these simple functions as well as their period in terms of frequency, wavelength, and many other laws related to them. The expression about the continuous functions by means of series is known in calculus, such as power series, Tyler series, Frobenius series, and others. The conclusion here is that the function which we are talking about it can be expressed by Fourier series like other series. Moreover, if the function is even or odd, the number of the given coefficients required will be reduced to one constant Only (as we will notice below) which makes it much easier to write the function and approximate it with Fourier series, which distinguishes it in this capacity from the rest of the series. Now, one of the problems that we face when expanding (or

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expressing) a function with a certain series is how this series converging to the required function? This is also an important matter. The last problem was solved by the Figer operator, which equals the arithmetic mean of Fourier series to different degrees, as we will notice. Of course, this operator represents best approximation for a given function, as latter, in the approximation theory. Fourier series is distinguished from other series by its wide uses in physics and engineering, as it is possible to solve the wave and heat equations in a way that is perhaps easier than solving with other methods of solving differential equations. It is also possible to treat noise in sounds, distortion of images when printing, and remove unwanted melodies and musical tones from musical pieces by means of representation these waves and images and processed it by Fourier series. In this article, I refer to Fourier series in terms of their definition, forms, and its uses in mathematics, physics, and engineering but I focus on the important conclusions for the uses of Fourier series which related to each part of this article. Of course, we will prove all these conclusions which I will mention in this article, some of it will be mathematical and others will be by Logicalis proof and giving necessary examples wherever it necessary. What is pointed out here is that these series, in addition to their uses in physics, nature, and other sciences, have many uses in mathematics as well. They are studied in applied mathematics in terms of their expansion of the given function, while they are studied in the pure mathematics in terms of studying it and its convergence in sequences of real numbers to obtaining from it the best approximation element of the function in question, and know the degree of this approximation, which will be taught here at Figure operator.

1.2 Historical review of Fourier Series

Jean-Baptiste Joseph Fourier **[4],[5]** a French mathematician, was born in the city of Auxerre. He received his education at the Monastery of Saint Benoît-sur-Loire and continued his education at the so-called École Normale, and taught there and at the Polytechnic School in Paris between 1795 and 1798. In the year 1798, he joined Napoleon Bonaparte's campaign against Egypt, and upon his return to France in 1802, he published important scientific news about Egyptian antiquities. Napoleon Bonaparte granted him the title of Baron in 1808, and in 1816 he was elected a member of the French Academy of Sciences, and then he was elected in 1827 members of the French Academy. Fourier is best known for his contributions to mathematics and physics. In 1822, he published an analytical theory of heat conduction in which he used triangular chains called Fourier Series, thanks to which any periodic function can be expressed as consisting of an infinite series of sines and cosines. A result of his study of the thermal conduction event, Fourier arrived at an important law that states that the thermal current through one area of an object, such as a wall or a window separating two media, is proportional to the temperature gradient to which the object is subject. The constant of proportionality is called the coefficient of thermal conductivity, and is symbolized by the letter K, which is a constant characteristic of matter.

Now, throughout the 18th century, leading mathematicians realized that power series were inadequate for representing functions, and that a different type of series was needed. The problems of mathematical physics led Bernoulli, Alembert, Euler, and Lagrange to strongly question the possibility of representing functions by trigonometric series (bounded by triangular functions).

This discussion sparked One of the crises that constituted an obstacle to the development of analysis. At that on 21, December 1807, the mathematician and physicist Joseph Fourier announced a thesis that began a new chapter in the history of mathematics. Fourier claimed that every function defined in any graph bounds a definite region. It can be decomposed into a sum consisting of periodic sine and cosine functions.

After that, the scientist Fourier discovered what is called the Fourier transform, which is a mathematical process used to convert mathematical functions from the time domain to the frequency domain. It is useful for analyzing signals and knowing the frequencies they contain, and it also has an application in solving differential equations. The name of the process is also derived from the name of the French scientist Fourier, thus completing all the requirements of this new invention.

2.Materials And Methods

2.1. Periodic functions

Let f be a real valued function and let T be a positive real number. We say that the real number T represents a period of the function f if the condition is true

$f\ (x\ +\ T) =\ f\ (x), \forall \mathrm{x} \in \mathrm{R}$ **[6]** . For functions $sinx$ and $cosx$ are periodic functions with period $2n\pi$ this means $\sin(x + 2n\pi) = \sin x$ and $\cos(x + 2n\pi) = \cos x$, $n = 0,1,2,3, \dots$.

The important conclusion here is that if T_1 be a period of f and T_2 be another period of f Then $T_1 + T_2$ be also period of f. To prove this claim:

Suppose that T_1 and T_2 be periods of the function f and let $y = f(x)$ and

$y_1 = f(x + T_1)$, $y_2 = f(x + T_2)$ and $y_3 = f(x + T_1 + T_2)$

This mean $f^{-1}(y) = x$, $f^{-1}(y_1) = x + T_1$ and $f^{-1}(y_2) = x + T_2$ and we must prove that $f\ (x\ +\ T_1 + T_2) =\ f\ (x), \forall \mathrm{x} \in \mathrm{R}$

So, since $y_3 = f(x + T_1 + T_2)$

So, $f^{-1}(y_3) = f^{-1}\big(f(x + T_1 + T_2)\big) = x + T_1 + T_2 = (x + T_1) + T_2$

This mean $f^{-1}(y_3) - T_2 = (x + T_1)$

And then $f(f^{-1}(y_3) - T_2) = f(x + T_1) = f(x)$ So, $f^{-1}(y_3) - T_2 = x$

Thus, $f^{-1}(y_3) = x + T_2$ and then $f(f^{-1}(y_3)) = f(x + T_2) = f(x)$

This mean $y_3 = f(x + T_1 + T_2) = f(x)$.

With same way we can prove that: aT and $T_1 - T_2$ be also period of f.

2.2. Definition and describe of Fourier Series

Suppose that f is a periodic function with period T then $f\ (x\ +\ T) =\ f\ (x), \forall x\ \in R$, also $f\ (x\ + n\, T) = f\ (x), \forall x\ \in R$, $n \in N$,so Fourier series of f is define as:

$f(x) = \frac{a_\circ}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{T} + b_n \sin\frac{n\pi x}{T})$ a_n, b_n **[1]** called Fourier Coefficients

Where $a_n = \frac{1}{T}\int_{-T}^{T} f(x) \cos\frac{n\pi x}{T} dx$ if we choose $n = 0$ we get $a_0 = \frac{1}{T}\int_{-T}^{T} f(x)\, dx$

Also, $b_n = \frac{1}{T}\int_{-T}^{T} f(x) \sin\frac{n\pi x}{T} dx$,where $n = 1,2,3, \dots \dots$ In a_n, b_n.

__**Example**__:

Find Fourier series for the function $f(x) = \begin{cases} 0 & if -3 < x < 0 \\ 4 & if \quad 0 < x < 3 \end{cases}$

Solution: $2T = 6$ so $T = 3$

$$a_0 = \frac{1}{T}\int_{-T}^{T} f(x)\, dx = \frac{1}{3}\left(\int_{-3}^{0} 0\, dx + \int_{0}^{3} 4\, dx\right) = \frac{1}{3}[4x]_0^3 = 4$$

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x) \cos\frac{n\pi x}{T} dx = \frac{1}{3}\left(\int_{-3}^{0} 0 \cos\frac{n\pi x}{3} dx + \int_{0}^{3} 4 \cos\frac{n\pi x}{3} dx\right)$$

$$= \frac{1}{3}\left(\int_0^3 4 \cos\frac{n\pi x}{3} dx\right) = \frac{4}{3}[\left(\frac{3}{n\pi}\sin\frac{n\pi x}{3}\right)]_0^3 = \frac{4}{n\pi}[\sin\frac{n\pi x}{3}]_0^3 = \frac{4}{n\pi} sinn\pi = 0\ if\ n \neq 0$$

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x) \sin\frac{n\pi x}{T} dx = \frac{1}{3}\left(\int_{-3}^{0} 0 \sin\frac{n\pi x}{3} dx + \int_0^3 4 \sin\frac{n\pi x}{3} dx\right)$$

$$= \frac{1}{3}\left(\int_0^3 4 \sin\frac{n\pi x}{3} dx\right) = \frac{4}{3}\left(\frac{3}{n\pi}\cos\frac{n\pi x}{3}\right) = \frac{4}{n\pi}[\cos\frac{n\pi x}{3}]_0^3 = \frac{4}{n\pi}(cosn\pi - 1)$$

So, $f(x) = \frac{4}{2} + \sum_{n=1}^{\infty}\left(0\cos\frac{n\pi x}{3} + \frac{4}{n\pi}(cosn\pi - 1)\sin\frac{n\pi x}{3}\right)$

$= \frac{4}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{n\pi}(cosn\pi - 1)\sin\frac{n\pi x}{3}\right) = \frac{4}{2} + \frac{4}{n\pi}(-2\sin\frac{\pi x}{3} + 0\sin\frac{2\pi x}{3} + (-2)\sin\pi x + 0\sin\frac{4\pi x}{3} + (-2)\sin\frac{5\pi x}{3} + \cdots$

$= \frac{4}{2} + \frac{4}{n\pi}(-2\sin\frac{\pi x}{3} + (-2)\sin\pi x + (-2)\sin\frac{5\pi x}{3} + \cdots$

$= \frac{4}{2} - \frac{8}{n\pi}(\sin\frac{\pi x}{3} + \sin\pi x + \sin\frac{5\pi x}{3} + \cdots$

Consolation here since Fourier series is an infinite sum of sine and cosine functions, which in the end will approximately equal the given function, and since these functions are smooth and have known differential and algebraic properties, so Fourier series represent an excellent tool for approximating the given function.

2.2. Fourier Series for odd and even Functions

Let f be a real function then we say that:**[1]**

1. A function $f(x)$ is even if $f(-x) = f(x)$), for all x in the domain of f.
2. A function f(x) is odd if $f(-x) = -f(x)$, for all x in the domain of f.it is clear that $\cos(-x) = \cos(x)$ so, *cosen* is an even function, also, $\sin(-x) = -\sin(x)$ so, *cosen* is a odd function.now by us our information in calculus :

$\int_{-a}^{a} f(x)dx = 0$ if f is odd and $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$ if f is even.

So, $a_n = 0$ for odd function and $b_n = 0$ for even function, and forier series for it is clear.

For example, of an even function suppose that $f(x) = x^2$ in $[-2,2]$

$$a_0 = \frac{1}{T}\int_{-T}^{T} f(x)\,dx = \frac{1}{2}\int_{-2}^{2} x^2\,dx = \frac{1}{2}[\frac{1}{3}x^3]_{-2}^{2} = \frac{1}{6}[8+8] = \frac{8}{3}.$$

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{n\pi x}{2}dx = \frac{1}{2}[\frac{1}{4}x^2n^2\pi^2\sin(n\pi x) + \frac{1}{2}n\pi\cos(n\pi x) + \sin(n\pi x)]_{-2}^{2}$$

$$= \frac{1}{4}x^2n^2\pi^2\sin(2n\pi) + \frac{1}{2}n\pi\cos(2n\pi) + \sin(n\pi) = \frac{1}{2}n\pi \; if \; n \neq 0$$

$$f(x) = \frac{4}{3} + \sum_{n=1}^{\infty}\left(\frac{1}{2}n\pi\cos\frac{n\pi x}{2}\right) = \frac{4}{3} + \frac{1}{2}\pi(\cos\frac{\pi}{2}x + \cos\pi x + \cos\frac{3\pi}{2}x + \cdots$$

Conclusion here is that because Fourier series for even or odd functions (Specifically, the odd ones)reduction of coefficients $a_n(res. b_n)$ what are called Fourier coefficients so, it provides an excellent and more concise picture of the given function compared with the same function with other types of series .

2.3. Types of Fourier Series

There are two common forms of the Fourier Series, Trigonometric form as above and Exponential form, Trigonometric form (standard form) as we showed previously is:

$$f(x) = \frac{a_\circ}{2} + \sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{T} + b_n\sin\frac{n\pi x}{T})$$

Trigonometric Fourier series uses integration of a periodic signal multiplied by sines and cosines at the fundamental and harmonic frequencies. If performed by hand, this can a painstaking process. Even with the simplifications made possible by exploiting waveform symmetries, there is still a need to integrate cosine and sine terms, now:

if we replace $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$ we get an exponential form as:

$F(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx}$ **[7]**

where $c_n = \frac{1}{2}(a_n + b_n), n = 1,2,3, \dots$ And $c_0 = \frac{a_2}{2}$

For example: if $f(x) = e^x, 0 \leq x \leq 0.5$ we need to find Exponential Fourier series, So:$T_0 = 0.5$, $w_0 = \frac{2\pi,}{T_0} = 4\pi$

$c_n = 2\int_0^{0.5} e^{-x} e^{-i4\pi nx}$ dx $== 2\left[\frac{e^{-(1+i4\pi n)x}}{-(1+i4\pi n)}\right]_0^{0.5} = \frac{2}{1+i4\pi n}\left[1 - e^{-0.5} e^{-i2\pi nx}\right]$

Since $e^{-i2\pi nx} = 1$ So, $c_n = \frac{0.79}{1+i4\pi n}$.

And then, $f(x) = \sum_{n=-\infty}^{\infty} \frac{0.79}{1+i4\pi n} e^{i4\pi nx}$

2.4. Fourier Transformation

From above we learn how to find a Fourier series for a periodic function. Now we have the right to ask if a function which we are talking about it is not a periodic, Is it can have a Fourier series in some way? The answer to this is yes, and the result is called the Fourier transform **[7،1]** which is found by assuming that the period of its time is infinity. This infinite period of time is symbolized by symbol T , since sum of countable n-parts when the time T gone to infinity will lead to the fact that its number n goes to infinity as well, So the Fourier series formula mentioned in the above will be transformed into the integral formula, and the result of this switching process is called Fourier transformation. Now we have the right to ask another question that is: What is the benefit of our derivation of this transformation? The answer is that this transformation used to convert the function from time domain to the frequency domain, As you did.

The stander form is $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$ .Also:

invers of this transformation must be the same function f i.e.$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} dt$ن

Here, I mention Dirichlet's conditions for the existence of the Fourier transform ,that is Fourier transform can be applied to any function if it satisfies the following conditions: 1. is absolutely integrable i.e. is convergent. 2. The function has a finite number of maxima and minima. 3. has only a finite number of discontinuities in any finite, Now, for this transformation let we take the last example:

suppose that $f(x) = \begin{cases} 1 & if\ |x| < a \\ 0 & if\ |x| > a \end{cases}$ find Fourier Transformation for f.

So,$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt = \int_{-\infty}^{\infty} 1. e^{-i\omega t} dt = \frac{.e^{-i\omega t}}{-i\omega} = \frac{.e^{i\omega t} - e^{-i\omega t}}{i\omega} = 2\frac{\sin \omega t}{\omega}, \omega \neq 0.$

For $\omega = 0$ we obtain $F(\omega) = 2\omega$.

Now, conclusion here is that Fourier transformation It converts the function (or signal) from the time domain to the frequency domain, which makes it much easier to study.

3.Results And Discussion

3.1. An application of Fourier Series in physics **[6]** ,[**7]**

There are many applications in physics, Here I limit these applications only to two applications here, namely the heat equation and the wave equation, for solve heat equation which is represented by a partial differential equation describing the distribution of heat over time.one spatial cases at one dimension, we symbol by $u(x,t)$ to the temperature which obeys the relation $\frac{\partial \mu}{\partial x} - \alpha \frac{\partial^2 \mu}{\partial x^2} = 0$.

Let be take this example:

A metal arm of length 2cm and its end temperature are kept at zero, and the initial temperature of the arm is $3x$ What is the temperature of the points of the arm at different times.
Solution: The temperature is represented by the equation:
$\frac{\partial U}{\partial t} = a^2 \frac{\partial^2 U}{\partial x^2}$.,$0 \le x \le 2$, we have $U(0,t) = 0, U(2,t) = 0, U(x,0) = 3x$.
The solution to this equation is given by:
$U(x,t) = \sum b_n e^{-a^2(n\pi/L)^2 .t} \sin(n\pi/L)x$ Where b_n are the sine Fourier coefficients of the function $3x$ on the period $(2,0)$.
Now,$bn = \frac{2}{L}\int_0^L f(x)\ sin\,(n\pi x/L)\,dx = 2/2\int_0^L 3x \sin\left(\frac{n\pi x}{L}\right)dx = 3(2/n\pi)\int_0^L x\,sin\,(n\pi x/2)(n\pi/2)\,dx$

$= 6/n\pi\,\{[-x\,cos\,n\pi x/2]_0^2 + \int_0^2 cos\,(n\pi x/2)\,dx\}$

$= -12/n\pi\,cos\,n\pi = -12/n\pi\,(-1)^n = 12/n\pi\,(-1)^{n+1}$

So, $U(x,t) = 12/\pi \sum_{n=1}^{\infty}(-\frac{1)^{n+1}}{n}.e^{-a^2(n\pi/4)^2 t}.\ sin\,(n\pi/2)\,x$.

Here there is more than one conclusion, as is known that solving differential equations, whether ordinary or partial by using by chains is a general solution. So, the first conclusion here is that Fourier series provide an easier method than other series when you take them in sine form, because constants a_0, a_n become zero. of course, There is no need to give examples here and solve them, as they are well-studied and appropriate. The other conclusion is that in cooling devices such as refrigerators and air conditioners, the heat radiator of these devices heats up and then loses this heat to the surroundings and then cools after the device stops working. Then this same phenomenon is repeated after a period of time. This is the case with the Earth as it heats up during the day due to... The heat of the sun then loses this heat to the ocean at night, and so on. The role of the Fourier series here is a double role, as it is used on the one hand to represent this phenomenon as it is a periodic phenomenon, and then to solve the problem of the spread of heat (its loss) to the ocean. Other conclusion, Fourier series is a mathematical process used to convert mathematical functions from the time domain to the frequency domain The validity of this claim can simply be proven through our understanding that the phenomena that occur in nature are all related to time, so they are in the time domain, and Fourier series work to transform them into sine and cosine functions, which are considered periodic functions, so they are in the frequency domain, which greatly facilitated the study of these functions. phenomena.

3.2. An application of Fourier Series in Engineering

There are many applications for series and Fourier transforms, and we will take just two of them as examples **[7],[8]**

I. Selective Filtering
Fourier series can be used to design filters that remove specific frequency components from a signal while preserving others. This is known as selective filtering. For example, we often design audio filters using the Fourier series to remove unwanted noise from an audio signal while preserving the desired frequencies.

II. Noise Filtering
The Fourier series can be used to remove unwanted noise from a signal. This is known as noise reduction or noise cancellation. For example, active noise cancellation headphones use the Fourier series to remove unwanted background noise from an audio signal.
We notice that in both previous applications, the Fourier transform is used to reduce the percentage of noise or distortion from wireless signals, as well as distortion in images and other distortions occurring in various fields. To know the reason for this useful use (proof), we say that since the Fourier transform transforms the wireless signal into a triangular polynomial consisting of an infinite number of sine and

cosine functions, and since both of them depend on a certain angle, then it is possible to isolate the polynomial that represents the noise or the sound of the sound by Isolate his corner after knowing it. Likewise, when the Fourier transformation transforms the signal from the time domain to the frequency domain, unwanted frequencies can be removed.

3.3. An application of Fourier Series in Approximation Theory

One of the important an applications of Fourier series in approximation theory in mathematics is to obtain the best approximation element of a selected function which be on polynomial form, Since the Fourier series is in the form of a trigonometric polynomial, $f(x) = \frac{a_\circ}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{T} + b_n \sin\frac{n\pi x}{T})$ for a periodic function f

Now, for the n-th sum of this series is:

$s_n = s_n(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin(2n+1)\frac{t-x}{2}}{2\sin\frac{t-x}{2}}$ where $s_n = n$-th sum of Fourier series. **[9]**

And then the norm of the operator $s_n(f)$ is equal to

: $A_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left|\frac{\sin(2n+1)\frac{t}{2}}{2\sin\frac{t}{2}}\right| dt = \frac{4}{\pi^2}\log n + O(1)$ **[9]**

AS we now notice that the value of A_n related with its degree n,and since $\log n$ is increasing function together with increasing value of n, So the difference between select function f and A_n in norm is related with its degree n.Now the most important conclusion here is that Fourier series does not considered as a best approximation element for the consider function f,

Now, how can Fourier series be used to obtain the best approximation of the chosen function, and what is the degree of this approximation? This is what we will discuss in the next section.

3.4. Fejer Operator

Fejer obtain arithmetic mean of an operator $s_n(f)$ which he symbols by $\sigma_n(f,x)$

So, $\sigma_n(f,x) = \frac{s_0+s_1+\cdots+s_{n-1}}{n} = \frac{1}{2n\pi}\int_{-\pi}^{\pi} f(t)(\frac{\sin\frac{n(t-x)}{2}}{2\sin\frac{t-x}{2}})dt$ **[9]**

Of course, the idea of this an operator is from integral operator $\int_{-\pi}^{\pi} K_n(x-t)f(t)dt$

Now , we mention the following proposition in **[10]**

For all $f \in L^2([-\pi;\ \pi])$, we have:

$$\sigma_n(f,x) = \int_{-\pi}^{\pi} K_n(x-t)f(t)dt = \begin{cases} \frac{n+1}{2\pi} & if\ x = 0 \\ \frac{1}{2\pi(n+1)}\left(\frac{\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}}\right) & other\ wise \end{cases}$$

The function $K_n(x)$is called the Fejér kernel, and it has the following properties: (1) $K_n(x) \geq 0$ and $K_n(x) = K_n(-x)$for all x, (2) KN is periodic with period 2π (3) $\int_{-\infty}^{\infty} K_n(x)dt = 1$, and (4) for any $\delta \in (0,\pi)$ and for all

$\delta \le |x| \le \pi$, we have$|K_n(x)| \le \frac{1}{2\pi(n+1)sin^2(\frac{\delta}{2})}$

From above an information, we get the following theorem in [9].[10]

Let $f \in L^2([-\pi;\ \pi])$, be 2π-periodic (so $f(-\pi) = f(\pi)$). Then $\sigma_n(f,x) \to f$uniformly on $[-\pi;\ \pi]$.

The important conclusion here, is that Figer operator can represent the best approximation of the given function Thus, the problem of not obtaining the element of the best approximation of the function by means of Fourier series was solved, and we obtained this element by taking the arithmetic mean of these operators (Fourier series).

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There is no conflict of interest

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