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A New Integral Transform “Dukani Transform” with Applications

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ABSTRACT

This paper describes a novel integral transform called the "Dukani Transform," which we can employ to precisely (analytically) solve the Volterra integral equations (VIE) of the first sort. To do this, first the Dukani transform of transcendental and elementary algebraic mathematical functions is derived by the authors. Next, the basic properties of the Dukani transform were discussed. Then, using a range of differential equations, including integral equations (IEs), ordinary differential equations (ODEs), and integro differential equations (IDEs), the exact (analytic) solution for first-class generic VIE was found by the authors. Numerical problems have been thoroughly examined and solved, in a step wise manner, demonstrating the applicability of the Dukani transform. The findings demonstrate that, without need for time-consuming calculations, the suggested "Dukani Transform" new integral transform produces accurate (VIEs.) of first-kind and second kind.

MSC.

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1. Introduction

Researchers currently have three main advantages over other mathematical methods when it comes to solving problems in science, social science, and engineering: simplicity, accuracy, and the ability to produce findings without the need for laborious calculations. Integral transformations are the method of choice among mathematical techniques because of these characteristics. Laplace[1], Elzaki[2], Aboodh [3], Mohand[4], Sumudu[5], Sawi[6], Kamal[7,8], Sadik [9], Sawi [10],Rishi[11, 12], Anuj[13] are only a few of the integral transforms that researchers have discovered most recently.

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Several researchers studied the well-known issues of decay and development and addressed modeling in biotechnology, health science, and mechanics using a variety of integral transforms. (Laplace [14], Sawi[15], [16]. Mohand [4], Kushare[17, 18], Soham [19] Higazy and Aggarwal used ordinary differential equations to create a model for the Sawi transform. [10]. were successful in figuring out the precise answers to the riddle of chemical kinetics. El-Mesady and collaborators [20]. applied the Jafari transform to a medical issue. And for the purpose of solving integral equations, partial differential equations, and ordinary differential equations, a variety of transformations are available. [2, 13, 17, 19, 21].

The goal of the study is to generate the "Dukani Transform," a new integral transform with basic features, and apply it to solve the first kind V.I.E. with a convolution type kernel (VIEs) and ordinary differential equations (ODEs), as well as the second kind VIEs and integro differential equations (IDEs). This will allow for precise solution with minimal computational effort. The suggested transform, known as the "Dukani transform," is superior to the other recognized transforms because it solves the problems accurately and efficiently without involving a great deal of time-consuming computation. Also, it is the twin of the integral "Laplace transform," which is widely used.

In Section 2 we provide information on this duality relation as well as a description of this transform. Because it uses this relation to yield all of the features of the Laplace transform, the Dukani transform is helpful. Several features have been inferred, such as linearity, scaling, shifting, and convolution theorem.

2. Definition of Dukani Transform

A function $w(t)$ represent the original function as:

1 $w(t) \equiv 0$ for $t < 0$,

2 $|w(t)| < Me^{t_0 t}$ for $t > 0$ with $M > 0, t_0 \in \mathbb{R}$.

Considering functions in the set A defined by: a currently created transform for functions of exponential order which is termed the Dukani transform

$$A = \left\{ w(t) : \exists\, N\,, k_1, k_2 > 0\,.\, w(t) < Me^{\frac{|t|}{k_j}}, if\ t \in (-1)^j \times [0, \infty) \right\}$$

The constant N should be finite number, for a stated function in set A, k_1, k_2 might be finite or infinite.

The Dukani transformation of exponential order piecewise continuous function, indicated by the operator $D(.)$, $w(t)$ defined in the interval, $(0, \infty)$ is provided by

$$D\{w(t)\} = \frac{1}{q^2} \int_0^\infty e^{-\frac{t}{q^3}} w(t) dt = W(q), \qquad t > 0 \ \ , k_1 < q < k_2\,.$$

In this transformation, the variable q is used for factoring the variable t in the argument of the function and the transform is associated with the Aboodh, Laplace, Elzaki, and Sadik transforms.

Remark 1: Dukani transform shares a dual relationship with "Laplace transform", which is a popular and widely used transform, if $L(f(t)) = \int_0^\infty f(t) e^{-qt} dt = L(q)$, where L is Laplace transform and $D(w(t)) = W(q)$ where D is Dukani transform, then $W(q) = \frac{1}{q^2} L(\frac{1}{q^3})$.

Theorem 1: [**Sufficient Condition for Existence of a Dukani Transformation**]: The Dukani transform $D\{w(t)\}$ exists when it has exponential order and $\int_0^b |w(t)|dt$ exists for $b > 0$.

Proof:

$$\frac{1}{q^2}\int_0^\infty \left|w(t)e^{-\frac{t}{q^3}}\right| dt = \frac{1}{q^2}\int_0^n \left|w(t)e^{-\frac{t}{q^3}}\right| dt + \frac{1}{q^2}\int_n^\infty \left|w(t)e^{-\frac{t}{q^3}}\right| dt, \text{ where } n \in (0,\infty)$$

$$\leq \frac{1}{q^2}\int_0^n |w(t)|dt + \frac{1}{q^2}\int_0^\infty |w(t)|e^{-\frac{t}{q^3}}dt \leq \frac{1}{q^2}\int_0^n |w(t)|dt + M\frac{1}{q^2}\int_0^\infty e^{\alpha t}e^{-\frac{t}{q^3}}dt,$$

$$= \frac{1}{q^2}\int_0^n |w(t)|dt + M\frac{1}{q^2}\int_0^\infty e^{-\left(\frac{1}{q^3}-\alpha\right)t}dt = \frac{1}{q^2}\int_0^n |w(t)|dt + \frac{M\frac{1}{q^2}}{-\left(\frac{1}{q^3}-\alpha\right)} \lim_{t\to\infty} e^{-\left(\frac{1}{q^3}-\alpha\right)t} |_0^t,$$

$$= \frac{1}{q^2}\int_0^n |w(t)|dt + \frac{M\frac{1}{q^2}}{\left(\frac{1}{q^3}-\alpha\right)}.$$

The first integral $\frac{1}{q^2}\int_0^n |w(t)|dt$ exists, and the second term $\frac{M\frac{1}{q^2}}{\left(\frac{1}{q^3}-\alpha\right)}$ is finite for $\frac{1}{q^3} > \alpha$"so the integral $\frac{1}{q^2}\int_0^\infty e^{-\frac{t}{q^3}}w(t)dt$ converges absolutely and the Dukani transform $D\{w(t)\}$ exists.

3. Dukani Transform for Some Functions

In this section we derive the DukaniTransform for most useable functions.

Case 1: Regarding the constant function $w(t) = c$, where $c \in W$, With utilizing the Dukani transform's definition, we have

$$D\{w(t)\} = \frac{1}{q^2}\int_0^\infty e^{-\frac{t}{q^3}}w(t)dt = W(q) \Rightarrow D\{c\} = \frac{1}{q^2}\int_0^\infty e^{-\frac{t}{q^3}}cdt$$

$$\Longrightarrow D\{c\} = c\,\frac{1}{q^2}\left\{-q^3e^{-\frac{t}{q^3}}\Big|_0^\infty\right\} \Rightarrow D\{c\} = cq\ .$$

Case 2: Regarding the function $w(t) = t$, With utilizing the Dukani transform's definition, we have
$$D\{w(t)\} = \frac{1}{q^2}\int_0^\infty e^{-\frac{t}{q^3}}w(t)dt = W(q) \Rightarrow D\{t\} = \frac{1}{q^2}\int_0^\infty e^{-\frac{t}{q^3}}tdt,$$

Let $u = t \Rightarrow du = dt$ and, let $dv = e^{-\frac{t}{q^3}}dt \Rightarrow v = -q^3e^{-\frac{t}{q^3}}$
$$\Rightarrow D\{t\} = \frac{1}{q^2}\left\{-tq^3e^{-\frac{t}{q^3}}\Big|_0^\infty + \int_0^\infty q^3e^{-\frac{t}{q^3}}dt\right\} \Rightarrow D\{t\} = \frac{1}{q^2}\left\{-q^6e^{-\frac{t}{q^3}}\right\}_0^\infty = \frac{q^6}{q^2} = q^4\,.$$

Case 3: Regarding the function $w(t) = t^2$, With utilizing the Dukani transform's definition, we have
$$D\{w(t)\} = \frac{1}{q^2}\int_0^\infty e^{-\frac{t}{q^3}}w(t)dt = W(q) \Rightarrow D\{t^2\} = \frac{1}{q^2}\int_0^\infty e^{-\frac{t}{q^3}}t^2dt,$$

$$\Rightarrow D\{t^2\} = -\frac{1}{q^2}\left\{e^{-\frac{t}{q^3}}\{t^2q^3 + 2tq^6 + 2q^9\}_0^\infty \Rightarrow D\{t^2\} = \frac{1}{q^2} * 2q^9 = 2!\,q^7,\right.$$

Case 4: Taking into consideration that the function $w(t) = t^n$, where $n > 0$ then, the Dukani transform becomes $D\{t^n\} = n!\,q^{3n+1}$. We can use mathematical induction for solve it.

Case 5: $w(t) = t^{\beta}$; $t > 0$; $\beta > -1$, then $\Rightarrow D\{t^{\beta}\} = \Gamma(\beta+1)q^{3\beta+1}$, With utilizing the Dukani transform's definition, we have

$$D\{w(t)\} = \frac{1}{q^2}\int_0^{\infty} e^{-\frac{t}{q^3}} w(t)dt = W(q) \Rightarrow D\{t^{\beta}\} = \frac{1}{q^2}\int_0^{\infty} e^{-\frac{t}{q^3}} t^{\beta} dt \Rightarrow D\{t^{\beta}\} = \frac{1}{q^2}\left\{-t^{\beta} q^3 e^{-\frac{t}{q^3}}\Big|_0^{\infty} + \beta q^3 \int_0^{\infty} t^{\beta-1} e^{-\frac{t}{q^3}} dt\right\} \Rightarrow D\{t^{\beta}\} = \frac{1}{q^2}\{\beta q^3 \cdot \Gamma(\beta) q^{3\beta}\} = \Gamma(\beta+1)q^{3\beta+1}.$$

Case 6: Regarding the function $w(t) = e^{at}$, With utilizing the Dukani transform's definition, we have

$$D\{w(t)\} = \frac{1}{q^2}\int_0^{\infty} e^{-\frac{t}{q^3}} w(t)dt = W(q) \Rightarrow D\{e^{at}\} = \frac{1}{q^2}\int_0^{\infty} e^{-\frac{t}{q^3}} e^{at} dt D\{e^{at}\} = \frac{1}{q^2}\int_0^{\infty} e^{-t\left(\frac{1}{q^3}-a\right)} dt \Rightarrow$$

$$D\{e^{at}\} = \frac{1}{q^2} \cdot \frac{-1}{\frac{1}{q^3}-a} e^{-t\left(\frac{1}{q^3}-a\right)}\Bigg]_0^{\infty} \Rightarrow D\{e^{at}\} = \frac{1}{q^2}\frac{q^3}{(1-aq^3)} = \frac{q}{(1-aq^3)}.$$

Case 7: Regarding the function $w(t) = sinat$, With utilizing the Dukani transform's definition, we have $D\{w(t)\} = \frac{1}{q^2}\int_0^{\infty} e^{-\frac{t}{q^3}} w(t)dt = W(q)$, And from case 6 we get

$$D\{sin(at)\} = D\left\{\frac{e^{ait}-e^{-ait}}{2i}\right\} \Rightarrow D\{sinat\} = \frac{1}{2i}\{D(e^{ait} - e^{-ait})\} \Rightarrow D\{sinat\} = \frac{1}{2i}\left\{\frac{q}{1-aiq^3} - \frac{q}{1+aiq^3}\right\} \Rightarrow$$

$$D\{sinat\} = \frac{1}{2i}\left\{\frac{q+aiq^4-q+aiq^4}{(1-aiq^3)(1+aiq^3)}\right\}, \Rightarrow D\{sinat\} = \frac{aq^4}{(1+a^2q^6)}.$$

Case 8: Regarding the function $w(t) = \cos(at)$, With utilizing the Dukani transform's definition, we have $D\{w(t)\} = \frac{1}{q^2}\int_0^{\infty} e^{-\frac{t}{q^3}} w(t)dt = W(q)$, And from case 6 we obtain $\Rightarrow D\{cos(at)\} = D\left\{\frac{e^{ait}+e^{-ait}}{2}\right\} \Rightarrow$

$$D\{cosat\} = \frac{1}{2}\{D(e^{ait}) + D(e^{-ait})\}$$

$$\Rightarrow D\{cosat\} = \frac{1}{2}\left\{\frac{q}{1-aiq^3} + \frac{q}{1+aiq^3}\right\} \Rightarrow D\{cosat\} = \frac{q}{(1+a^2q^6)}.$$

Case 9: Regarding the function $w(t) = sinh(at)$,With utilizing the Dukani transform's definition, we have $D\{w(t)\} = \frac{1}{q^2}\int_0^{\infty} e^{-\frac{t}{q^3}} w(t)dt = W(q)$, And from case 6 we obtain

$$\Rightarrow D\{sinh(at)\} = D\left\{\frac{e^{at}-e^{-at}}{2}\right\} \Rightarrow D\{sinh(at)\} = \frac{1}{2}\left\{\{D\{e^{at}\} - D\{e^{-at}\}\right\} \Rightarrow D\{sinh(at)\} = \frac{1}{2}\left\{\frac{q}{1-aq^3} - \frac{q}{1+aq^3}\right\} \Rightarrow D\{sinh(at)\} = \frac{1}{2}\left\{\frac{q+aq^4-q+aq^4}{(1-aq^3)(1+aq^3)}\right\} = \frac{aq^4}{(1-a^2q^6)}.$$

Case 10: Regarding the function $w(t) = cosh(at)$, With utilizing the Dukani transform's definition, we have $D\{w(t)\} = \frac{1}{q^2}\int_0^{\infty} e^{-\frac{t}{q^3}} w(t)dt = W(q)$, And from case 6 we get

$$\Rightarrow D\{cosh(at)\} = D\left\{\frac{e^{at}+e^{-at}}{2}\right\} \Rightarrow D\{coshat\} = \frac{1}{2}\{D(e^{at}) + D(e^{-at})\} \Rightarrow D\{coshat\} = \frac{1}{2}\left\{\frac{q}{1-aq^3} + \frac{q}{1+aq^3}\right\}, \Rightarrow$$

$$D\{coshat\} = \frac{q}{(1-a^2q^6)}.$$

4. Dukani Transformation's Properties

In this section we present some important properties of Dukani transform that useful for solving differential equations and integral equations.

4.1 Linearity of Dukani Transform

Theorem 2: if $D\{w(t)\} = W(q)$ then $D\{\sum_{i=1}^{n} \alpha_i w_i(t)\} = \alpha_i \sum_{i=1}^{n} W_i(q)$, where α_i is randomly constants

Proof: From the definition of Dukani transform we obtain

$D\{w(t)\} = \sqrt{q}\int_0^\infty e^{-tq}w(t)dt = W(q) \Rightarrow D\{\sum_{i=1}^{n} \alpha_i w_i(t)\} = \frac{1}{q^2}\int_0^\infty \sum_{i=1}^{n} \alpha_i w_i(t)e^{-\frac{t}{q^3}}dt \Rightarrow$

$D\{\sum_{i=1}^{n} \alpha_i w_i(t)\} = \alpha_i \sum_{i=1}^{n} \frac{1}{q^2}\int_0^\infty e^{-\frac{t}{q^3}}w_i(t)dt,$

$\Rightarrow D\{\sum_{i=1}^{n} \alpha_i w_i(t)\} = \alpha_i \sum_{i=1}^{n} W_i(q).$

4.2. Scaling Property of Dukani Transformatiom

Theorem 3: If $D\{w(t)\} = W(q)$, then $D\{w(kt)\} = \frac{1}{\sqrt[3]{k}}W(\sqrt[3]{k}q)$.

Proof: From the definition of Dukani transform $D\{w(kt)\} = \frac{1}{q^2}\int_0^\infty e^{-\frac{t}{q^3}}w(kt)dt,$

Let $kt = u \Rightarrow kdt = du \Longrightarrow$

$D\{w(kt)\} = \frac{1}{k}\frac{1}{q^2}\int_0^\infty e^{-\frac{u}{kq^3}}w(u)du \Longrightarrow D\{w(kt)\} = \frac{(\sqrt[3]{k})^2}{k}\frac{1}{(\sqrt[3]{k}q)^2}\int_0^\infty e^{-\frac{t}{(\sqrt[3]{k}\,q)^3}}w(u)du \Longrightarrow D\{w(kt)\} = \frac{1}{\sqrt[3]{k}}W(\sqrt[3]{k}q).$

4.3. Shifting Property of Dukani Transform

Theorem 4: If $D\{w(t)\} = W(q)$, then $D\{e^{at}w(t)\} = \frac{1}{(1-aq^3)^{\frac{2}{3}}}\, W(\frac{v}{\sqrt[3]{(1-aq^3)}})$.

Proof: Let $D\{w(t)\} = W(q)$, from the definition of Dukani transform

$D\{w(kt)\} = \frac{1}{q^2}\int_0^\infty e^{-\frac{t}{q^3}}w(kt)dt \Longrightarrow D\{e^{at}w(t)\} = \frac{1}{q^2}\int_0^\infty e^{-\frac{t}{q^3}}e^{at}w(t)dt \Longrightarrow D\{e^{at}w(t)\} =$

$\frac{1}{q^2}\int_0^\infty e^{-t(\frac{1}{q^3}-a)}w(t)dt \Longrightarrow D\{e^{at}w(t)\} = \frac{1}{q^2}\frac{\frac{q^2}{(1-aq^3)^{\frac{2}{3}}}}{\left(\frac{v}{\sqrt[3]{(1-aq^3)}}\right)^2}\int_0^\infty e^{\frac{-t}{\left(\frac{q}{\sqrt[3]{1-aq^3}}\right)^3}}w(t)dt,$

$$D\{e^{at}w(t)\} = \frac{1}{(1-aq^3)^{\frac{2}{3}}}\, W(\frac{v}{\sqrt[3]{(1-aq^3)}}).$$

4.4. (Convolution) Property of Dukani Transformation

Theorem 5: If $D\{w(t)\} = W(q)$ and $D\{g(t)\} = G(q)$ then $D\{w(t) * g(t) = q^2 W(q)G(q)$ where convolution of $w(t)$ and $g(t)$ represented by

$$w(t) * g(t) = \int_0^t w(t-t)g(t)dt = \int_0^t w(t)g(t-t)dt.$$

Proof: From the definition of convolution type and definition of Dukani transform we obtain

$$D\{w(t) * g(t)\} = \frac{1}{q^2}\int_0^\infty e^{-\frac{t}{q^3}}\int_0^t w(s)g(t-s)dsdt = \frac{1}{q^2}\int_0^\infty \, w(s)\int_s^\infty g(t-s)e^{-\frac{t}{q^3}}dt\Big)ds,$$

Let $t-s=u \Rightarrow dt=du \Longrightarrow D\{w(t) * g(t)\} = \frac{1}{q^2}\int_0^\infty \, w(s)\int_0^\infty \, g(u)e^{-\frac{(s+u)}{q^3}}duds,$

$$= \frac{1}{q^2}\int_0^\infty \, w(s)e^{-\frac{s}{q^3}}ds \cdot \int_0^\infty \, g(u)e^{-uq}du = q^2\left[\int_0^\infty \frac{1}{q^2}e^{-\frac{s}{q^3}}w(t)dt\right] * \left[\int_0^\infty \frac{1}{q^2}\, e^{-\frac{s}{q^3}}g(u)du\right],$$

$$= q^2\, W(q)\, G(q).$$

5. Dukani Transform of Derivatives

Theorem 6: Let $D\{w(t)\} = W(q)$ then

a) $D(w'(t)) = q^{-3}W(q) - q^{-2}w(0),$

b) $D\{w''(t)\} = q^{-6}W(q) - q^{-5}w(0) - q^{-2}w'(0),$

c) $D\{w'''(t)\} = q^{-9}W(q) - q^{-8}w(0) - q^{-5}w'(0) - q^{-2}w''(0),$

d) $D\{w^n(t)\} = q^{-3n}W(q) - \sum_{k=0}^{n-1} q^{-3k-2}w^{n-1-k}(0).$

Proof:

a) $D\{w'(t)\} = \frac{1}{q^2}\int_0^\infty \, e^{-\frac{t}{q^3}}w'(t)dt$

let $w = e^{-\frac{t}{q^3}} \Rightarrow dw = -q^{-3}e^{-\frac{t}{q^3}}dt$, and $dv = w'(t)dt \Rightarrow v = w(t)$

$$D\{w'(t)\} = \frac{1}{q^2}\left\{e^{-\frac{t}{q^3}}w(t)\Big|_0^\infty + q^{-3}\int_0^\infty \, e^{-\frac{t}{q^3}}w(t)dt\right\},$$

$$D\{w'(t)\} = -\frac{1}{q^2}\, w(0) + \frac{q^{-3}}{q^2}\int_0^\infty \, e^{-tq}w(t)dt,$$

$$D\{w'(t)\} = q^{-3}W(q) - q^{-2}w(0).$$

b) $D\{w''(t)\} = D\{(w'(t))'\}$

$$D\{w''(t)\} = q^{-3}D\{w'(t)\} - q^{-2}w'(0),$$

$$D\{w''(t)\} = q^{-3}\{q^{-3}W(q) - q^{-2}w(0)\} - q^{-2}w'(0),$$

$$D\{w''(t)\} = q^{-6}W(q) - q^{-5}w(0) - q^{-2}w'(0).$$

c) $D\{w'''(t)\} = q^{-3}D\{w''(t)\} - q^{-2}w''(0)$

$$D\{w'''(t)\} = q^{-3}\{q^{-6}\, W(q) - q^{-5}w(0) - q^{-2}w'(0)\} - q^{-2}w''(0)$$

$$D\{w'''(t)\} = q^{-9}W(q) - q^{-8}w(0) - q^{-5}w'(0) - q^{-2}w''(0).$$

d) mathematical induction can be used for solving part four

$$D\{w^n(t)\} = q^{-3n}W(q) - \sum_{k=0}^{n-1} q^{-3k-2}w^{n-1-k}(0)$$

6. Dukani Transformation of Integral of a Function

Theorem 7: Let $D\{w(t)\} = W(q)$ then $\left\{\int_0^t \, w(s)ds\right\} = \frac{W(q)}{q}.$

Proof: Let $G(t) = \int_0^t w(s)ds$ so $G'(t) = w(t)$ and $D\{G'(t)\} = D(w(t))$,

$q^{-3}D\{G(t)\} - q^{-2}G(0) = W(q)$,

$q^{-3}D\{G(t)\} = W(q) \rightarrow D\{G(t)\} = q^3W(q) \rightarrow D\left\{\int_0^t w(s)ds\right\} = q^3W(q)$.

7. Dukani Transform of Variable Coefficients

Theorem 8: If $D\{w(t)\} = W(q)$, then

a) $D\{tw(t)\} = \frac{q^4}{3}\frac{d}{dq}\big(W(q)\big) + \frac{2}{3}q^3W(q)$.

b) $D\{t^2w(t)\} = \frac{q^8}{9}\frac{d^2}{dq^2}\big(W(q)\big) - \frac{8}{9}q^7\frac{d}{dq}\big(W(q)\big) + \frac{22}{9}q^6W(q)$.

c) $D\{tw'(t)\} = \frac{q}{3}\frac{d}{dq}\big(W(q)\big) - \frac{1}{3}\big(W(q)\big)$.

Proof:

a) $D\{w(t)\} = \frac{1}{q^2}\int_0^\infty w(t)e^{-\frac{t}{q^3}}dt = W(q)$

Now $\frac{d}{dq}\big(W(q)\big) = \frac{1}{q^2}\int_0^\infty \frac{3t}{q^4}w(t)e^{-\frac{t}{q^3}}dt - \frac{2}{q^3}\int_0^\infty w(t)e^{-\frac{t}{q^3}}dt$

$\frac{d}{dq}\big(W(q)\big) = \frac{3}{q^4}D\{tw(t)\} - \frac{2}{q}W(q) \Longrightarrow \frac{3}{q^4}D\{tw(t)\} = \frac{d}{dq}\big(W(q)\big) + \frac{2}{q}W(q)$.

$D\{tw(t)\} = \frac{q^4}{3}\frac{d}{dq}\big(W(q)\big) + \frac{2}{3}q^3W(q)$.

b) $D\{w(t)\} = \frac{1}{\sqrt{q}}\int_0^\infty w(t)e^{-\frac{t}{q^3}}dt = W(q)$ Now

$\frac{d^2}{dq^2}\big(W(q)\big) = \frac{9}{q^{10}}\int_0^\infty t^2w(t)e^{-\frac{t}{q^3}}dt - \frac{18}{q^7}\int_0^\infty tw(t)e^{-\frac{t}{q^3}}dt - \frac{6}{q^7}\int_0^\infty tw(t)e^{-\frac{t}{q^3}}dt + \frac{6}{q^4}\int_0^\infty w(t)e^{-\frac{t}{q^3}}dt$,

$\frac{d^2}{dq^2}\big(W(q)\big) = \frac{9}{q^8}D\{t^2w(t)\} + \frac{6}{q^2}W(q) - \frac{24}{q^5}D\{tw(t)\}$

$\frac{d^2}{dq^2}\big(W(q)\big) = \frac{9}{q^8}D\{t^2w(t)\} + \frac{6}{q^2}W(q) - \frac{24}{q^5}\left\{\frac{q^4}{3}\frac{d}{dq}\big(W(q)\big) + \frac{2}{3}q^3W(q)\right\}$,

$\frac{9}{q^8}D\{t^2w(t)\} = \frac{d^2}{dq^2}\big(W(q)\big) + \frac{8}{q}\frac{d}{dq}W(q) + \frac{22}{q^2}\big(W(q)\big)$

$D\{t^2w(t)\} = \frac{q^8}{9}\frac{d^2}{dq^2}\big(W(q)\big) - \frac{8}{9}q^7\frac{d}{dq}\big(W(q)\big) + \frac{22}{9}q^6W(q)$.

c) $D\{tw'(t)\} = \frac{1}{q^2}\int_0^\infty w(t)e^{-\frac{t}{q^3}}dt = W(q)$ Now, by using part (a) $D\{tw'(t)\} = \frac{q^4}{3}\frac{d}{dq}(D\{w'(t)\}) + \frac{2}{3}q^3D\{w'(t)\}$ and from theorem 6 we obtain $D\{tw'(t)\} = \frac{q^4}{3}\frac{d}{dq}(q^{-3}W(q) - q^{-2}w(0)) + \frac{2}{3}q^3\{q^{-3}W(q) - q^{-2}w(0)\}$,

$D\{tw'(t)\} = \frac{q^4}{3}\left\{\left(q^{-3}\frac{d}{dq}W(q) - 3q^{-4}W(q) + 2q^{-3}w(0)\right)\right\} + \left\{\frac{2}{3}W(q) - \frac{2}{3}qw(0)\right\}$,

$D\{tw'(t)\} = \frac{q}{3}\frac{d}{dq}W(q) - W(q) + \frac{2}{3}qw(0) - \frac{2}{3}qw(0) + \frac{2}{3}W(q)$,

$D\{tw'(t)\} = \frac{q}{3}\frac{d}{dq}W(q) - \frac{1}{3}W(q)$.

8. The Inverse Dukani Transformation

If $D\{w(t)\} = W(q)$, then $w(t)$ is known inverse Dukani transform of $W(q)$, and it is defined as $w(t) = D^{-1}\{W(q)\}$

9. Linearity of Inverse Dukani Transform

Theorem 8: If $D\{w_i(t)\} = W_i(q)\ and\ D^{-1}\{W_i(q)\} = w_i(t)$ then the inverse Dukani transform is also linear operator $D^{-1}\{\sum_{i=1}^{n} \alpha_i W_i(q)\} = \sum_{i=1}^{n} \alpha_i D^{-1}\{W_i(q)\}$, where α_i are arbitrary constants.

10. Illustrative Examples Applications

The six issues in this section demonstrate the practical applications of the Dukani transformation. These examples show how to solve first-kind Volterra integral equations and ordinary differential equations precisely.

Example 1:[22] Observe the integral equation $t + t^2 = \int_0^t cot(t-s)\, w(s)ds.$

Solution: The given integral equation is V.I.E of first kind $t - t^2 = \int_0^t cos(t-s)\, w(s)ds$

By taking Dukani transform operator for both sides, we acquire

$D\{t - t^2\} = D\left\{\int_0^t cos(t-s)\, w(s)ds\right\}$ from the convolution property in theorem 5 we obtain

$q^4 - 2q^7 = q^2 D\{cost\}D\{w(t)\} \Longrightarrow \qquad q^4 - 2q^7 = q^2 \frac{q}{q^6+1} W(q) \Longrightarrow (q - 2q^4)(q^6 + 1) = W(q) \qquad \Longrightarrow$ $W(q) = q^7 + q - 2q^{10} - 2q^4$ by take inverse Dukani transform we acquire

$D^{-1}\{W(q)\} = D^{-1}\{q - 2q^4 + q^7 - 2q^{10}\}$ so the exact solution is $w(t) = 1 - 2t + \frac{t^2}{2} - \frac{t^3}{3}$.

Example 2:[21] Observe the second kind integral equation

$w(t) = 1 - t + \int_0^t (t-u)w(u)du$

Solution: The given integral equation is V.I.E of second kind

$$w(t) = 1 - t + \int_0^t (t-u)w(u)du$$

By taking Dukani transform operator for both sides we acquire

$D\{w(t)\} = D\{1 - t\} + D\left\{\int_0^t (t-u)w(u)\right\}$ from the convolution property in theorem 5 we obtain $W(q) = D\{1 - t\} + q^2 D\{t\}D\{w(t)\} \Longrightarrow W(q) = q - q^4 + q^2 q^4\, W(q)$

$W(q)(1 - q^6) = q(1 - q^3) \Longrightarrow W(q) = \frac{q(1-q^3)}{1-q^6} = \frac{q}{1+q^3}$, by take inverse Dukani transform we acquire $D^{-1}\{W(q)\} = D^{-1}\left\{\frac{q}{1+q^3}\right\}$ Hence, the exact solution is $w(t) = e^{-t}$.

Example 3:[22] Observe the first kind V.I. with difference kernel defined as

$$sint = \int_0^t e^{t-s} w(s)ds.$$

Solution: The I.E is $sint = \int_0^t e^{t-s} w(s)ds$

By operating Dukani transformation for each side, we acquire

$D\{sint\} = D\left\{\int_0^t e^{t-s} w(s)ds\right\}$ and by using theorem 5 of convolution property, we obtain

$\frac{q^4}{(q^6+1)} = q^2\ D\{e^t\}D\{w(t)\} \Longrightarrow \frac{q^4}{(q^6+1)} = q^2\ \frac{q}{(1-q^3)} W(q) \rightarrow W(q) = \frac{q(1-q^3)}{(q^6+1)}$

Taking the inverse Dukani transform for each side, we acquire

$$D^{-1}\{W(q)\} = D^{-1}\left\{\frac{(q-q^4)}{(q^6+1)}\right\} = D^{-1}\left\{\frac{q}{(1+q^6)}\right\} - D^{-1}\left\{\frac{q^4}{(1+q^6)}\right\}.$$

Then, the exact solution is $w(t) = cost - sint$.

Remark: The above three problem has been solved in [22] and the same exact solution obtained with large computational work that we obtained by Dukani transform easily.

Example 4:[23] Observe the second order differential equation

$$w'' - 2w' - 3w = 0; w(0) = 1\,, w'(0) = 2$$

Solution: We can solve the differential equation by Dukani transform

Apply Dukani transformation for each side we acquire

$D\{w'' - 2w' - 3w\} = D\{0\}$, from linearity property in theorem 2 and from property Dukani transformation for derivative in theorem 6 we obtain

$q^{-6}W(q) - q^{-5}w(0) - q^{-2}\ w'(0) - 2q^{-3}W(q) + 2q^{-2}\ w(0) - 3W(q) = 0$

$(q^{-6} - 2q^{-3} - 3)W(q) = q^{-5} \rightarrow W(q) = \frac{q^{-5}}{q^{-6}-2q^{-3}-3} = \frac{q}{(1-3q^3)(1+q^3)} = \frac{3}{4}\frac{q}{(1-3q^3)} + \frac{1}{4}\frac{q}{(1+q^3)}$

Apply inverse Dukani transformation for each side we acquire

$D^{-1}\{W(q)\} = D^{-1}\left\{\frac{3}{4}\frac{q}{(1-3q^3)} + \frac{1}{4}\frac{q}{(1+q^3)}\right\}$, Then, the exact solution is $w(t) = \frac{3}{4}e^{3t} + \frac{1}{4}e^{-t}$.

Example 5: Observe the first order differential equation $w' - 2w = 0,\ \ w(0) = 1$

Solution: We can solve the differential equation by Dukani transform

Apply Dukani transformation for each side we acquire

$D\{w' - 2w\} = D\{0\}$, from linearity property in theorem 2 and from property Dukani transformative for derivative in theorem 6 we obtain

$q^{-3}W(q) - q^{-2}w(0) - 2W(q) = 0 \Longrightarrow (q^{-3} - 2)W(q) = q^{-2} \rightarrow W(q) = \frac{q}{(1-2q^3)}$

Apply inverse Dukani transformation for each side we acquire $D^{-1}\{W(q)\} = D^{-1}\left\{\frac{q}{(1-2q^3)}\right\}$

Then, the exact solution is $w(t) = e^{2t}$.

Example 6:[23] Observe the first order differential equation $w'(t) + w(t) = 2,\ \ w(0) = 1$

Solution: We can solve the differential equation by Dukani transform

Apply Dukani transformation for each side we acquire

$D\{w' + w\} = D\{1\}$ from linearity property in theorem 2 and from property Dukani transformation for derivative in theorem 6 we obtain

$q^{-3}W(q) - q^{-2}w(0) + W(q) = 2q$

$(q^{-3}+1)W(q) = 2q + q^{-2} \rightarrow W(q) = \frac{2q}{q^{-3}+1} + \frac{q^{-2}}{q^{-3}+1}$

Apply inverse Dukani transform for both sides we acquire

$D^{-1}\{W(q)\} = D^{-1}\left\{2q - \frac{2q}{1+q^3} + \frac{q}{1+q^3}\right\}$,

Then, the exact solution is $w(t) = 2 - e^{-t}$.

Remark 2: The above two example has been solved in [21] and the same exact solution obtained.

Example 7:[21] Observe the integro differential equations

$$w'' = -1 - t + \int_0^t (t-u)w(u)du \ ; w(0) = w'(0) = 1 \ .$$

Solution: The given integral equation is Volterra integral equation of first kind

$$w'' = -1 - t + \int_0^t (t-u)w(u)du \ ; w(0) = w'(0) = 1$$

By taking Dukani transform operator for both sides we acquire

$D\{w''\} = -D\{1\} - D\{x\} + D\left\{\int_0^t (t-u)w(u)du\right\}$ from the convolution property in theorem 5 we obtain $q^{-6}W(q) - q^{-5}w(0) - q^{-2}w'(0) = -q - q^4 + q^2q^4W(q)$

$(q^{-6} - q^6)W(q) = q^{-2} + q^{-5} - q - q^4 \Longrightarrow (1 - q^{12})W(q) = q^4 + q - q^7 - q^{10}$

$(1 - q^6)(1 + q^6)W(q) = q(1 + q^3) - q^7(1 + q^3) = q(1 + q^3)(1 - q^6)$

$(1 + q^6)W(q) = q(1 + q^3)$

$W(q) = \frac{q}{(1+q^6)} + \frac{q^4}{(1+q^6)}$ by take inverse Dukani transform we acquire $D^{-1}\{W(q)\} = D^{-1}\left\{\frac{q}{(1+q^6)} + \frac{q^4}{(1+q^6)}\right\}$,
Then, the exact solution is $w(t) = sint + cost$.

Example 8:[21] Observe the integro differential equations $w' = 1 + \int_0^t w(u)du \ ; w(0) = 1$.

Solution: The given integral equation is Volterra integral equation of first kind

$w' = 1 + \int_0^t w(u)du \ ; w(0) = 1$, By taking Dukani transform operator for both sides we acquire $D\{w'\} = D\{1\} + D\left\{\int_0^t w(u)\, du\right\}$ from the convolution property in theorem 7 we obtain $q^{-3}W(q) - q^{-2}w(0) = q + q^3W(q) \Longrightarrow (q^{-3} - q^3)W(q) = q + q^{-2}$

$W(q) = \frac{q^4}{(1-q^3)} + \frac{q}{(1-q^3)}$ by take inverse Dukani transform we acquire

$D^{-1}\{W(q)\} = D^{-1}\left\{\frac{q^4}{(1-q^3)} + \frac{q}{(1-q^3)}\right\}$Then, the exact solution is $w(t) = sinht + cosht = e^t$.

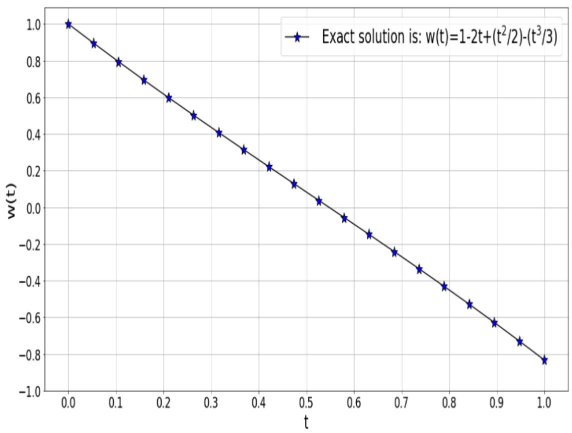


Fig. 1 Exact solution for example 1, where $t \in [0,1]$

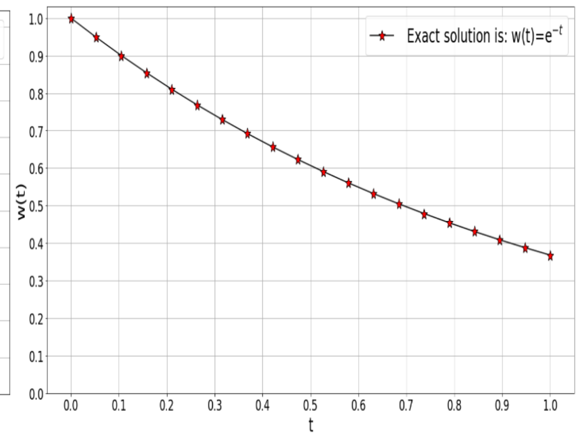


Fig. 2 Exact solution for example 2, where $t \in [0,1]$

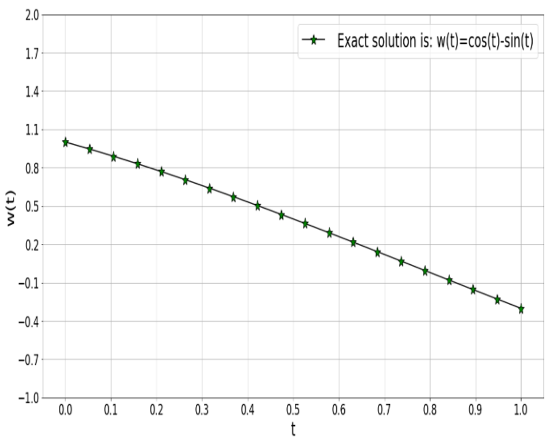


Fig. 3 Exact solution for example 3, where $t \in [0,1]$

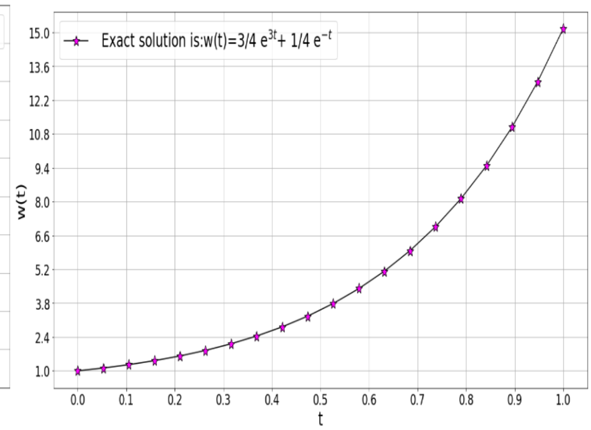


Fig. 4 Exact solution for example 4, where $t \in [0,1]$

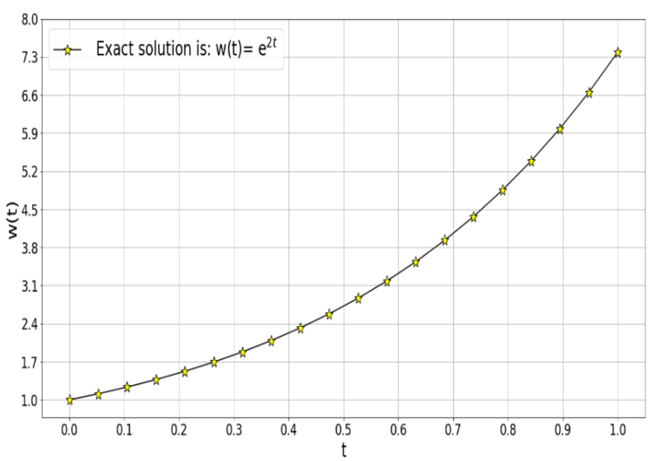


Fig. 5 Exact solution for example 5, where $t \in [0,1]$

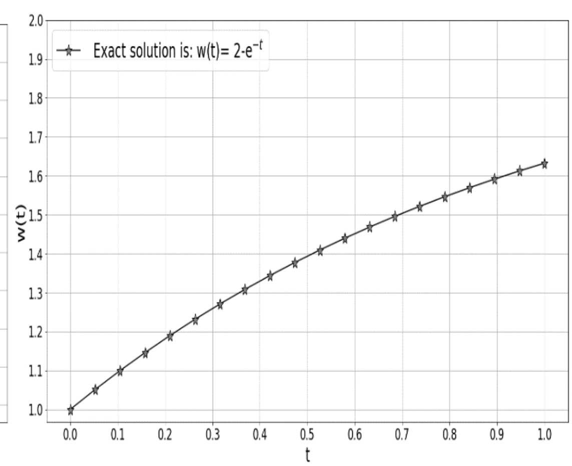
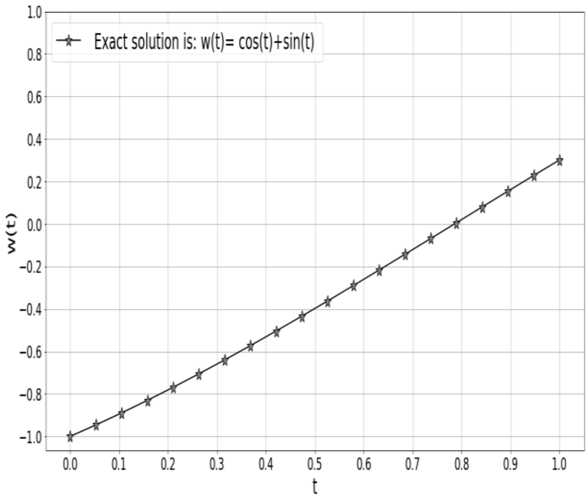


Fig. 6 Exact solution for example 6, where $t \in [0,1]$



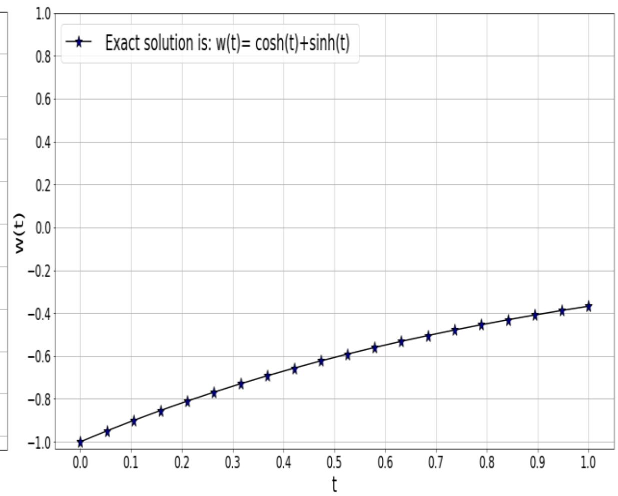


Fig. 7 Exact solution for example 7, where t ∈ [0,1]   Fig. 8 Exact solution for example 8, where t ∈ [0,1]

Figure 1, 2 and 3 illustrate the curve solutions of example 1, 2 and 3 respectively, within the time space $t \in [0,1]$. In this case, it is evident that when the space variable grows, $w(t)$ decreases, establishing a reverse relationship between both of them. Figures 4,5,6,7 and 8 show the curve solutions of example 4,5,6,7 and 8 respectively, with time interval $t \in [0,1]$. The graph shows that as the space variable grows, so does the function $w(t)$, indicating a positive association between the space variable and $w(t)$.

Conclusions

At last, we present a new transformation which is called the Dukani transformation, whose scaling, linearity, shifting, and convolutional properties were all verified. First- and higher-order ordinary differential equations, as well as first-kind Volterra integra equations, have been analyzed using the Dukani transformation. A few examples are included with the results and illustrated graphically by using Python. The Dukani transform will be applied to the analysis of fractional differential equations and specific models in the future.

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