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Supplement -Hollow and Supplement-Lifting Modules

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ABSTRACT

The principle idea of our work is to present newly generalization for two major types of modules, which are hollow module and lifting module they are supplement –hollow module and supplement –lifting module respectively. We consider supplement - hollow module as. Let C be a non-zero unital left R-module and R be a ring with identity then C is known as the supplement-hollow module (indicates sp-hollow). If each proper submodule of C is a supplement−small submodule in C. Furthermore, an R-module C is known as supplement –lifting module (indicates sp−lifting).If there is a submodule H of U for each sub module U of C such as $C = H\oplus L$ and $U \cap L \ll_{sp} C$,where L is a submodule of C. After studying these ideas, we came up with some connected findings

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1.Introduction

All throughout work. R indicates the commutative ring with unity and modules will be unitary left $R-$module. A proper submodule L of a module C is called small (notation $L \ll C$), if for every submodule H of a module C, with $C = L + H$, implies that $H = C$ see [1]. A submodule P of a module C is called supplement in C, if and only if $C = T + P$ and $T \cap P \ll P$ whose $T \leq C$ see [2] . C be a non-zero module is referred to as hollow module if every proper submodule of C is small see [3].The $R-$module C is referred to as lifting whatever a sub module N of C ,there is a direct summand W of C such as $W \leq N$ and $\frac{N}{W} \ll \frac{C}{W}$ see [4]. More points around lifting modules see in [5],[6]. A previous study [7] presented the idea of a supplement-small submodule, which is a new generalization of a small submodule, such that a supplement−small (sp−small) submodule of C is a proper submodule W of a module

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C, as denoted by $(W \ll_{sp} C)$. If $W + H = C$ whose $H \leq C$, then H is a supplement sub module of C. Several authors have constructed and studied several kinds of small submodules as a generalization show [8–12]. In this paper, we provide the following definition as a generalization of hollow module. A non zero R −module C is referred to as supplement−hollow (sp−hollow) if each proper submodule of C is a supplement−small submodule in C. Different properties of this type of module are investigated .As a generalization of lifting module , the notation for a supplement –lifting module has been established such that an R −module C is referred to as supplement-lifting (sp−lifting) . For each sub module N of C there is a sub module K of N in the case of $C = K \oplus H$ and $N \cap H \ll_{sp} C$,where H is a sub module of C .Several characteristics of this idea will be determined.

Now, start with the following two lemmas that we need it in this paper.

Lemma (1.1) [7] : Assume that C is R −module . Next statements are hold.

1) Let H and V are sub modules of , such that $H \leq V \leq C$, if $V \ll_{sp} C$, then $H \ll_{sp} C$.

2) Let $f: C_1 \to C_2$ be an isomorphism where C_1 and C_2 be R −modules, H a submodule of C_1 ,if $H \ll_{sp} C_1$,

then $f(H) \ll_{sp} C_2$.

3) Let an R −module C has (SUSP) and V, L are sub modules of C, such that $V \leq L \leq C$, and L is direct

summand of C , if $V \ll_{sp} C$, then $V \ll_{sp} L$.

Where in [13] the following concept have been presented: the summation of any two supplement of an R −module

C is again supplement , and then the module C has the supplement sum property , for short (SUSP).

Lemma (1.2) [7] :

1) Let H , V be two submodules of C such that $H \leq V \leq C$, if $V \ll_{sp} C$, then $\frac{V}{H} \ll_{sp} \frac{C}{H}$

2) Let H be a submodule of C ,where C has (SUSP) , H is a supplement submodule in C , such as $H \leq V \leq C$,

if $\frac{V}{H} \ll_{sp} \frac{C}{H}$,then $V \ll_{sp} C$.

We prove the following lemma that we used in this paper .

Lemma (1.3) [7]:

Assume that $C = C_1 \oplus C_2$ such as $R = Ann(C_1) + Ann(C_2)$, if $P_1 \ll_{sp} C_1$ and $P_2 \ll_{sp} C_2$ then $P_1 \oplus P_2 \ll_{sp} C_1 \oplus C_2$.

Proof:

Suppose that T be sub module of C ,such as $P_1 \oplus P_2 + T = C$. Because $R = Ann(C_1) + Ann(C_2)$, then $T = T_1 \oplus T_2$ for some $T_1 \leq C_1$ and $T_2 \leq C_2$, then $P_1 \oplus P_2 + T_1 \oplus T_2 = C_1 \oplus C_2$. So $(P_1 + T_1) \oplus (P_2 + T_2) = C_1 \oplus C_2$ and hence $P_1 + T_1 = C_1$ and $P_2 + T_2 = C_2$ and since $P_1 \ll_{sp} C_1$ and $P_2 \ll_{sp} C_2$, then $T_1 \leq_{sp} C_1$ and $T_2 \leq_{sp} C_2$, therefore $T_1 \oplus T_2 \leq_{sp} C_1 \oplus C_2$ by [14], and hence $T = T_1 \oplus T_2 \leq_{sp} C_1 \oplus C_2 = C$, so $T \leq_{sp} C$, hence $P_1 \oplus P_2 \ll_{sp} C_1 \oplus C_2$.

2. Supplement −Hollow Module

This part is devoted to express the idea of supplement−hollow ,with some examples and main properties .

Definition (2.1) : Assuming C is a non –zero module ,it is referred to as **a supplement−hollow** (sp−hollow) module .If every proper sub module of C is a supplement−small sub module in C.

Remarks and Examples (2.2)

1) Since each small submodule is sp−small [7], therefore each hollow module is sp−hollow. Thus , in Z_4 as Z −module is a hollow and hence sp-hollow module.

The opposite of (1) is not true, Thus in the following example: in the $Z-$ module Z_6 all proper sub modules are sp−small , since $\{\bar{0}, \bar{3}\}$ is supplement –small sub module in Z_6 as the only submodule L such as, $\{\bar{0}, \bar{3}\} + L = Z_6$ are $\{\bar{0}, \bar{2}, \bar{4}\}$ and Z_6 as well $\{\bar{0}, \bar{2}, \bar{4}\}$ are supplement .Then $\{\bar{0}, \bar{3}\} \ll_{sp} Z_6$, in similar way $\{\bar{0}, \bar{2}, \bar{4}\} \ll_{sp} Z_6$, then it is supplement – hollow ,but Z_6 as the $Z-$ module is not hollow.

2)Every simple module is hollow, so it is sp−hollow module.

3)In Z_{12} as the Z −module:$< \bar{3} > + < \bar{2} > = Z_{12}$ but $< \bar{2} >$ is not supplement in Z_{12} , so $< \bar{3} >$ is not sp−small in Z_{12} .hence Z_{12} as the $Z-$ module is not sp- hollow module.

4) If C is semi−simple module ,then C is sp−hollow. As in Z_6 as the $Z-$ module

5) In Z as the Z −module is not sp−hollow. Thus in the following, suppose $U = nZ$ is a proper submodule in Z , such that $nZ + mZ = Z$,but mZ is not supplement in Z , so nZ is not supplement –small in Z similarly mZ. and since all proper submodules are not sp- small submodules then, it is not supplement –hollow .

Proposition (2.3): Assuming $g: C_1 \to C_2$ be an isomorphism where C_1 and C_2 be an R −modules . If C_1 is sp−hollow , then C_2 is sp−hollow.

Proof:

Suppose that U is proper submodule of C_2. Thus $g^{-1}(U)$ is proper sub module of C_1. If not , $g^{-1}(U) = C_1$, then $U = C_2$ and that is contradiction , since C_1 is sp−hollow, hence $g^{-1}(U) \ll_{sp} C_1$. By lemma (1.1) we get $gg^{-1}(U) = U$, so $U \ll_{sp} C_2$ and hence C_2 is sp-hollow.

Proposition (2.4): Assume C is sp−hollow module with (SUSP) ,then the direct summand of C is sp−hollow.

Proof:

Suppose that U, H are proper sub modules of C, were $H \leq U$ and U is direct summand of C since C is sp−hollow so we get $H \ll_{sp} C$, and since U is a direct summand of C .By lemma (1.1) ,then $H \ll_{sp} U$ and U is sp-hollow.

Theorem (2.5) : Assume C_1 and C_2 be an R-modules and let $C = C_1 \oplus C_2$ such that $R = Ann(C_1) + Ann(C_2)$, then C is sp−hollow if and only if C_1 , C_2 are sp−hollow, providing that $U \cap C_x \neq C_x$ for $x = 1,2$ and $U \leq C$.

Proof:

It is clearly from Proposition (2.3) that C_1 , C_2 are sp − hollow .

The other side ,suppose that U is proper submodule of C , as well C_1, C_2 are sp-hollow. Because $R = Ann\,(C_1) + Ann\,(C_2)$, then $U = U_1 \oplus U_2$ for some $U_1 \leq C_1$ and $U_2 \leq C_2$, then $U = (U_1 \cap C_1) \oplus (U_2 \cap C_2)$, hence $U_1 \cap C_1$ and $U_2 \cap C_2$ are proper submodules of C_1 and C_2, and since C_1 and C_2 are sp-hollow, then $U_1 \cap C_1 \ll_{sp} C_1$ as well $U_2 \cap C_2 \ll_{sp} C_2$.By lemma (1.3) , we get $(U_1 \cap C_1) \oplus (U_2 \cap C_2) \ll_{sp} C_1 \oplus C_2$ and hence $U \ll_{sp} C$.

Recall that distributive R −module is defined as follows : If each H, L as well U are sub modules of C , then $H \cap (L + U) = (H \cap L) + (H \cap U)$.[15]

Proposition (2.6) : Assume $C = C_1 \oplus C_2$ is R-module has (SUSP) with C_1 , C_2 are sub modules of C as well C is distributive thus ,C is sp-hollow if and only if, C_1 as well C_2 are sp−hollow providing that $U \cap C_x \neq C_x$ considering $x = 1,2$ and $U \leq C$.

Proof:

It is clearly from theorem (2.5) that C_1 as well C_2 are sp−hollow

The other side , because C is distributive, then $U = (U \cap C_1) + (U \cap C_2)$ whose U is a proper sub module of C. hence from the similar proof of theorem (2.5), we get C is sp−hollow.

3. Supplement –Lifting module

The Supplement-lifting module present in this part.

Definition (3.1): Consider C to be an $R -$ module , C is named **supplement-lifting** for short (sp−lifting), if for each sub module U of C there is a submodule H of U such as $C = H \oplus L$ and $U \cap L \ll_{sp} C$,where L is a submodule of C.

Remarks and Examples (3.2):

1)Since each small sub module is sp−small [7] , then each lifting module is sp−lifting.

However, the converse is untrue, in the example that follows: In $C = Z_8 \oplus Z_2$ as Z −module . The submodules of C are $A_1 = <(\bar{1},\bar{0})>$,, $A_2 = <(\bar{2},\bar{0})>$, $A_3 = <(\bar{4},\bar{0})>$,$A_4 = <(\bar{0},\bar{1})>$,$A_5 = <(\bar{1},\bar{1})>$, $A_6 = <(\bar{2},\bar{1})>$, $A_7 = <(\bar{4},\bar{1})>$. $A_8 = \{(\bar{2},\bar{0}),(\bar{2},\bar{1}),(\bar{0},\bar{1})(\bar{4},\bar{0}),(\bar{4},\bar{1}),(\bar{6},\bar{0})(\bar{6},\bar{1}),(\bar{0},\bar{0})\}$.$A_9 = \{(\bar{4},\bar{0}),(\bar{4},\bar{1}),(\bar{0},\bar{1}),(\bar{0},\bar{0})\}$, $A_{10} = <(\bar{0},\bar{0})>$, $A_{11} = M$

So ,we obtain the following : for all submodules A_1 , A_2 , A_3, A_4, A_5, A_6, A_7 ,there exists $\{(\bar{0},\bar{0})\}$ is direct summand of C,so that $C = \{(\bar{0},\bar{0})\} \oplus L$ and $L \leq C$, $A_x \cap L \ll_{sp} C$, $x = 1,2,3,4,5,6,7$.

For $A_8 = \{(\bar{2},\bar{0}),(\bar{4},\bar{0}),(\bar{6},\bar{0}),(\bar{2},\bar{1}),(\bar{4},\bar{1}),(\bar{6},\bar{1}),(\bar{0},\bar{1}),(\bar{0},\bar{0})\}$ .There exists $A_4 = <(\bar{0},\bar{1})>$direct summand of M , so that $C = A_4 \oplus A_1$ and $A_8 \cap A_1 = A_2 \ll_{sp} C$ .Similarly ,with a submodule A_9 . For $A_{10} = <(\bar{0},\bar{0})>$ if $K = 0$, so that $C = 0 \oplus C$ and $C \cap A_{10} = A_{10} \ll_{sp} M$. So C is sp- lifting . But C is not lifting since , if C is lifting then $C = \{(\bar{0},\bar{0})\} \oplus A_1$,but A_1 not small in C since $A_1 + A_5 = C$, $A_5 \neq C$ in $C = Z_8 \oplus Z_2$ as Z-module.

2) Z-module Z_{24} is not sp-lifting since, assume $U = Z_{24}$ the only direct summand of Z_{12} are $\{\bar{0}\}$, $3Z_{12}$ and $8Z_{12}$ such that $Z_{24} = H\oplus L$. If $H = \{\bar{0}\}$ thus $L = Z_{24}$ and $U \cap L = Z_{24} \cap Z_{24} = Z_{24}$ which is not sp-small in Z_{24}, if $H = 3Z_{12}$ so $L = 8Z_{24}$ and $Z_{24} \cap 8Z_{24} = 8Z_{24}$ which is not sp-small in Z_{12}, if $H = 8Z_{12}$ so $L = 3Z_{12}$ and $Z_{24} \cap 3Z_{24} = 3Z_{24}$ which is not sp-small in Z_{24}.

3) Each sp−hollow module is the sp-lifting module.

4) Each local module is hollow hence sp-hollow by (2.2), hence it is sp-lifting by 3.

Proposition (3.3) : Suppose C is indecomposable, if C is sp-lifting module, then C is sp-hollow.

Proof:

Assuming that C is sp−lifting and that U is a proper submodule of C ,let $P \leq U$. For $T \leq C$ and $U \cap T \ll_{sp} C$, such that $C = P\oplus T$ since C is indecomposable, then either $P = 0$ or $P = C$. If $P = C$ we have $U = C$,and this contradiction so $P = 0$ .Therefore, $C = T$, so $U = U \cap C = U \cap T \ll_{sp} C$ which means that , $U \ll_{sp} C$ and that C is sp-hollow .

Theorem (3.4): Assume C is an R-module , if C is sp-lifting module ,then each submodule U of C can be written as $U = P\oplus T$ where P is direct summand of C and $T \ll_{sp} C$.

Proof:

Suppose that U be submodule of C, so there exists a sub module W of U such that $C = W\oplus V$ and $U \cap V \ll_{sp} C$ whose V is a sub module of C, so by modular law we have $U = U \cap C = U \cap (W\oplus V) = W\oplus(U \cap V)$. Now let $W = P$ and $T = U \cap V$, so $U = P\oplus T$ whose P is direct summand of M as well $T \ll_{sp} C$.

Theorem (3.5): Assume C is an R-module ,if every submodule U of C can be written as $U = K\oplus L$ whose K direct summand of C as well $L \ll_{sp} C$ then for every sub module U of C, there exists a direct summand H of C ,such as $H \leq U$ and $\frac{U}{H} \ll_{sp} \frac{C}{H}$.

Proof :

Let U be sub module of C as well $U = K\oplus T$ whose K is direct summand of C , $T \ll_{SP} C$. It is enough to see , $\frac{U}{K} \ll_{sp} \frac{C}{K}$, suppose $\frac{W}{K} \leq \frac{C}{K}$ such as $\frac{U}{K} + \frac{W}{K} = \frac{C}{K}$ thus $\frac{K\oplus T}{K} + \frac{W}{K} = \frac{C}{K}$,then $C = K + T + W = T + W$ and because $T \ll_{sp} C$, so $W \leq_{sp} C$, so by[4,p.238],we get $\frac{W}{K} \leq_{sp} \frac{C}{K}$ and then $\frac{U}{K} \ll_{sp} \frac{C}{K}$.

Theorem (3.6): Assume C is an R-module with (SUSP) .If there exists direct summand H of C, for each sub module U of C such as $H \leq U$ and $\frac{U}{H} \ll_{sp} \frac{C}{H}$ then C is sp-lifting module .

Proof :

Suppose U be its sub module of C then there exists a submodule H of U ,such as $C = H\oplus L$ and $\frac{U}{H} \ll_{sp} \frac{C}{H}$, so by lemma(1.2) we get $U \ll_{sp} C$ and since $U \cap L \leq U \leq C$, hence by lemma (1.1) we have $U \cap L \ll_{sp} C$.

Proposition (3.7): Assume that C is sp−lifting module as well U, H the submodules of C so that $C = U + H$, then there is W is direct summand of C that is $U + W \leq_{sp} C$.

Proof:

Suppose C is sp−lifting then by theorem(3.4) we get , $H = W + L$ where W direct summand of C and $T \ll_{sp} C$, since $C = U + H$ hence $C = U + H = U + W + T$ and since $T \ll_{sp} C$,therefore $U + W \leq_{sp} C$.

3.Conclusions

In this work two classes of an R –modules whose supplement –hollow and supplement -lifting modules are present with many properties such as the image supplement –hollow module, is again a supplement –hollow. Also, C is R −module has (SUSP) , then any direct summand of sp−lifting module is sp−lifting is studied.

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