APPROXIMATION METHODS IN THE THEORY OF OPERATOR INCLUSIONS

HISHAMR. Rahman Mohammed

E-mail: hishamrehman@yahoo.com

Abstract. In the present paper we use approximation methods for the study of operator inclusions of the form $a(x) \in \Phi(x)$, where a is a closed linear surjective operator from a Banach space onto another one, and Φ is a multimap being a composition of a multimap with "good" values and a continuous singlevalued map. As application we consider the solvability of an integro- differential system which may be treated as a control object with an integral feedback. Key Words and Phrases: multivalued map, fixed point, coincidence point, continuous selection, operator inclusion, closed linear operator, integro-differential system.

Introduction

The topological and geometrical properties of values of multivalued maps (multimaps) play an important role in the theory of fixed points and in the study of solvability of operator inclusions (see, e.g. [5], [12], [13]).At the present time we may recognize various approaches to these directions: metric (see, e.g. [18]),homological (see, e.g. [5],[7],[12]) and approximation(see,e.g. [1]-[7],[11],[12],[13],[15],[17]). Starting from the research of A.D. Myshkis [17],in a number of works(see, e.g. [1]-[4],[7],[8],[15]and others) the approximation methods were applied to various classes of multimaps with nonconvex values.

In the present paper we use the approximation methods for the study of operator inclusions of the form $a(x) \in \Phi(x)$,where a is a closed linear surjective operator from a Banach space onto another one, and Φ is a multimap being a composition of a multimap with "good" values and a continuous single valued map. The property of a value to be "good" means that this set belongs to some family of subsets described by a suitable collection of axioms. We prove the existence theorem for such inclusions and present conditions under which the solutions set is unbounded. It should be mentioned that for convex-valued multimaps Φ, the inclusions of that form were studied in the paper of the first author[10].

As application we consider the existence result for an integro - deferential system which may be treated as a control object with an integral feedback.

1. Approximate families of sets and Michael systems

For a metric space Y, we denote by $P(Y)$ the collection of all nonempty subsets of Y, by $C(Y)$ the collection of all nonempty closed subsets of Y, and by $K(Y)$ the collection of all nonempty compact subsets of Y. If Y is a subset of a normed space, by $C_v(Y)$ we denote the collection of all nonempty closed convex subsets of Y, and by $K_\upsilon(Y)$ the collection of all nonempty compact convex subsets of Y.

In this section we will cite some notions and results of the paper[11].

Let $(Y,\cdot)$ be a metric space; for any $\varepsilon > 0$,by $U_\varepsilon(B)$ we will denote the ε-neighborhood of a set $B \in P(Y)$.

1.1.Definition. A family $A(Y) \subset C(Y)$ is said to be approximate if there exists a map $\lambda : P(Y) \to A(Y)$ such that:

(A1) $\lambda(B) = B$ *for each* $B \in A(Y)$;

(A2) *if* $B, C \in P(Y)$ *and* $B \subset C$, *then* $\lambda(B) \subset \lambda(C)$;

(A3) for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for each $B \in P(Y)$ the following inclusion holds: $\lambda(U_\delta(B)) \subset U_\varepsilon(\lambda(B))$.

(A4) for each set $B \in P(Y)$,each point $y \in \lambda(B)$ and evry $\varepsilon > 0$,there exist a compact subset $B' \subset B$ and a point $y' \in \lambda(B')$ such that $\rho(y, y') < \varepsilon$.

Consider some examples of approximate families.

Define the map λ: $P(Y) \to C(Y)$ by $\lambda(B) = \overline{B}$. It is easy to see that conditions (A1)-(A4) are satisfied and hence the collection $C(Y)$ is approximate.

For a closed convex subset Y of a normed space, the collection $C_\upsilon(Y)$ is approximate. In fact, the map $\lambda : P(Y) \to C_\upsilon(Y)$ may be defined as $\lambda(B) = \overline{co}(B)$.

Suppose that Z is a closed convex subset of a Banach space; (Y, ρ) is a metric space and there exists a homeomorphism g: Z → Y satisfying the following condition: there exist positive numbers c_1 and c_2 such that

$$c_1\|x-y\| \le \cdot\,(g(x),g(y)) \le c_2\|x-y\|.$$

for each x, y ∈ Z. Consider the collection of sets

$$Ag(Y) = \{g(B) | B \in C_\upsilon(Z)\}.$$

It is easy to verify that the system Ag(Y)is approximate.

Let X and Y be metric spaces. Let us recall(see, e.g. [5],[6],[13]) that a multimap F: X→P(Y) is said to be: upper semi continuous [lower semi continuous] if $F_+^{-1}(V) = \{x \in X: F(x) \subset V$ [respectively $F_-^{-1}(V) = \{x \in X: F(x) \cap V = \varnothing\}$] is open in X for each open V ⊂ Y. The set Γ(F) ⊂X×Y,Γ(F)= {(x , y): y ∈ F(x)} is called the graph of F.A continuous map f: X →Y is said to be:

(i) a continuous selection of a multimap F: $X \to P(Y)$ provided $x \in F(x)$ for each $x \in X$; (ii) a single-valued ε-approximation of F, $\varepsilon > 0$,if $\Gamma(f) \subset U\varepsilon(\Gamma(F))$.

1.2. Definition. A lower semi continuous multimap $F\varepsilon : X \to P(Y)$,$\varepsilon>0$,is said to be a lower semi continuous ε-approximation of a multimap F: $X \to P(Y)$,if:

(i) $F(x) \subset F\varepsilon(x)$ for each $x \in X$;

(ii) $\Gamma(F\varepsilon) \subset U\varepsilon(\Gamma(F))$.

1.3. Theorem. Let A(Y) be an approximate family in a metric space Y, and F: $X \to A(Y)$ an upper semi continuous multimap. Then for every $\varepsilon>0$ there exists a lower semi continuous ε-approximation $F\varepsilon : X \to A(Y)$ such that $F\varepsilon(X) \subset \lambda(F(X))$.

1.4. Definition. A family of nonempty subsets M(Y) of a metric space Y is said to be the Michael system if the following condition holds true: (M) for each metric space X, lower semi continuous multimap F: $X \to M(Y)$, closed subset $A \subset X$ and continuous selection $f : A \to Y$ of the restriction $F|A$, there exists a continuous selection $\tilde{f} : X \to Y$ of a multimap F such that $\tilde{f}|A = f$. A Michael system M(Y) which is also the approximate family will be called a strong Michael system and it will be denoted by AM(Y).

From the classical Michael theorem (see[16]) it follows that the collection of all nonempty convex closed subsets of a Banach space is a strong Michael system. Other examples of strong Michael systems are presented by the collections Ag(Y) (see[11]).

The notion of a strong Michael system is closely related to the existence of single-valued ε-approximations for multimaps. In fact, the next statement follows from Theorem 1.3.

1.5. Theorem. Let F: $X \to AM(Y)$ be an upper semi continuous multimap, then for every $\varepsilon>0$ there exists a single-valued ε-approximation $f\varepsilon$ of F such that $f\varepsilon(X) \subset \lambda(F(X))$.

1.6. Definition. A strong Michael system AM(Y) is called regular if for every compact $K \subset Y$, the set $\lambda(K) \in AM(Y)$ is also compact. If Y is a closed convex subset of a normed space, collections $C_v(Y)$ and Ag(Y) may be considered as examples of a strong Michael system. From Theorem 1.5 we obtain the following statement.

1.7. Corollary. Let the system AM(Y) be regular and an upper Semi continuous multimap $F : X \to AM(Y)$ is compact(i.e. F(X) is relatively compact). Then for every $\varepsilon > 0$ there exists a single-valued compact ε-approximation $f\varepsilon$ of F.

A multimap F: $X \to K(Y)$ will be called completely continuous if it is upper semi continuous and the set $F(\Omega)$is relatively compact for each bounded subset $\Omega \subset X$.

2. On affixed point theorem

We need the following statements that may be easily verified. The first one is the refinement of the theorem on uniform continuity.

2.1. Lemma. Let (X, $\cdot_X$),(Y, $\cdot_Y$)be metric spaces; f: X → Y a continuous map and K ⊂ X a compact set. Then for every ε > 0 there exists δ > 0 such that x', x'' ∈ U_δ (K) and ρ_X (x',x'') < δ imply ρ_Y (f(x'),f(x''))<ε. The second one describes the connection between the fixed points of single-valued approximations and a fixed point of a multimap.

2.2. Lemma. Let M be a closed convex bounded subset of a Banach space E, F : M → K(E) a completely continuous multimap. If there exists ε_0 >0 such that each single-valued ε-approximation f_ε of a multimap F with 0 < ε ≤ ε_0 has a fixed point, then F also has a fixed point.

Let X be a metric space, E a Banach space, Φ : X → K(E) an upper Semi continuous multimap.

2.3. Definition. A multimap Φ is said to be superpositionally approximable (SA-multimap) if there exist a metric space Y, a regular Michael system AM(Y), an upper semi continuous multimap F : X → AM(Y), and acontinuous map p : Y → E such that Φ may be presented as the composition Φ=pF We will say that a SA-multimap Φ = pF is normal if the multimap F is completely continuous.

2.4. Theorem. Let M be a closed convex bounded subset of a Banach space E , Φ : M → K(E) is a normal SA-multimap. If Φ(M) ⊂ M, then Φ has a fixed point

Proof. Let τ : E → M be a continuous retraction and η an arbitrary positive number. Since the set M is bounded, there exists a number R > 0 such that Uη(M) ⊂ B_R where B_R ⊂ E is a closed ball of radius R. Let Φ = pF be the representation of the SA-multimap Φ. Consider a continuous map p_1 = τp : Y → M.

By virtue of the boundedness of M, the set N = F(M) ⊂ Y is compact, hence by Lemma 2.1, for every δ ∈ (0,η)there exists ε > 0 such that

$$\rho(p_1(x'), p_1(x'')) < \delta$$

whenever ρ (x',x'')< ε and x',x'' ∈ Uε(N). Without loss of generality we will assume that ε < δ.

By virtue of Corollary 1.9 the multimap F has a completely continuous ε-approximation f : M → Y. Let us demonstrate that that the composition f_1 = p_1 . f is a completely continuous δ-approximation of the multimap Φ_1 = p_1 .F. In fact, let x ∈ M be an arbitrary point, then there exist points x' ∈ M and y ∈ F(x′) such that ||x − x′|| < ε and ρ(f(x),y) < ε. Hence f(x) ∈ Uε (N).Then || p_1 (f(x)) − p_1 (y)|| < δ. Since p_1 (y) ∈ p_1 (F(x′)) = Φ_1 (x′), the map f_1 is a continuous δ-approximation. The compactness of a map f_1 follows from the compactness of f.

Let us demonstrate now that $f_1(B_R) \subset B_R$. In fact, for each point x ∈ B_R we have:

$$f_1(x) = \tau(p(f(x))) \in M \subset B_R .$$

So, by Schauder theorem, the map f_1 has a fixed point. Applying Lemma 2.2, we conclude that the multimap Φ_1 has a fixed point. Let $x_* \in \Phi_1(x_*) = \tau(\Phi(x_*))$. Since $x_* \in$ M, we obtain that $\tau(\Phi(x_*)) = \Phi(x_*)$. □

3. On a class of operator inclusions

Let E_1, E_2 be Banach spaces, a : D(a) ⊂ $E_1 \to E_2$ a closed linear surjective operator. By $a^{-1} : E_2 \to C_v(E_1)$ we denote the multivalued linear operatorbeing the inverse to a (see, e.g. [9]).Denote L = Ker(a) and E = E_1/Ker(a). It is known that the norm in E can be defined in the following way: if [x] = x+ Ker(a) ∈ E, then

$$\|[x]\| = \inf_{u \in Ker(a)} \|x+u\|$$

Let $p : E_1 \to E$ be a natural projection. Consider the linear operator a_1 : D(a_1) ⊂ E → E_2, where D(a_1) = p(D(a)) and a_1 ([x]) = a(x). It is easyto see that a_1 is closed surjective operator with the trivial kernel. It means that the operator a1 has a bounded inverse. Then we have:

$$\|a_1^{-1}\| = \sup_{y \in E_2} \frac{\|a_1^{-1}(y)\|}{\|y\|} = \sup_{y \in E_2}\left(\frac{\inf\{\|x\| \, x \in E_1 a(x) = y\}}{\|y\|}\right).$$

By definition, the value $\|a_1^{-1}\|$ will be called the norm $\|a^{-1}\|$of the multioperator a^{-1}.

Consider the following example. Let C= C([a,b];R^n)be the space of con-tinuous functions and D denote the subspace of continuously differentiable functions. Let us evaluate $\|d^{-1}\|$for the operator of differentiation d: D⊂C→C.

3.1. Proposition. $\|d^{-1}\| = \frac{b-a}{2}$

Proof. For y∈C consider

$$d^{-1}(y) = \{x \in C \mid x(t) = \alpha + \int_a^t y(s)ds, \alpha \in E^n\}$$

Then

$$\inf_{\alpha\in E^n}\{\|x\|_C \mid x\in d^{-1}(y) = \inf_{\alpha\in E^n} \alpha + \left\|\alpha + \int_a^t y(s)ds\right\|_C \le$$

$$\le \left\|\int_a^t y(s)ds - \int_a^{\frac{a+b}{2}} y(s)ds\right\|_C = \left\|\int_{\frac{a+b}{2}}^t y(s)ds\right\|_C \le$$

$$\le \max_{a\le t\le b}\left|\int_{\frac{a+b}{2}}^t \|y(s)\|ds\right| = \frac{b-a}{2}\|y\|_C$$

Hence

$$\|d^{-1}\| \le \frac{b-a}{2}$$

Consider a function $y_0 \in C, y_0(t) \equiv (1,0,...,0).$ We have

$$d^{-1}(y_0) = \{x\in C \mid x(t) = (\alpha_1 + t - a, \alpha_2, ..., \alpha_n), \alpha_i \in R\}.$$

Therefore

$$\inf_{\alpha\in E^n}\|x\|_C = \inf_{\alpha\in E^n}\max_{a\le t\le b}\sqrt{(\alpha_1+t-a)^2 = \sum_{i=2}^n \alpha_i^2} = \inf_{\alpha_1\in R^1}\max_{a\le t\le b}|\alpha_1 + t - a| =$$

$$= \inf_{\alpha_1\in R^1}\max\{|b-a+\alpha_1|,|\alpha_1|\} = \frac{b-a}{2}$$

From the other side , $\|y_0\|_C = 1.$ So

$$\frac{b-a}{2}\|y_0\|_C = \inf_{\alpha\in E^n}\{\|x\|_C\ x\in d^{-1}(y_0)\}$$

and therefore $\|d^{-1}\| = \dfrac{b-a}{2}$ □

3.2. Lemma For every k, $\|a^{-1}\|<k$, there exists a continuous map q: $E_2 \to E_1$, such that:

1) a(q(y))= y for each y$\in E_2$;

2) $\|q(y)\|\le k\|y\|$.

For a multimap F: $E_1 \to P(E_2)$ we will consider the solvability of the following operator inclusion

$$a(x)\in F(x) \qquad (1)$$

Solutions of inclusion(1) are called coincidence points of the pair (a,F). The coincidence points set of the pair (a,F) will be denoted by Coin(a,F).

For a map q satisfying conditions of Lemma 3.2 define a multimap

$$F_1 : E_2 \times Ker(a) \to AM(E_2)\ , F_1(y,u) = F(q(y)+u)$$

Consider the following inclusion:

$$y \in F_1\ (y,u). \qquad (2)$$

3.3. Lemma There exists a one-to-one correspondence between Coin (a,F) and the solutions set of inclusion(2).

Proof. In fact, let x_0 ∈Coin(a,F), i.e. y_0 = a(x_0) ∈F(x_0). Then u_0 = x_0 −q(y_0) ∈Ker(a). Hence y_0 ∈F(x_0)= F_1 (y_0, u_0),i.e. the couple (y_0, u_0) is the solution of inclusion (2).

From the other side, if the pair(y_0, u_0)is a solution of(2),let us denote x_0 = q(y_0)+u_0. Then y_0 ∈F(x_0), and a(x_0)= a(q(y_0))+a(u_0)= y_0. □

We will need one more auxiliary statement.

Let E be a Banach space, the norm in the Banach space E_0 = E×R^1 will be defined in the following way:

$$\|(x,t)\| = \sqrt{\|x\|^2 + t^2}$$

Let $S_r^{\ 0} \subset E_0$ be the sphere of the radius rcentered at zero; F: $S_r^{\ 0}$ →K(E)a completely continuous SA-multimap. Consider the inclusion

$$x \in F(x,t). \qquad (3)$$

3**.4. Lemma.** If

$$\|F(x,t)\| := \max_{u \in F(x,t)} \|u\| \le r$$

for each point (x,t)∈$S_r^{\ 0}$,then inclusion (3) has a solution in $S_r^{\ 0}$.

Proof. Let B be a closed ball of the radius r in the space E. Consider the multimap G: B→K(E) defined by

$$G(x)=F(x,\sqrt{r^2 - \|x\|^2}).$$

It is clear that G is a completely continuous SA-multimap. Notice also that G(B) ⊂B. Hence, by virtue of Theorem 2.4 the multimap G has a fixed point. It remains to mention that if x_0 is a fixed point of G, then for $t_0 = \sqrt{r^2 - \|x_0\|^2}$, we see that(x_0, t_0)∈$S_r^{\ 0}$ is the solution of inclusion (3) □

Let E_1, E_2 be Banach spaces, a: D(a)⊂ $E_1 \to E_2$ a closed surjective linear operator. Let Y be a metric space and a multimap F: X$\subset E_1 \to$C(Y)is upper semicontinuous.

3.5. Definition. A multimap F is completely continuous modulo a (or a-completely continuous), if for each bounded sets A$\subset E_2$ and B$\subset$X the set $\overline{F(B \cap a^{-1}(A))}$ is compact in Y.

It is known that that the set D(a)may be regarded as a Banach space E endowed with the graph norm:

$$\|x\|_{D(a)} = \|x\|_{E_1} + \|a(x)\|_{E_2}$$

It is clear that the inclusion map j: E$\to E_1$ is continuous. For X$\subset$D(a) denote $\tilde{X} = j^{-1}(X)$ and consider the multimap $\tilde{F} : \tilde{X} \to K(E_2), \tilde{F}(x) = F(j(x))$

We have the following criterion.

3.6. Proposition. The multimap F is a-completely continuous iff the multimap $\tilde{F}$ is completely continuous.

Proof. (i)Let F be a-completely continuous. If C$\subset \tilde{X}$ is abounded setin E, then the set B= j(C) is bounded in E_1, and the set A= a(j(C))= a(B) is bounded in E_2.Thentheset $\tilde{F}$ (C)= F(j(C))= F(B∩a^{-1}(A))is relatively compact.

(ii) Let the multimap $\tilde{F}$ is completely continuous. Consider bounded subsets A$\subset E_2$ and B$\subset$X. Let C= j^{-1}(B∩a^{-1}(A))$\subset$E. It is clear that C is a bounded subset of $\tilde{X}$. Then F(B∩a^{-1}(A))= $\tilde{F}$ (C)is relatively compact. □

Let Φ : $E_1 \to$C(E_2)be a SA-multimap, i.e. there exist a metric space Y, a regular Michael system AM(Y) in the space Y, an upper semicontinuous multimap F: X→AM(Y), and a continuous map p: Y→E such that Φ= pF.

3.7. Definition. SA-multimap Φ= pF is said to be a-completely continuous if the multimap F is a-completely continuous.

3.8. Theorem. Let Φ: $E_1 \to$C(E_2) be a SA-multimap satisfying the following conditions:

1) Φ is a-completely continuous;

2) there exist nonnegative numbers D_1 and D_2 such that

$$\|\Phi(x)\| = \max_{y \in \Phi(x)} \|y\| \le D_1 \|x\| + D_2$$

for each x$\in E_1$.If

$$D_1 < \frac{1}{\|a^{-1}\|}$$

then Coin $(a, \Phi) \neq \varnothing$

Proof. If dim(Ker(a))=0 then a^{-1} is a continuous linear operator. Then, using the conditions of the theorem, we can construct a ball $B_R \subset E_2$ centered at the origin such that for each point y$\in B_R$ we have $\hat{\Phi}(y) = \Phi(a^{-1}(y)) \subset B_R$.

Since the multimap $\hat{\Phi}$ is completely continuous, by Theorem 2.4 it has a fixed point y_*. It is clear that the point $x_* = a^{-1}(y_*)$ is the solution of inclusion (1).

Now consider the case dim(Ker(a)) >0. Let k be an arbitrary number satisfying

$$\|a^{-1}\| < k < \frac{1}{D_1}$$

and q: $E_2 \to E_1$ a map given by Lemma 3.2

Let us choose in the subspace Ker(a) a non-zero vector such that

$$\|e\| < \frac{1 - D_1 k}{D_1}$$

Consider the space $E_0 = E_2 \times R^1$ with the norm $\|(y,t)\| = \sqrt{\|y\|^2 + t^2}$. Let Φ= p∘F where F: E_1 →AM(Y). Consider the multimap F_1 : E_0 →AM(Y) defined as

$$F_1(y,t) = F(q(y)+te).$$

Let us show that this multimap is completely continuous. Let A$\subset E_0$ be an arbitrary bounded set, then there exists a number R>0 such that for every point(y,t)∈Awehave ||(y,t)||≤R. Let the map $\hat{q} : A \to E_1$ be defined by the relation $\hat{q}(y,t) : q(y) + te$. Denote B= y_* (A), then for each point x∈B the following estimate holds

$$\|x\| = \|q(y) + te\| \leq (k + \frac{1 - D_1 k}{D_1})R \ ,$$

i.e. B is also a bounded set. Notice that B⊂a^{-1}(A). Then, by virtue of a-complete continuity of the multimap F, the set F_1 (A)= F(B∩a^{-1}(A))is relatively compact. So the multimap F_1 is completely continuous.

Denote $\hat{\Phi} = p \circ F_1$. Let $S_r^0 \subset E_0$ be the sphere of the radius rcentered at the origin. Let us demonstrate that, for sufficiently large, the estimate

$$\left\|\hat{\Phi}\right\| = \max_{u \in \in \hat{\Phi}(y,t)} \left\|u\right\| \le r$$

holds for each $(y,t) \in S_r^0$. In fact, if $u \in \hat{\Phi}(y,t)$ then

$$\left\|u\right\| \le D_1 \left\|q(y) + te\right\| + D_2 < D_1 k \left\|y\right\| + D_1 \left|t\right| \left\|e\right\| + D_2 .$$

If

$$r > \frac{D_2}{1 - D_1 k 1 - D_1 \left\|e\right\|}$$

then

$$\left\|\hat{\Phi}(y,t)\right\| < D_1 kr + D_1 r \left\|e\right\| + D_2 \le r .$$

Now we may apply Lemma3.3 and conclude that the inclusion $y \in \hat{\Phi}(y,t)$ has a solution(y_0, t_0)∈$S_r{}^0$. Then $x_0 = q(y_0) + t_0 e$ ∈Coin(a,Φ). □

3.9. Theorem. In conditions of Theorem 3.8 let, additionally,

dim(Ker(a))>0.

Then the set Coin(a,Φ) is unbounded.

Proof. Supposing the contrary, we will have a number α>0, such that ||x||≤α for each point x∈Coin(a,Φ). Then the set Φ(Coin(a,Φ)) is also bounded, i.e. there exists such number β>0, that ||y||≤β for each point y∈Φ(Coin(a,Φ)). Consider a sequence of numbers rn →∞such that

$$r_n > \frac{D_2}{1 - D_1 k 1 - D_1 \left\|e\right\|}$$

where the number k and the vector e are defined in the course of the proof of Theorem3.8 Then there exists a sequence of points $(y_n, t_n) \in S_{r_n}^0 \subset E_0$ such that the points $x_n = q(y_n) + t_n e$. belong to the set Coin(a,Φ).

Then $\left\|x_n\right\| \le \alpha$ for each n .Fromthe other side,since $a(x_n) = y_n \in \Phi(x_n)$,we have $y_n \in \Phi(Coin(a,\Phi))$ $y_n \in \Phi(Coin(a,\Phi))$ .Hence $\left\|y_n\right\| \le \beta$ for each n .Then

$$\left|t_n\right| \le \frac{\left\|x_n\right\| + \left\|q(y_n)\right\|}{\left\|e\right\|} \le \frac{\alpha + k\beta}{\left\|e\right\|}$$

So, the sequence { t_n }is alsobounded. Then apoint(y_n, t_n)does not belong to the sphere $S_{r_n}^0$ for a sufficiently large n, giving the contradiction. □

4. On a classofintegro-differential inclusions

4.1. **Multioperator of superposition and integral multioperator.** We shall start with some preliminary remarks (details can be found, e.g. in [6],[13]).

Let I⊂ ℝ be a compact interval endowed with Lebesgue measure; E a separable Banach space.

4.1. Definition. A multifunction F: I→K(E) is said to be measurable if for each open set V⊂E the set F_+^{-1} (V)is measurable.

It is known that every measurable multifunction F: I→K(E) has a measurable selection φ: I→E, φ(t)∈F(t)for a.e. t∈I

Let E, E_0 be separable Banach spaces.

4.2. Proposition. Suppose that a multimap F: I× E_0 →K(E) satisfies conditions:

F**1**) the multifunction F(.,x): I→K(E)has a measurable selection for each x∈ E_0;

F**2**) the multimap F(t,.): E_0 →K(E)is upper semicontinuous for a.e. t∈I.

Then the multimap F is superpositionally selectable, i.e. for each measur-ablefunction q: I→ E_0, the multifunction Φ: I→K(E),Φ(t)= F(t,q(t)) has a measurable selection.

Let a multimap F: I× E_0 →K(E)additionally to(F1) and(F2) satisfies also the following condition:

F**3**) there exists a measurable function α: I→R^1 such that

$$\|F(t,x)\| = \max_{y\in F(t,x)} \|y\| \le \alpha(t)(1+\|x\|)$$

for all x∈ E_0 and a.e. t∈I.

Then the multimap

$$\mathrm{P}_F : C(I, E_0) \to P(L^1(I, E)),$$

assigning to every continuous function q∈C(I; E_0) the set of all summable selections of the multifunction Φ : I→K(E),

$$\Phi(t) = F(t, q(t))$$

is said to be a superposition multioperator generated by F.

Let us mention the following property of the superposition multioperator.

4.3. Proposition. Let a multimap F: I× E_0 → K_v (E)satisfy conditions (F1)-(F3)and a: L^1(I;E)→ E_1 a continuous linear operator to a normed space E_1. Then the composition a∘ P_F : C(I; E_0)→$C_v(E_1)$ is a closed multimap.

Now let[a,b]⊂ $ℝ^1$, L($ℝ^n$, $ℝ^n$)be the space of continuous linear operators in $ℝ^n$ and k:[a,b]×[a,b]

$\to L(\mathbb{R}^n, \mathbb{R}^n)$ a continuous map. Then the linear integral operator j_k : $L^1([a,b]; \mathbb{R}^n) \to C([a,b]; \mathbb{R}^n)$defined as

$$j_k(\varphi)(t) = \int_a^b k(t,s)\varphi(s)ds$$

is completely continuous(see, e.g. [19]). Itis alsoknown(see, e.g. [14])that

$$\|j_k\| = \max_{a \le t, s \le b} \|k(t,s)\|$$

Suppose that a multimap $F : [a,b] \times \mathbf{R}^n \to k_v(\mathbf{R}^n)$ satisfies conditions(F1) -(F3).

4.4. Definition. The composition

$$j_k \circ \mathrm{P}_F : C([a,b];\mathbf{R}^n) \to C_v(C([a,b];\mathbf{R}^n))$$

is said to be the Hammerstein integral multioperator, generated by F.It will be denoted by $\int_a^b (k \circ \mathrm{P}_F)$.

Proposition 4.3 yields the following statement

4.5. Theorem. Let a multimap

$$F : [a,b] \times \mathbf{R}^n \to k_v(\mathbf{R}^n)$$

satisfy conditions(F1)-(F3). Then the Hammerstein integral multioperator

$$\int_a^b (k \circ \mathrm{P}_F) : C([a,b];\mathbf{R}^n) \to C_v(C([a,b];\mathbf{R}^n))$$

is completely continuous.

4.2**. Existence theorem for a class of integro-differential inclusions.**

For T>0,let $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a continuous map satisfying the following con$\int_a^b (k \circ \mathrm{P}_F) : C([a,b];\mathbf{R}^n) \to C_v(C([a,b];\mathbf{R}^n))$ dition:(f)there exist positive numbers C_1, C_2 and C_3 such that

$$\|f(t,u,v)\| \le C_1\|u\| + C_2\|v\| + C_3$$

for all $(t,u,v) \in [0,T] \times \mathbf{R}^n \times \mathbf{R}^m$.

Let $k : [0,T] \times [0,T] \to L(\mathbf{R}^m \times \mathbf{R}^m)$ be a continuous map. Denote

$$K_0 = \max_{s,t \in [0,T]} |k(t,s)|$$

Let a multimap $F : [a,b] \times \mathbf{R}^n \to k_v(\mathbf{R}^m)$ satisfy conditions(F1) -(F3).

We will consider the following problem:

$$x'(t) = f(t, x(t), y(t)) \qquad (4)$$

$$y \in \int_0^h (k \circ \mathrm{P}_F)(x) \qquad (5)$$

Where h$\in [0,T]$.

4.6.Definition. A pair of function x$\in$C([0,h],$\mathbb{R}^n$) , y$\in$C([0,h],$\mathbb{R}^m$) satisfying relations (4) and (5) for all t$\in$[0,h] is said to be a solution of problem(4),(5).

Notice that (4), (5) may be interpreted as the control problem where x is the trajectory of the system, and y is the control satisfying the feedback condition(5).

Let us givethe operator treatment of problem(4),(5). Consider the super-position operator

$$\hat{f} : C([0,h],\mathbb{R}^n) \times C([0,h],\mathbb{R}^m) \to C([0,h],\mathbb{R}^n),$$

generated by f and the multimap

$$\hat{F} : C([0,h], \mathbb{R}^n) \to C_v\,(C([0,h], \mathbb{R}^n) \times C([0,h],\mathbb{R}^m)),$$

given by

$$\hat{F}(x) = (x, \int_0^h (k \circ \mathrm{P}_F)(x))$$

Let d:D(d)$\subset$C([0,h], $\mathbb{R}^n$)$\to$C([0,h], $\mathbb{R}^n$)be the differentiation operator whose domain D(d) is the subspace of continuously differentiable functions on [0,h].

It is easy to verify the following statement.

4.7. Lemma. Problem (4), (5) is equivalent to the following operator inclusion:

$$d(x) \in \hat{f}\,(\hat{F}(x)), \qquad (6)$$

i.e. if a pair (x,y) is the solution of problem(4),(5), then x is the solution of inclusion(6) and, conversely,if x is the solution of(6),then there exists

$$y \in \int_0^h (k \circ \mathrm{P}_F)(x)$$

such that the pair (x,y)is the solution ofproblem(4),(5). Consider the multimap

Φ : C([0,h], $\mathbb{R}^n$)$\to$C(C([0,h], $\mathbb{R}^n$))defined as

$$\Phi(x) = \hat{f}\,(\hat{F}(x)).$$

4.8. Lemma. Φ is a d-completely continuous SA-multimap.

Proof. Since $\hat{f}$ is continuous and $\hat{F}$has closed convex values, Φ is a SA-multimap.

Now noticethatthegraph normforthedifferentiation operator dcoincides with thenormof thespace

$C^1([0,h], \mathbb{R}^n)$,and theembedding of thisspaceinto the space $C([0,h], \mathbb{R}^n)$ is completely continuous. Applying Theorem 4.5 and Proposition 3.6 we conclude that the multimap $\hat{F}$is d-completely continuous. □

4.9. Theorem. If

$$h(C_1 + C_2 K_0 \int_0^h \alpha(s)ds < 2\,, \tag{7}$$

then:

(a) the set of solutions {x,y} of problem(4),(5) defined on the interval [0,h] is nonempty;

(b) the set of trajectories {x} is unbounded in the space $C([0,h], \mathbb{R}^n)$.

Proof. For an arbitrary $x \in C^1([0,h], \mathbb{R}^n)$,let us estimate ||y|| for $y \in \Phi(x)$.

We have

$$y(t) = f(t, x(t), \int_0^h k(t,s)z(s)ds) \,,$$

where $z(s) \in F(s,x(s))$for a.e. $s \in [0,h]$. Then

$$\|y(t)\| \le C_1 \|x(t)\| + C_2 \left\| \int_0^h k(t,s)z(s)ds \right\| + C_3$$

$$\le C_1 \|x\| + C_2 k_0 \int_0^h \alpha(s)(\|x(s)\| + 1)ds + C_3$$

$$\le (C_1 + C_2 k_0 \int_0^h \alpha(s)ds) \|x\| + C_4$$

Where $C_4 = C_2 k_0 \int_0^h \alpha(s)ds + C_3$

Hence

$$\max_{y \in \Phi(x)} \|y\| \le (C_1 + C_2 k_0 \int_0^h \alpha(s)ds) \|x\| + C_4$$

From Proposition 3.1 we know that $\|d^{-1}\| = \dfrac{h}{2}$, and therefore, from condition (7)we have

$$C_1 + C_2 k_0 \int_0^h \alpha(s)ds < \frac{1}{\|d^{-1}\|}$$

Applying Theorem 3.8 we obtain the nonemptyness of the solutions set for inclusion(6) andfromLemma4.7 we deduce conclusion(a).

Since dimKer(d)= n>0 we may apply Theorem3.9 and obtain conclusion (b). □

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