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Laguerre-Krall polynomials for nonlinear integral equations of the first kind

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ABSTRACT

In this article, numerical method based on Laguerre-Krall polynomials for solving nonlinear integral equations (NIE) of the first type which is a malignant problem is discussed. This method reduces the operation of solving the problem and turns it into simple systems of obvious algebraic equations that are easily solvable. Four examples were presented to analyze the numerical method, applicability degree and accuracy of the method. Numerical results showed that the accuracy of this method is acceptable and good compared to other methods.

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1. Introduction

One of the most widely used mathematical tendencies, which is widely used in various engineering, medical and physics problems, is integral equations. Today, solving and checking an equation with a special method apart from other available methods is almost without validity and there is a need to check the relationship between methods in different sources. Various schemes have been derived to solve one-dimensional and multidimensional linear and nonlinear integral equations, for instance including operational matrix method *, covariance method *, use of wavelet, square method *, and discrete image method *. Two-dimensional ordinary integral equations are an important tool for modeling many problems in engineering. The most important problem for solving different equations in higher dimensions is one of the most complicated of mathematical operations.

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The present study proposes a new method for the numerical solution of nonlinear integral equations. It should be mentioned that the given method is based on the approximation of unknown functions by using the Laguerre-Krall polynomial.

Nonlinear multidimensional equations of Volterra with Legendre wavelets and singular integral differential equations are discussed in the articles [1, 2]. Delay and fractional equations with polynomials and orthogonal wavelets have been investigated in [4, 5, 6]. Piecewise fuzzy interpolation and rabid wavelet are used for delayed differential and integral equations in articles [7, 8, 9]. Volterra functional equations are also used with Euler and trapezoid discretization methods in [10]. For differential equations with different orders and integral equations, coexistence methods are mentioned in articles [12, 16]. Spectral Legendre method and Runge-Kutta method for multidimensional equations are given in articles [17, 18]. Nonlinear equations of Volterra and Fredholm have been analyzed and approximately solved in papers [14, 22] such as Hammerstein by quadrature Nystrom-based methods.

Also in new research, we can refer to article [25] in which smoothness properties and regularity of solutions to nonlinear second kind Volterra integral equations on a bounded interval [0,b] with probable singularities of the derivatives of the equation answer at near the zero point of the interval [0,b]. Article [26] is investigated to the solution of nonlinear second kind Volterra equations with high oscillatory kernel by using discrete collocation equation and a Filon-type quadrature rule. In [27], the authors organized a numerical scheme for an overall type of nonlinear Volterra equations. They discussed conditions that under them the equation has solutions by employing Quasilinearization scheme in which solving a nonlinear equation is reduced a sequence of linear system. An iterative numerical model based upon Nystrom method and Quasilinearization procedure to find the approximate acceptable solution of the nonlinear second kind Fredholm equations under some suitable assumptions is reduced to a sequence of linear Fredholm equations in [32]. The application of Laguerre-Krall polynomials in optimal control problems is proposed in [33]. It involves transforming the original integral equation problem into a manageable form that can be solved using prescribed optimization procedures.

In this paper, the important aim is to present an effective direct numerical method, using operational Laguerre-Krall matrices obtained based on properties of these polynomials, to solve the following non-linear ill-posed Volterra equations of the first type:

$$f(t) = \lambda \int_{t_0}^{t} k(x,t) h\big(u(x)\big) dx, \quad t \epsilon D \coloneqq [t_0, T\ ]\ * \tag{1}$$

where the given functions K,f are smooth and a non-linear function is in terms of sentences of unknown functions.

This article is organized and written as follows: we brought a brief introduction of CP and HCP polynomials and their properties. As a key idea, of this type of polynomials and their operational vectors. a general scheme for solving the nonlinear Volterra equations is presented. the error analysis is done and an upper bound for the error of the proposed method is obtained. some numerical examples have been presented and in these examples, the numerical results of presented method have been calculated in comparison with two other methods, including the Taylor series method and the Violet Legendre method, which is a confirmation of the appropriate error level of the existing method.

2. Laguerre-Krall Polynomials

Laguerre-Krall polynomials $L_n(x)$ of degree n in the some articles are defined as follows:

$$L_n(x) = \sum_{i=0}^{n} \frac{(-1)^i}{(i+1)!} \binom{n}{i} [i(\alpha + n + 1) + \alpha] x^i \tag{2}$$

A family of $\{L_n(x)\}_{n=0}^{\infty}$ polynomials is also orthogonal to the measure of ω .

where

$$d\omega = w(x)dx \tag{3}$$

Therefore, the weight function is:

$$w(x) = \frac{1}{\alpha}\delta(x) + e^{-x}H(x) \tag{4}$$

Where *H(x)* is the heavy side step function and measure ω refers to the weight of the Laguerre e^{-x} on the interval $[0, \infty)$.

The first six terms of this polynomial are listed as follows:

$L_0(x) = 1,$
$L_1(x) = 2 - 3x,$
$L_2(x) = 2 - 7x + 2x^2,$
$L_3(x) = 2 - 12x + 7x^2 - \frac{5x^3}{6},$
$L_4(x) = 2 - 18x + 16x^2 - \frac{23x^3}{6} + \frac{x^4}{4},$

$$L_5(x) = 2 - 25x + 30x^2 - \frac{65x^3}{6} + \frac{17x^4}{12} - \frac{7x^5}{120} \ ,$$

For any function $f: [0,1] \to R$ the Krall- Laguerre approximation $L_n(f)$ is considered as follows:

$$L_n(f)(x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) p_{n,i}(x)\ , \tag{5}$$

Where $p_{n,i}(x)$ is the polynomial of degree m as follows:

$$p_{n,i}(x) = \sum_{i=0}^{n} \frac{(-1)^i}{(i+1)!} \binom{n}{i} [i(\alpha + n + 1) + \alpha] x^i, \qquad i = 0,1, \dots, n. \tag{6}$$

3. Solving a Nonlinear Volterra Integral Equation of First Kind

In this section, the aim is to provide an efficient direct numerical method, using operational Laguerre-Krall matrices, for solving the following Volterra nonlinear first type integral equations:

$$f(t) = \lambda \int_{t_0}^{t} k(x,t) h\big(u(x)\big) dx, \quad t \epsilon D = [t_0, T\,]\,. \tag{7}$$

where the given functions $\boldsymbol{K}, \boldsymbol{f}$ are smooth and a non-linear function h is in terms of sentences of unknown functions *u(x)*.

We also assume that

$$u(x) = \sum_{i=0}^{n} u\left(\frac{i}{n}\right) \frac{(-1)^i}{(i+1)!} \binom{n}{i} [i(\alpha + n + 1) + \alpha] x^i\ , \tag{8}$$

Also by vector representation, we can write

$$u(x) = \sum_{n=1}^{N} c_n\, L_n(x) = C^T L(x) = C^T L\, X_x \ , \tag{9}$$

in which T is the transduction and C is vector coefficients of Laguerre-Krall unknown coefficients and vector $L(x)$ will be as follows:

$$\mathrm{C} = [c_1, c_1, \dots, c_N]^T, \quad L(x) = [L_1(x), L_2(x), \dots, L_N(x)]^T = L\, X_x \,, \tag{10}$$

In the last relation, X_x is the following vector:

$$X_x = [1, x, x^2, \dots, x^n]^T, \tag{11}$$

and L is a $n \times n$ coefficients matrix that can be approximately derived by X_x.

If we put, $g(x) = h\big(u(x)\big), \quad 0 \leq x \leq 1$, then we will have:

$$f(t) = \lambda \int_{t_0}^{t} k(x,t) g(x) dx, \quad t \epsilon D \coloneqq [t_0, T\,]\,. \tag{12}$$

And also,

$$u(t) \cong u_N(t) = u^t l(t) \;, \tag{13}$$

$$k(s,t) \cong k_N(s,t) = l^t(t) k l(t), \tag{14}$$

$$h(u(t)) \simeq h^t l(t)\,, \tag{15}$$

By inserting in the equation(12) , we will have:

$$F^T L(t) = L^T(t) \int_{t_0}^{t} K\, L(s)\, H^T\, L(s) ds, \tag{16}$$

Using the operational matrix and the product, we will have:

$$F^T L(t) = L^T(t) K\, H^T P L(t), \tag{17}$$

Suppose:

$$Q = K\, H^T P, \tag{18}$$

As a result, we will have: $F^T = Q,$

It can be said that without the operation vector, the usual collocation procedure is also used to solve above equation is unable.

Here we consider h to be invertible, then obtained from H and by

$$h\big(u(t)\big) \cong H^T\, L(t) \tag{19}$$

The unknown function can be obtained.

4. 4. Analysis of Error

In this section, we obtain the error estimate of the desired approximate solution to find the error bounds of the new numerical approach by applying the Logger-Kroll polynomial. Consider Volterra's type of nonlinear integral equations from the form of equation (1).

Here we can suppose that:

$$\Delta = L^2[0,1]\,,\ \{L_1(t), L_2(t), \dots, L_N(t)\} \subset \Delta, S = Span\{L_1(t), L_2(t), \dots, L_N(t)\}. \tag{20}$$

We consider the arbitrary function $y(t)$ as the best approximation of S.

Let $y_n \epsilon S$, so we can write:

$$\exists y_n \epsilon S : \forall g \in S \Rightarrow \|y - y_n\|_2^2 \leq \|y - g\|_2^2, \tag{21}$$

Where $\|y - y_n\|_2^2 = \int_0^1 |y(t)|^2 dt.$ (22)

Here we find an upper bound for the relation (16) that is error term.

We set, $e_n(t) = y(t) - y_n(t)$, where $y(t)$ is the exact analytical solution and $y_n(t)$ is the approximate proposed solution of equation (1).

Therefore, we can write as follows:

$$\|e_n(t)\|_2^2 = \|y(t) - y_n(t)\|_2^2 = \int_0^1 |y(t) - y_n(t)|^2 dt \tag{23}$$

$$= \int_0^1 \left| \int_{t_0}^t k(x,t)h(u(x)) - k(x,t)h(u_n(x))dx \right|^2 dt$$

(24)

$$= \int_0^1 \left| \int_{t_0}^t k(x,t)(h(u(x)) - h(u_n(x)))dx \right|^2 dt$$

(25)

On the other hand, $h(x)$ is continuous in considered interval and so it is locally Lipchitz continuous in $x\epsilon\mathbb{R}$, then, there is a constant C such that

$$|h(u(x)) - h(u_n(x))| \leq C|u(x) - u_n(x)|. \tag{26}$$

Then by relations (25) and (26) we can write,

$$\|e_n(t)\|_2^2 \leq \int_0^1 \left| \int_{t_0}^t k(x,t)C|u(x) - u_n(x)|dx \right|^2 dt \tag{27}$$

$$= \int_0^1 \left| \int_{t_0}^t k(x,t)C\left|u(x) - \sum_{n=1}^N c_n L_n(x)\right|dx \right|^2 dt \tag{28}$$

$$= \int_0^1 \left| \int_{t_0}^t k(x,t)C\left|\sum_{n=N+1}^\infty c_n L_n(x)\right|dx \right|^2 dt \tag{29}$$

$$\leq \int_0^1 \left| \int_{t_0}^t k(x,t)C \sum_{n=N+1}^\infty |c_n||L_n(x)|\, dx \right|^2 dt \tag{30}$$

$$\leq \int_0^1 \left| \int_{t_0}^t k(x,t)C \sum_{n=N+1}^\infty |c_n| \sum_{i=0}^n \frac{(-1)^i}{(i+1)!}\binom{n}{i}[i(\alpha+n+1)+\alpha]x^i\, dx \right|^2 dt\,, \tag{31}$$

And so,

$$\|e_n(t)\|_2^2 \leq \sqrt{\sum_{n=N+1}^\infty \sum_{i=0}^n \frac{(-1)^i}{(i+1)!}|c_n|\binom{n}{i}[i(\alpha+n+1)+\alpha]c^i}\ . \tag{32}$$

5. Results and Numerical Examples

Example 1. The first example is a non-linear integral problem with a trigonometric kernel:

$$\int_0^t \cos(t-s)\, y^n(s)\, ds = 6\left(1-\cos(s)\right), \quad y(0) = y'(0) = 0, \tag{33}$$

This equation has the following exact answer: $y(s) = s^3$.

The approximate solution using the proposed design by orthogonal bases is in high precision compared to the exact solution. The approximate results of this example using the Legendre wavelet method, Taylor Series method and proposed method are listed in Table 1 for N=4, 6, 8, 10,12,14,16.

Table 1. Numerical results of example 1

N	$e_N = \lVert y - y_N \rVert_2$ (Presented Method)	$e_N = \lVert y - y_N \rVert_2$ (*Taylor Series Method*)	$e_N = \lVert y - y_N \rVert_2$ (*Legendre Wavelet Method*)
4	3.415×10^{-8}	2.159×10^{-4}	3.202×10^{-3}
6	1.325×10^{-8}	2.834×10^{-4}	2.834×10^{-2}
8	3.914×10^{-9}	6.162×10^{-4}	3.366×10^{-3}
10	2.369×10^{-9}	8.024×10^{-5}	4.369×10^{-4}
12	3.251×10^{-9}	3.214×10^{-5}	3.321×10^{-4}
14	5.243×10^{-9}	6.251×10^{-5}	5.661×10^{-4}
16	5.254×10^{-9}	6.187×10^{-5}	4.277×10^{-4}

Example 2. The second example is a nonlinear equation with an exponential kernel:

$$\int_0^t e^{(t-s)}\, ln(y(s))\, ds = e^s - s - 1\,, \tag{34}$$

This equation has the following exact answer: $y(s) = e^s$.

The approximate results of this example using the Legendre wavelet method, Taylor Series method and our desired Laguerre-Krall method are listed in Table 2 for $N = 4, 6, 8, 10, 12, 14, 16$. In the numerical results we see that the error of proposed method has better than both Legendre wavelet scheme and Taylor Series method.

Table 2. Numerical results of example 2

N	$e_N = \lVert y - y_N \rVert_2$ (Presented Method)	$e_N = \lVert y - y_N \rVert_2$ (*Taylor Series Method*)	$e_N = \lVert y - y_N \rVert_2$ (*Legendre Wavelet Method*)
4	3.421×10^{-9}	3.579×10^{-5}	3.822×10^{-3}
6	1.645×10^{-9}	3.147×10^{-5}	6.353×10^{-3}

8	4.510×10^{-9}	2.714×10^{-4}	5.313×10^{-4}
10	5.326×10^{-9}	2.198×10^{-4}	1.676×10^{-4}
12	8.522×10^{-9}	4.212×10^{-5}	2.612×10^{-4}
14	2.137×10^{-10}	2.411×10^{-5}	1.676×10^{-4}
16	3.354×10^{-10}	2.012×10^{-5}	2.525×10^{-4}

Example 3. The third example is also a nonlinear equation with an exponential kernel:

$$\int_0^t e^{(t-x)}\, u^2(s)dx = e^{2s} - e^s, \quad (35)$$

This equation has exact solution as $u(s) = e^s$.

The approximate results of this example using the Legendre wavelet method, Taylor Series method and proposed Laguerre-Krall method are listed in Table 3 for $N = 4, 6, 8, 10,12,14,16$. In this example, The error of the mentioned method is more suitable than the other two methods. Also, Legendre wavelet 's method has closer error results to our research method.

Table 3. Numerical results of example 3

N	$e_N = \|y - y_N\|_2$ (Presented Method)	$e_N = \|y - y_N\|_2$ (Taylor Series Method)	$e_N = \|y - y_N\|_2$ (Legendre Wavelet Method)
4	2.425×10^{-8}	1.354×10^{-4}	5.214×10^{-7}
6	3.545×10^{-8}	2.645×10^{-4}	6.184×10^{-7}
8	2.587×10^{-9}	3.154×10^{-4}	3.310×10^{-7}
10	6.179×10^{-9}	2.159×10^{-5}	1.798×10^{-7}
12	3.547×10^{-10}	6.012×10^{-5}	3.626×10^{-7}
14	2.223×10^{-10}	2.197×10^{-5}	3.365×10^{-8}
16	4.421×10^{-10}	3.210×10^{-5}	1.932×10^{-8}

Example 4. The fourth example is also a non-linear equation with a sine trigonometric kernel:

$$\int_0^t (\sin(t-s)+1)\cos(u(s))\,ds = \frac{sins}{2} + sins,$$

(36)

The exact solution of this equation is $u(s) = s$.

The approximate results of this example using the Legendre wavelet method, Taylor Series method and proposed Laguerre-Krall method are listed in Table 4 for $N = 4, 6, 8, 10,12,14,16$. In this example, the absolute error of the obtained results and comparing them with other selected methods is a confirmation of the high accuracy of our method. We can see that as the value of N increases, the absolute error decreases.

Table 4. Numerical results of example 4

N	$e_N = \|y - y_N\|_2$ (Presented Method)	$e_N = \|y - y_N\|_2$ (*Taylor Series Method*)	$e_N = \|y - y_N\|_2$ (*Legendre Wavelet Method*)
4	2.155×10^{-8}	6.152×10^{-4}	3.364×10^{-3}
6	3.261×10^{-8}	6.264×10^{-4}	1.021×10^{-2}
8	4.415×10^{-9}	4.153×10^{-4}	2.004×10^{-3}
10	3.251×10^{-9}	6.131×10^{-5}	3.108×10^{-4}
12	6.258×10^{-9}	1.212×10^{-5}	3.058×10^{-4}
14	6.951×10^{-9}	2.369×10^{-5}	3.099×10^{-4}
16	7.357×10^{-9}	1.745×10^{-5}	2.084×10^{-4}

Conclusions

Explicit formulas for approximating operation vectors for different equations have been proposed with Laguerre-Krall polynomials. These vectors allow us to find our suitable and simple numerical scheme that can be used for all types of integral equations. Finally, the problem is reduced to solving a set of algebraic equations with few operations. The main advantages of this method include ease of implementation, simplicity of understanding, high accuracy and appropriate convergence rate. The absolute error of the obtained results and comparing them with other selected methods is a confirmation of the high accuracy of our mentioned method.

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