Journal of Al-Qadisiyah for Computer Science and Mathematics Vol.16.(4) 2024,pp.Math 52–63

Available online at www.qu.edu.iq/journalcm
JOURNAL OF AL-QADISIYAH FOR COMPUTER SCIENCE AND MATHEMATICS
ISSN:2521-3504(online) ISSN:2074-0204(print)

Applications of Quasi-Subordination on Subclasses of Bi-Univalent Functions Associated with Generalized Differential Operator

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ARTICLE INFO

Article history:
Received: 28 /11/2024
Rrevised form: 16 /12/2024
Accepted : 29 /12/2024
Available online: 30 /12/2024

Keywords: Analytic functions, Quasi-subordination, Bi-univalent functions, Generalized differential operator, Coefficient estimates.

Each keyword to start on a new line

ABSTRACT

This paper introduces and defines subclasses of the function class $\sum$ of analytic and bi-univalent functions associated with the operator $\mathcal{N}_{\lambda}^{n}(\alpha,\beta,\mu)\, f(z)$ within the open unit disk through quasi-subordination. We derive results concerning the corresponding bound estimations of the coefficients a_2 and a_3.

https://doi.org/10.29304/jqcsm.2024.16.41796

1-Introduction:

Let $\mathcal{A}$ be the class of all normalized analytic functions f in an open unit disk $U=\{z: z\in\mathbb{C}, |z|<1\}$ of the form:

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n,\ \ (z\in U). \tag{1.1}$$

A function f has an inverse f^{-1} is satisfying $f^{-1}\big(f(z)\big)=z, (z\in U)$, and $f\big(f^{-1}(w)\big)=w,\ \left(|w|<r_0(f), r_0(f)\geq\frac{1}{4}\right)$, where

$g(w)=f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-$

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$$(5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots, (w \in U). \tag{1.2}$$

If f and f^{-1} are univalent functions in U, then f is classified as bi-univalent in U, and the category of bi-univalent functions defined in U is represented by $\sum$, (see [26])

Let f and g are analytic functions in $\mathcal{A}$. Then f is said to be quasi-subordinate to g in U and written as follows:

$$f(z) \prec_q g(z), \ (z \in U),$$

if there is $\theta(z)$ also $w(z)$ be two analytic functions in U, with $w(0) = 0$ in such a manner that $|\theta(z)| < 1, |w(z)| < 1$ also $f(z) = \theta(z)g(w(z))$. If $\theta(z) = 1$, then $f(z) = g(w(z))$, therefore $f(z) \prec g(z)$ in U. If $w(z) = z$, then $f(z) = \theta(z)g(z)$, also it is said that f is majorized by g and written $f(z) \ll g(z)$ in U. (see [5], [25])

Ma and Manda [21] established a category of starlike and convex functions through the application of subordination and studied classes $S^*(\phi)$ and $\mathcal{G}^*(\phi)$ which is defined by

$$S^*(\phi) = \left\{f \in \mathcal{A}: \frac{zf'(z)}{f(z)} \prec \phi(z), z \in U\right\},$$

and

$$\mathcal{G}^*(\phi) = \left\{f \in \mathcal{A}: 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), z \in U\right\}.$$

By $S_{\Sigma}^*(\phi)$ and $\mathcal{G}_{\Sigma}^*(\phi)$, we denote to bi-starlike and bi-convex functions f is bi-starlike and bi-convex of Ma-Minda type respectively [21].

In the sequal, it assumed that ϕ of the form :

$$\phi(z) = 1 + \sum_{n=1}^{\infty} a_n z, \ (\ a_1 > 0, z \in U), \tag{1.3}$$

where $\phi(0) = 1$ and $\phi'(0) > 0$, also

$$\varphi(z) = a_0 + \sum_{i=1}^{\infty} a_i z^i, \tag{1.4}$$

which are analytic and bounded in U. However, there are only a few works determining the general coefficient bounds $|a_2|$ and $|a_3|$ ([2,7,8,12,13,16,17,18,20,22,23,24,27,30,31,32,33,34] and [20]) for the analytic bi-univalent functions in the literature. ([7,9]).

For a function $f \in \mathcal{A}$, $\alpha, \beta, \mu, \lambda \geq 0$ and $n \in \mathbb{N}_0$ we define the differential operator, as follows [11]:

$$\mathcal{N}^0 \ f(z) = f(z),$$

$$\mathcal{N}_{\lambda}^1(\alpha,\beta,\mu)f(z) = \left(\frac{\alpha - \mu + \beta - \lambda}{\alpha + \beta}\right) f(z) + \left(\frac{\mu + \lambda}{\alpha + \beta}\right) zf'(z),$$

$$\mathcal{N}_{\lambda}^2(\alpha,\beta,\mu)f(z) = \mathcal{N}\left(\mathcal{N}_{\lambda}^1(\alpha,\beta,\mu)f(z)\right),$$

$$\vdots$$

$$\mathcal{N}_{\lambda}^n(\alpha,\beta,\mu)f(z) = \mathcal{N}\left(\mathcal{N}_{\lambda}^{n-1}(\alpha,\beta,\mu)f(z)\right). \tag{1.5}$$

We note that if $f \in \mathcal{A}$ is given by (1.1), then by (1.5), we have

$$\mathcal{N}_{\lambda}^n(\alpha,\beta,\mu)f(z) = z + \sum_{k=2}^{\infty} \mathfrak{X}(\kappa,\alpha,\beta,\mu)\ a_k z^k,$$

where

$$\mathfrak{X}(\kappa,\alpha,\beta,\mu)=\left(\frac{\alpha+(\mu+\lambda)(\kappa-1)+\beta}{\alpha+\beta}\right)^{n}.$$

Figure 1 below describe the geometric changes on $f(z)=\mathrm{z}+6z^2$ under $\boldsymbol{\mathcal{N}}_2^1(0{,}1{,}3)$ and $\boldsymbol{\mathcal{N}}_3^1(0.5{,}0.5{,}10)$, respectively.

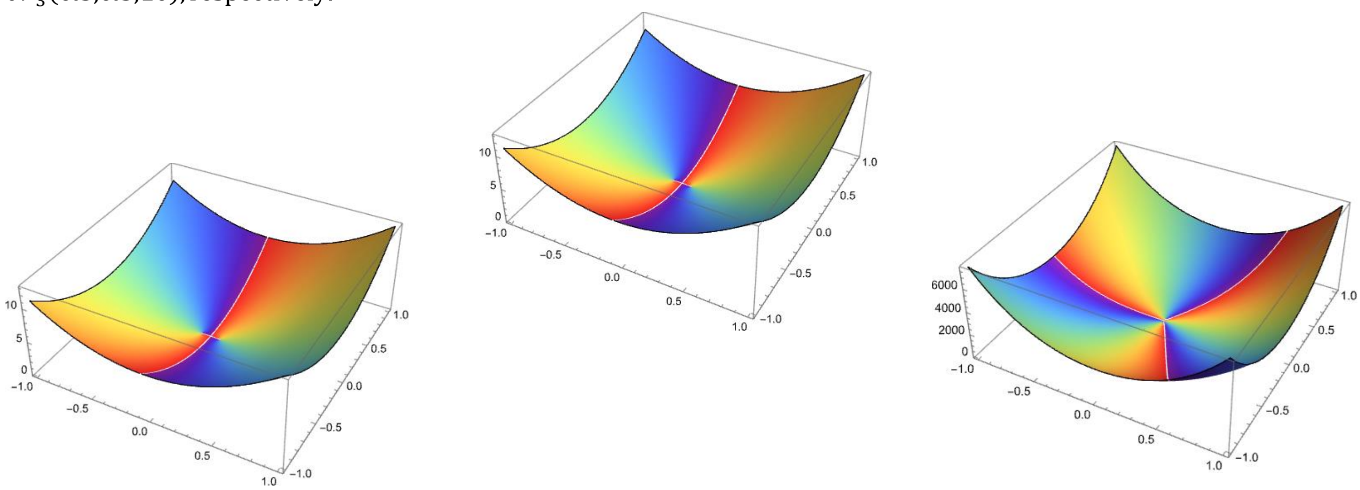

Fig. 1: Complex plot of $\boldsymbol{\mathcal{N}}_2^1(0,1,3)\ f(z)$, and $\boldsymbol{\mathcal{N}}_3^1(0.5,0.5,10)\ f(z)$, respectively.

Remark 1.1. By suitably specializing the parameters, the operator $\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)f(z)$ in (1.5) reduces to many known and new differential operators :

- When $\beta=1$ also $\mu=0$, we obtain
  $\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,1{,}0)f(z)=\boldsymbol{\mathcal{N}}^n f(z)=z+\sum_{n=2}^{\infty}\left(\frac{\alpha+\lambda(\kappa-1)+1}{\alpha+1}\right)^n a_n z^n$ of Aouf, El-Ashwah and El-Deeb differential operator [6].

- When $\alpha=1,\ \beta=0$ also $\mu=0$, we obtain
  $\boldsymbol{\mathcal{N}}_\lambda^n(1{,}0{,}0)f(z)=\boldsymbol{\mathcal{N}}^n f(z)=z+\sum_{n=2}^{\infty}\left(1+\lambda(\kappa-1)\right)^n a_n z^n$ of Al-Oboudi differential operator [1].

- When $\alpha=1,\ \beta=0,\ \lambda=1$ also $\mu=0$, we obtain
  $\boldsymbol{\mathcal{N}}_1^n(1{,}0{,}0)f(z)=\boldsymbol{\mathcal{N}}^n f(z)=z+\sum_{n=2}^{\infty}(\kappa)^n a_n z^n$ of Sălăgean's differential operator [26].

- When $\alpha=1,\ \beta=1,\ \lambda=1$ also $\mu=0$, we obtain
  $\boldsymbol{\mathcal{N}}_1^n(1{,}1{,}0)f(z)=\boldsymbol{\mathcal{N}}^n f(z)=z+\sum_{n=2}^{\infty}\left(\frac{\kappa+1}{2}\right)^n a_n z^n$ of Uralegaddi and Somanatha differential operator [28].

- When $\beta=1,\ \lambda=1$ also $\mu=0$, we obtain
  $\boldsymbol{\mathcal{N}}_1^n(\alpha,1{,}0)f(z)=\boldsymbol{\mathcal{N}}^n f(z)=z+\sum_{n=2}^{\infty}\left(\frac{\kappa+\alpha}{\alpha+1}\right)^n a_n z^n$ of Cho and Srivastava differential operator [9,10].

Lemma (1.1) [14]. Let $h(z)=1+h_1z+h_2z^2+\cdots\in P$, where P is the family of all functions h, analytic in U, for which $Re\{h(z)\}>0,(z\in U)$, then $|h_i|\le 2$ for $i=1{,}2{,}3,\cdots$.

2-Coefficient Estimates for the Class $\mathcal{W}_{\Sigma,q}^{\alpha,\beta,\mu,n}(\sigma,\gamma,\rho,\lambda;\phi)$.

Definition 2.1. A function $f \in \sum$ is assumed as in (1.1), then $f \in \mathcal{W}_{\Sigma,q}^{\alpha,\beta,\mu,n}(\sigma,\gamma,\rho,\lambda;\phi)$ if the subsequent two quasi-subordinations are satisfied:

$$\frac{1}{\gamma}\left[(1-\rho)\left(\frac{\mathcal{N}_{\lambda}^{n}(\alpha,\beta,\mu)f(z)}{z}\right)^{\sigma}+\rho\big(\mathcal{N}_{\lambda}^{n}(\alpha,\beta,\mu)f(z)\big)'\left(\frac{\mathcal{N}_{\lambda}^{n}(\alpha,\beta,\mu)f(z)}{z}\right)^{\sigma-1}-1\right]^{\gamma}+$$

$$(1-\gamma)\left[\frac{z(\mathcal{N}_{\lambda}^{n}(\alpha,\beta,\mu)f(z))''}{(\mathcal{N}_{\lambda}^{n}(\alpha,\beta,\mu)f(z))'}\right]\prec_{q}(\phi(z)-1)\,, \tag{2.1}$$

and

$$\frac{1}{\gamma}\left[(1-\rho)\left(\frac{\mathcal{N}_{\lambda}^{n}(\alpha,\beta,\mu)g(w)}{w}\right)^{\sigma}+\rho\big(\mathcal{N}_{\lambda}^{n}(\alpha,\beta,\mu)g(w)\big)'\left(\frac{\mathcal{N}_{\lambda}^{n}(\alpha,\beta,\mu)g(w)}{w}\right)^{\sigma-1}-1\right]^{\gamma}+$$

$$(1-\gamma)\left[\frac{w(\mathcal{N}_{\lambda}^{n}(\alpha,\beta,\mu)g(w))''}{(\mathcal{N}_{\lambda}^{n}(\alpha,\beta,\mu)g(w))'}\right]\prec_{q}(\phi(w)-1)\,, \tag{2.2}$$

for $z,w \in U, \gamma \in \mathbb{C}\setminus\{0\}$, $\sigma,\rho \geq 0$ and $g = f^{-1}$ is given by (1.2) .

Remark (2.1). The class $\mathcal{W}_{\Sigma,q}^{\alpha,\beta,\mu,n}(\sigma,\gamma,\rho,\lambda;\phi)$ is a generalization of several known classes considered in earlier investigations, which are being recalled below.

- For $\gamma = 1$ and $n = 0$, we have

$$\mathcal{W}_{\Sigma,q}^{\alpha,\beta,\mu,0}(\sigma,1,\rho,\lambda;\phi)=\mathcal{B}_{\Sigma,q}^{\rho,\sigma}(\phi)$$

where the class $\mathcal{N}_{\Sigma,q}^{\rho,\sigma}(\phi)$ introduced by Orhan et al. [22]

- For $\gamma = 1, \sigma = 1$ and $n = 0$, we have

$$\mathcal{W}_{\Sigma,q}^{\alpha,\beta,\mu,0}(1,1,\rho,\lambda;\phi)=\mathcal{R}_{\Sigma,q}^{\rho}(\phi)\ (\rho\geq 0),$$

where the class $\mathcal{R}_{\Sigma,q}^{\sigma}(\phi)$ introduced by Goyal et al. [17]

- For $\gamma = 1, \rho = 1$ and $n = 0$, we have

$$\mathcal{W}_{\Sigma,q}^{\alpha,\beta,\mu,0}(\sigma,1,1,\lambda;\phi)=\mathcal{F}_{\Sigma,q}^{\sigma}(\phi)\ (\sigma\geq 0),$$

where the class $\mathcal{F}_{\Sigma,q}^{\sigma}(\phi)$ introduced by Goyal et al. [18]

- For $\gamma = 1, \rho = 1, \sigma = 0$ and $n = 0$, we have

$$\mathcal{W}_{\Sigma,q}^{\alpha,\beta,\mu,0}(0,1,1,\lambda;\phi)=S^{*}(\phi).$$

- For $\gamma = 1, \rho = 1, \sigma = 1$ and $n = 0$, we have

$$\mathcal{W}_{\Sigma,q}^{\alpha,\beta,\mu,0}(1,1,1,\lambda;\phi)=\mathcal{H}_{\Sigma,q}(\phi).$$

Theorem 2.1. If f is assumed as in (1.1) and $f \in \mathcal{W}_{\Sigma,q}^{\alpha,\beta,\mu,n}(\sigma,\gamma,\rho,\lambda;\phi)$, then

$$|a_2| \leq min\{\zeta_1,\zeta_2\}, \tag{2.5}$$

and

$$|a_3| \le min\{\xi_1, \xi_2\}, \tag{2.6}$$

for

$$\zeta_1 = \frac{\delta_0 \tau_1}{|[(\rho+\sigma)+2(1-\gamma)]||\mathcal{X}|},$$

$$\zeta_2 = \frac{2\,\delta_0 \tau_1 \sqrt{\tau_1}}{|\mathcal{X}|\sqrt{\left|\left[\tau_0 \delta_1^2\,[(2\rho+\sigma)(2(\sigma+1))+2(\gamma-1)(\rho+\sigma)^2+8(1-\gamma)]-16(\tau_1-\tau_2)\left((\rho+\sigma)+2(1-\gamma)\right)^2\right]\right|}},$$

and

$$\xi_1 = \frac{\delta_0^2 \tau_1^2}{\left((\rho+\sigma)+2(1-\gamma)\right)^2|\mathcal{X}|^2} + \frac{2\tau_1\delta_1}{|((2\rho+\sigma)+6(1-\gamma))||\mathcal{Y}|},$$

$$\xi_2 = \frac{16\delta_0\tau_2}{|[(2\rho+\sigma)(2(\sigma+1))+2(\gamma-1)(\rho+\sigma)^2+8(1-\gamma)]||\mathcal{X}|^2} + \frac{2\tau_1\delta_1}{|((2\rho+\sigma)+6(1-\gamma))||\mathcal{Y}|},$$

where

$$\mathcal{X} = \mathfrak{X}(2,\alpha,\beta,\mu), \qquad \mathcal{Y} = \mathfrak{X}(3,\alpha,\beta,\mu).$$

Proof. Since $f \in \mathcal{W}_{\Sigma,q}^{\alpha,\beta,\mu,n}(\sigma,\gamma,\rho,\lambda;\phi)$ and $g = f^{-1}$. Then, there are analytic functions $u, v : U \to U$ with $u(0) = 0,\ v(0) = 0$ such that

$$\frac{1}{\gamma}\left[(1-\rho)\left(\frac{\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)f(z)}{z}\right)^\sigma + \rho\left(\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)f(z)\right)'\left(\frac{\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)f(z)}{z}\right)^{\sigma-1} - 1\right]^\gamma +$$

$$(1-\gamma)\left[\frac{z(\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)f(z))''}{(\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)f(z))'}\right] \prec_q (\phi(z)-1), \tag{2.7}$$

and

$$\frac{1}{\gamma}\left[(1-\rho)\left(\frac{\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)g(w)}{w}\right)^\sigma + \rho\left(\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)g(w)\right)'\left(\frac{\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)g(w)}{w}\right)^{\sigma-1} - 1\right]^\gamma +$$

$$(1-\gamma)\left[\frac{w(\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)g(w))''}{(\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)g(w))'}\right] \prec_q (\phi(w)-1). \tag{2.8}$$

Define the functions $u(z)$ *and* $v(z)$ by

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n,\ \ q(w) = \frac{1+v(w)}{1-v(w)} = 1 + \sum_{n=1}^{\infty} e_n w^n, (z, w \in U). \tag{2.9}$$

Or equivalent,

$$u(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots, (z \in U), \tag{2.10}$$

and

$$v(w) = \frac{q(w)-1}{q(w)+1} = \frac{1}{2}[e_1 w + \left(e_2 - \frac{e_1^2}{2}\right)w^2 + \cdots, (w \in U). \tag{2.11}$$

It is clear that $p(z)$ *and* $q(w)$ are analytic in U with $p(0) = q(0) = 1$. Since $u, v : U \to U$, the functions $p(z)$ *and* $q(w)$ have a positive real part in U, and $|c_j(z)| \le 2$ *and* $|e_j(w)| \le 2$ ($j = 1,2,$).

Then, by (2.10),(2.11) and (1.3), we have

$\phi\big(u(z)\big) = 1 + \frac{\tau_1 c_1}{2} z + \left[\frac{\tau_1}{2}\left(c_1 - \frac{c_1{}^2}{2}\right) + \frac{\tau_2 c_1{}^2}{4}\right] z^2 + \cdots, (z \in U),$

and

$\phi\big(v(w)\big) = 1 + \frac{\tau_1 e_1}{2} w + \left[\frac{\tau_1}{2}\left(e_2 - \frac{e_1{}^2}{2}\right) + \frac{\tau_2 e_1{}^2}{4}\right] w^2 + \cdots, (w \in U).$

Furthermore, we see that

$$\varphi(z)[\phi(u(z) - 1)] = \frac{\delta_0 \tau_1 c_1}{2} z + \left[\frac{\delta_1 \tau_1 c_1}{2} + \frac{\delta_0 \tau_1}{2}\left(c_1 - \frac{c_1{}^2}{2}\right) + \frac{\delta_0 \tau_2 c_1{}^2}{4}\right] z^2 + \cdots, \qquad (2.12)$$

and

$$\varphi(w)[\phi(v(w) - 1)] = \frac{\delta_0 \tau_1 e_1}{2} w + \left[\frac{\delta_1 \tau_1 e_1}{2} + \frac{\delta_0 \tau_1}{2}\left(e_2 - \frac{e_1{}^2}{2}\right) + \frac{\delta_0 \tau_2 e_1{}^2}{4}\right] w^2 + \ldots, \qquad (2.13)$$

Expanding the left hand sides of (2.7) and (2.8), we obtain that

$$\frac{1}{\gamma}\left[(1-\rho)\left(\frac{\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)f(z)}{z}\right)^{\sigma} + \rho\big(\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)f(z)\big)'\left(\frac{\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)f(z)}{z}\right)^{\sigma-1} - 1\right]^{\gamma} +$$

$$(1-\gamma)\left[\frac{z(\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)f(z))''}{(\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)f(z))'}\right] = \big((\rho+\sigma) + 2(1-\gamma)\big)\mathcal{X} a_2 z +$$

$$\left[\big((2\rho+\sigma) + 6(1-\gamma)\big)\mathcal{Y} a_3 + \frac{(2\rho+\sigma)(\sigma-1) + (\gamma-1)(\rho+\sigma)^2 - 8(1-\gamma)\mathcal{X}^2}{2} a_2^2\right] z^2 + \cdots, \quad (2.14)$$

and

$$\frac{1}{\gamma}\left[(1-\rho)\left(\frac{\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)g(w)}{w}\right)^{\sigma} + \rho\big(\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)g(w)\big)'\left(\frac{\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)g(w)}{w}\right)^{\sigma-1} - 1\right]^{\gamma} +$$

$$(1-\gamma)\left[\frac{w(\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)g(w))''}{(\boldsymbol{\mathcal{N}}_\lambda^n(\alpha,\beta,\mu)g(z))'}\right] =$$

$$-\big((\rho+\sigma) + 2(1-\gamma)\big)\mathcal{X} a_2 z + \left[\frac{(2\rho+\sigma)(\sigma+3) + (\gamma-1)(\rho+\sigma)^2 + 8(1-\gamma)\mathcal{X}^2}{2} a_2^2 - \big((2\rho+\sigma) + 6(1-\gamma)\big)\mathcal{Y} a_3\right] z^2 + \cdots. \qquad (2.15)$$

Furthermore, from (2.12),(2.13) and (2.14),(2.15), by the coefficient comparison method, we know that

$$\big((\rho+\sigma) + 2(1-\gamma)\big)\,\mathcal{X} a_2 = \frac{\delta_0 \tau_1 c_1}{2}, \qquad (2.16)$$

$$\big((2\rho+\sigma) + 6(1-\gamma)\big)\mathcal{Y} a_3 + \frac{[(2\rho+\sigma)(\sigma-1) + (\gamma-1)(\rho+\sigma)^2 - 8(1-\gamma)]\mathcal{X}^2}{2} a_2^2$$

$$= \frac{\delta_1 \tau_1 c_1}{2} + \frac{\delta_0 \tau_1}{2}\left(c_1 - \frac{c_1{}^2}{2}\right) + \frac{\delta_0 \tau_2 c_1{}^2}{4}, \qquad (2.17)$$

and

$$-\big((\rho+\sigma) + 2(1-\gamma)\big)\mathcal{X} a_2 = \frac{\delta_0 \tau_1 e_1}{2}, \qquad (2.18)$$

$$\frac{[(2\rho+\sigma)(\sigma+3) + (\gamma-1)(\rho+\sigma)^2 + 16(1-\gamma)]\mathcal{X}^2}{2} a_2^2 - \big((2\rho+\sigma) + 6(1-\gamma)\big)\mathcal{Y} a_3$$

$$= \frac{\delta_1 \tau_1 e_1}{2} + \frac{\delta_0 \tau_1}{2}\left(e_2 - \frac{e_1{}^2}{2}\right) + \frac{\delta_0 \tau_2 e_1{}^2}{4}. \qquad (2.19)$$

From the two equations are equal (2.16) and (2.18), we get

$$c_1 = -e_1,$$

and

$$8\big((\rho+\sigma)+2(1-\gamma)\big)^2 \mathcal{X}^2 a_2^2 = \delta_0^2\, \tau_1^2\, ({c_1}^2 + {e_1}^2). \qquad (2.20)$$

By (2.17) and (2.19), we have that

$$[(2\rho+\sigma)\big(2(\sigma+1)\big)+2(\gamma-1)(\rho+\sigma)^2+8(1-\gamma)]\mathcal{X}^2 a_2^2 = 4\delta_0\tau_1(c_2+e_2)+2\delta_0({c_1}^2+{e_1}^2)(\tau_2-\tau_1). \qquad (2.21)$$

Therefore, by (2.20) and (2.21), we obtain that

$$a_2^2 = \frac{4\tau_1^3(c_2+e_2)\ \delta_0^2}{\Big[\delta_0\tau_1^2\,[(2\rho+\sigma)\big(2(\sigma+1)\big)+2(\gamma-1)(\rho+\sigma)^2+8(1-\gamma)]-16(\tau_1-\tau_2)\big((\rho+\sigma)+2(1-\gamma)\big)^2\Big]\mathcal{X}^2}. \qquad (2.22)$$

Hence, from (2.20),(2.22) and Lemma 1.1 ,we know that

$$|a_2| \le \frac{\delta_0\tau_1}{|[(\rho+\sigma)+2(1-\gamma)]||\mathcal{X}|},$$

and

$$|a_2| \le \frac{2\ \delta_0\tau_1\sqrt{\tau_1}}{|\mathcal{X}|\sqrt{\Big|\Big[\tau_0\delta_1^2\,[(2\rho+\sigma)\big(2(\sigma+1)\big)+2(\gamma-1)(\rho+\sigma)^2+8(1-\gamma)]-16(\tau_1-\tau_2)\big((\rho+\sigma)+2(1-\gamma)\big)^2\Big]\Big|}}.$$

Correspondingly, by detracting (2.19) and (2.17), it infers that

$$2\big((2\rho+\sigma)+6(1-\gamma)\big)(\mathcal{Y}a_3-\mathcal{X}^2a_2^2) = 2\delta_1\tau_1c_1+\delta_0\tau_1(c_2-e_2). \qquad (2.23)$$

Hence, by (2.20) and (2.23), it trails that

$$a_3 = \frac{\delta_0^2\,\tau_1^2\,({c_1}^2+{e_1}^2)}{8\big((\rho+\sigma)+2(1-\gamma)\big)^2\mathcal{X}^2} + \frac{2\tau_1\delta_1c_1+\delta_0\tau_1(c_2-e_2)}{2\big((2\rho+\sigma)+6(1-\gamma)\big)\mathcal{Y}}.$$

So, we attain from Lemma 1.1 that

$$|a_3| \le \frac{\delta_0^2\tau_1^2}{\big((\rho+\sigma)+2(1-\gamma)\big)^2|\mathcal{X}|^2} + \frac{2\tau_1\delta_1}{|((2\rho+\sigma)+6(1-\gamma))||\mathcal{Y}|}.$$

On the other hand, by (2.22) and (2.23) we infer that

$$a_3 = \frac{4\delta_0\tau_1(c_2+e_2)+2\delta_0({c_1}^2+{e_1}^2)(\tau_2-\tau_1)}{[(2\rho+\sigma)\big(2(\sigma+1)\big)+2(\gamma-1)(\rho+\sigma)^2+8(1-\gamma)]\mathcal{X}^2} + \frac{2\tau_1\delta_1c_1+\tau_0\delta_1(c_2-e_2)}{2\big((2\rho+\sigma)+6(1-\gamma)\big)\mathcal{Y}}.$$

Thus, from Lemma1.1 ,we obtain that

$$|a_3| \le \frac{16\delta_0\tau_2}{|[(2\rho+\sigma)\big(2(\sigma+1)\big)+2(\gamma-1)(\rho+\sigma)^2+8(1-\gamma)]||\mathcal{X}|^2} + \frac{2\tau_1\delta_1}{|((2\rho+\sigma)+6(1-\gamma))||\mathcal{Y}|}.$$

Then, we complete the proof of Theorem 2.1.

By putting $\rho = 0$ From Theorem (2.1), we get the subsequent Corollary:

Corollary 2.1. Consider f defined by (1.1) belongs to the class $\mathcal{W}_{\Sigma,q}^{\alpha,\beta,\mu,n}(\gamma,\sigma,\lambda;\phi)$. Then

$$|a_2| \le min\left\{\frac{\delta_0\tau_1}{|[\sigma+2(1-\gamma)]||\mathcal{X}|}, \frac{2\,\delta_0\tau_1\sqrt{\tau_1}}{|\mathcal{X}|\sqrt{\left|\left[\tau_0\delta_1^2\left[\sigma(2(\sigma+1))+2(\gamma-1)\sigma^2+8(1-\gamma)\right]-16(\tau_1-\tau_2)(\sigma+2(1-\gamma))^2\right]\right|}}\right\},$$

and

$$|a_3| \le min\left\{\frac{\delta_0^2\tau_1^2}{(\sigma+2(1-\gamma))^2|\mathcal{X}|^2}+\frac{2\tau_1\delta_1}{|(\sigma+6(1-\gamma))||\mathcal{Y}|}, \frac{16\delta_0\tau_2}{|[\sigma(2(\sigma+1))+2(\gamma-1)\sigma^2+8(1-\gamma)]||\mathcal{X}|^2}+\frac{2\tau_1\delta_1}{|(\sigma+6(1-\gamma))||\mathcal{Y}|}\right\}.$$

Putting $\sigma=0$ in Theorem (2.1) yields the following Corollary:

Corollary 2.2. Consider f defined by (1.1) belongs to the class $\mathcal{W}_{\Sigma,q}^{\alpha,\beta,\mu,n}(\gamma,\rho,\lambda;\phi)$. Then

$$|a_2| \le min\left\{\frac{\delta_0\tau_1}{|[\rho+2(1-\gamma)]||\mathcal{X}|}, \frac{2\,\delta_0\tau_1\sqrt{\tau_1}}{|\mathcal{X}|\sqrt{\left|\left[\tau_0\delta_1^2\left[2\rho(2(\sigma+1))+2(\gamma-1)\rho^2+8(1-\gamma)\right]-16(\tau_1-\tau_2)(\rho+2(1-\gamma))^2\right]\right|}}\right\},$$

and

$$|a_3| \le min\left\{\frac{\delta_0^2\tau_1^2}{(\rho+2(1-\gamma))^2|\mathcal{X}|^2}+\frac{2\tau_1\delta_1}{|(2\rho+6(1-\gamma))||\mathcal{Y}|}, \frac{16\delta_0\tau_2}{|[4\rho+2(\gamma-1)\rho^2+8(1-\gamma)]||\mathcal{X}|^2}+\frac{2\tau_1\delta_1}{|(2\rho+6(1-\gamma))||\mathcal{Y}|}\right\}.$$

3- Coefficients Bounds for the Class $\mathcal{M}_{\Sigma,q}^{\alpha,\beta,\mu,n}(s,t,r,\lambda;\phi)$

Definition 3.1. A function $f\in\sum$ is assumed as in (1.1), then $f\in\mathcal{M}_{\Sigma,q}^{\alpha,\beta,\mu,n}(s,t,r,\lambda;\phi)$ if the subsequent two quasi-subordinations are satisfied :

$$\frac{1}{r}\left[\left(\frac{(1-s)\mathcal{N}_\lambda^n(\alpha,\beta,\mu)f(z)+sz\left(\mathcal{N}_\lambda^n(\alpha,\beta,\mu)f(z)\right)'}{z}+tz\left(\mathcal{N}_\lambda^n(\alpha,\beta,\mu)f(z)\right)''-1\right)+(1-r)\left(\frac{1+\frac{z\left(\mathcal{N}_\lambda^n(\alpha,\beta,\mu)f(z)\right)''}{\left(\mathcal{N}_\lambda^n(\alpha,\beta,\mu)f(z)\right)'}}{\frac{z\left(\mathcal{N}_\lambda^n(\alpha,\beta,\mu)f(z)\right)'}{\mathcal{N}_\lambda^n(\alpha,\beta,\mu)f(z)}}-1\right)\right]\prec_q(\phi(z)-1), \tag{3.1}$$

and

$$\frac{1}{r}\left[\left(\frac{(1-s)\mathcal{N}_\lambda^n(\alpha,\beta,\mu)g(w)+sw\left(\mathcal{N}_\lambda^n(\alpha,\beta,\mu)g(w)\right)'}{w}+tw\left(\mathcal{N}_\lambda^n(\alpha,\beta,\mu)g(w)\right)''-1\right)+(1-r)\left(\frac{1+\frac{w\left(\mathcal{N}_\lambda^n(\alpha,\beta,\mu)g(w)\right)''}{\left(\mathcal{N}_\lambda^n(\alpha,\beta,\mu)g(w)\right)'}}{\frac{w\left(\mathcal{N}_\lambda^n(\alpha,\beta,\mu)g(w)\right)'}{\mathcal{N}_\lambda^n(\alpha,\beta,\mu)g(w)}}-1\right)\right]\prec_q(\phi(w)-1), \tag{3.2}$$

where $t\ge 0$, $0\le s\le 1$, $0<r\le 1$ and $(z,w\in U)$.

Remark (3.1). If we take $r=1$ and $n=0$ in defined (3.1), the class $\mathcal{M}_{\Sigma,q}^{\alpha,\beta,\mu,n}(s,t,r,\lambda;\phi)$ diminish to the category $\mathcal{H}_\Sigma^q(\lambda,\delta;\phi)$, which was studied recently by Yalcin et al. $(see[29])$.

Remark (3.2). If we take $r = 1$, $t = 0$ and $n = 0$ in defined (3.1), the class $\mathcal{M}_{\Sigma,q}^{\alpha,\beta,\mu,n}(s,t,r,\lambda;\phi)$ reduce to the class $\mathcal{R}_{\Sigma}^{q}(\lambda;\phi)$, which was studied recently by by Patil and Naik (*see*[23]).

Theorem 3.1. If f is assumed as in (1.1) and $f \in \mathcal{M}_{\Sigma,q}^{\alpha,\beta,\mu,n}(s,t,r,\lambda;\phi)$, then

$$|a_2| \le min\{\mathcal{G}_1, \mathcal{G}_2\}, \tag{3.3}$$

and

$$|a_3| \le min\{\mathcal{E}_1, \mathcal{E}_2\}, \tag{3.4}$$

for

$$\mathcal{G}_1 = \frac{\mathcal{C}_0\mathfrak{S}_1}{|(2t+s-r+2)||\mathcal{X}|},$$

$$\mathcal{G}_2 = \sqrt{\frac{2\,\mathcal{C}_0\mathfrak{S}_2}{|[2(6t+2s+1)+(r-1)(2t+s-r+2)^2]\mathcal{X}^2|}},$$

and

$$\mathcal{E}_1 = \frac{4\,\mathcal{C}_1\mathfrak{S}_1}{|[(6t+2s+1)+4(1-r)]\mathcal{Y}|} + \frac{\mathcal{C}_0^2{\mathfrak{S}_1}^2}{|(2t+s-r+2)^2\mathcal{X}^2|},$$

$$\mathcal{E}_2 = \frac{4\,\mathcal{C}_1\mathfrak{S}_1}{|[(6t+2s+1)+4(1-r)]\mathcal{Y}|} + \frac{8\,\mathcal{C}_0\mathfrak{S}_2}{|[2(6t+2s+1)+(r-1)(2t+s-r+2)^2]\mathcal{X}^2|},$$

where

$$\mathcal{X} = \mathfrak{X}(2,\alpha,\beta,\mu), \qquad \mathcal{Y} = \mathfrak{X}(3,\alpha,\beta,\mu).$$

Proof. Steps the proof of the theorem is the same in the Theorem (2.1), we can get the relations as follows:

$$(2t+s-r+2)\mathcal{X}a_2 = \frac{\mathcal{C}_0\mathfrak{S}_1\mathfrak{c}_1}{2}, \tag{3.5}$$

$$\left[\left((6t+2s+1)+4(1-r)\right)\mathcal{Y}a_3 + \frac{(r-1)(2t+s-r+2)^2-8(1-r)}{2}\mathcal{X}^2a_2^2\right]$$
$$= \frac{\mathcal{C}_1\mathfrak{S}_1\mathfrak{c}_1}{2} + \frac{\mathcal{C}_0\mathfrak{S}_1}{2}\left(\mathfrak{c}_2 - \frac{{\mathfrak{c}_1}^2}{2}\right) + \frac{\mathcal{C}_0\mathfrak{S}_2{\mathfrak{c}_1}^2}{4}, \tag{3.6}$$

and

$$-(2t+s-r+2)\mathcal{X}a_2 = \frac{\mathcal{C}_0\mathfrak{S}_1\mathfrak{d}_1}{2}, \tag{3.7}$$

$$\left[\frac{4(6t+2s+1)+8(1-r)+(r-1)(2t+s-r+2)^2}{2}\mathcal{X}^2a_2^2 - \left((6t+2s+1)+4(1-r)\right)\mathcal{Y}a_3\right]$$
$$= \frac{\mathcal{C}_1\mathfrak{S}_1\mathfrak{c}_1}{2} + \frac{\mathcal{C}_0\mathfrak{S}_1}{2}\left(\mathfrak{c}_2 - \frac{{\mathfrak{c}_1}^2}{2}\right) + \frac{\mathcal{C}_0\mathfrak{S}_2{\mathfrak{c}_1}^2}{4},. \tag{3.8}$$

In view of (3.5) and (3.7), we express that

$$a_2 = \frac{\mathcal{C}_0\mathfrak{S}_1\mathfrak{c}_1}{(2t+s-r+2)\mathcal{X}} = -\frac{\mathcal{C}_0\mathfrak{S}_1\mathfrak{d}_1}{(2t+s-r+2)\mathcal{X}}, \tag{3.9}$$

such that

$$\mathfrak{c}_1 = -\mathfrak{d}_1, \tag{3.10}$$

and

$$8(2t + s - r + 2)^2 \mathcal{X}^2 a_2^2 = \mathcal{C}_0^2\, \mathfrak{S}_1^2 \left({\mathfrak{c}_1}^2 + {\mathfrak{d}_1}^2\right). \tag{3.11}$$

By (3.6) and (3.8), we obtain that

$$(2(6t + 2s + 1) + (r - 1)(2t + s - r + 2)^2)\mathcal{X}^2 a_2^2 = 2\, \mathcal{C}_0 \mathfrak{S}_1(\mathfrak{c}_1 + \mathfrak{d}_1)$$
$$+ \mathcal{C}_0 \left((\mathfrak{S}_2 - \mathfrak{S}_1)\left({\mathfrak{c}_1}^2 + {\mathfrak{d}_1}^2\right)\right). \tag{3.12}$$

Applying Lemma1.1, for the coefficients $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{d}_1$ and $\mathfrak{d}_2$ it is follows from (3.10) and (3.11), we get

$$|a_2| \le \frac{\mathcal{C}_0 \mathfrak{S}_1}{|(2t+s-r+2)||\mathcal{X}|}, \tag{3.13}$$

and

$$|a_2| \le \sqrt{\frac{2\, \mathcal{C}_0 \mathfrak{S}_2}{|(2(6t+2s+1)+(r-1)(2t+s-r+2)^2)\mathcal{X}^2|}}. \tag{3.14}$$

Similarly, from (3.6),(3.8) and (3.10), it also implies that

$$2[(6t + 2s + 1) + 4(1 - r)]\mathcal{Y}(a_3 - a_2^2) = 2\, \mathcal{C}_1 \mathfrak{S}_1 \mathfrak{c}_1 + \mathcal{C}_0 \mathfrak{S}_1(\mathfrak{c}_2 - \mathfrak{d}_2). \tag{3.15}$$

Upon substituting from (3.11) and (3.12) putting in (3.15) and by Lemma1.1, we get that

$$|a_3| \le \frac{4\, \mathcal{C}_1 \mathfrak{S}_1}{|[(6t+2s+1)+4(1-r)]\mathcal{Y}|} + \frac{\mathcal{C}_0^2 {\mathfrak{S}_1}^2}{|(2t+s-r+2)^2 \mathcal{X}^2|}, \tag{3.16}$$

and

$$|a_3| \le \frac{4\, \mathcal{C}_1 \mathfrak{S}_1}{|[(6t+2s+1)+4(1-r)]\mathcal{Y}|} + \frac{8\, \mathcal{C}_0 \mathfrak{S}_2}{|(2(6t+2s+1)+(r-1)(2t+s-r+2)^2)\mathcal{X}^2|}. \tag{3.17}$$

This complete the proof of Theorem (3.1).

By putting $t = 0$ into Theorem (3.1) yields the following Corollary:

Corollary 3.2. Consied f defined by (1.1) belongs to the class $\mathcal{M}_{\Sigma,q}^{\alpha,\beta,\mu,n}(s, r, \lambda; \phi)$,. Then

$$|a_2| \le min\left\{\frac{\mathcal{C}_0 \mathfrak{S}_1}{|(s-r+2)||\mathcal{X}|}, \sqrt{\frac{2\, \mathcal{C}_0 \mathfrak{S}_2}{|(2(2s+1)+(r-1)(s-r+2)^2)\mathcal{X}^2|}}\right\},$$

and

$$|a_3| \le min\left\{\frac{4\, \mathcal{C}_1 \mathfrak{S}_1}{|[(2s+1)+4(1-r)]\mathcal{Y}|} + \frac{\mathcal{C}_0^2 {\mathfrak{S}_1}^2}{|(s-r+2)^2 \mathcal{X}^2|}, \frac{4\, \mathcal{C}_1 \mathfrak{S}_1}{|[(2s+1)+4(1-r)]\mathcal{Y}|} + \frac{8\, \mathcal{C}_0 \mathfrak{S}_2}{|(2(2s+1)+(r-1)(s-r+2)^2)\mathcal{X}^2|}\right\}.$$

By putting $s = 1$ into Theorem (3.1) yields the following Corollary:

Corollary 3.2. Consider f defined by (1.1) belongs to the class $\mathcal{M}_{\Sigma,q}^{\alpha,\beta,\mu,n}(t, r, \lambda; \phi)$,. Then

$$|a_2| \leq min\left\{\frac{\mathcal{C}_0\mathfrak{S}_1}{|(2t-r+3)||\mathcal{X}|}, \sqrt{\frac{2\,\mathcal{C}_0\mathfrak{S}_2}{|(2(6t+3)+(r-1)(2t-r+3)^2)\mathcal{X}^2|}}\right\},$$

and

$$|a_3| \leq min\left\{\frac{4\,\mathcal{C}_1\mathfrak{S}_1}{|[(6t+3)+4(1-r)]\mathcal{Y}|} + \frac{\mathcal{C}_0^2{\mathfrak{S}_1}^2}{|(2t-r+3)^2\mathcal{X}^2|}, \frac{4\,\mathcal{C}_1\mathfrak{S}_1}{|[(6t+3)+4(1-r)]\mathcal{Y}|} + \frac{8\,\mathcal{C}_0\mathfrak{S}_2}{|(2(6t+3)+(r-1)(2t-r+3)^2)\mathcal{X}^2|}\right\}.$$

Conclusion

We present novel subclasses $\mathcal{W}_{\Sigma,q}^{\alpha,\beta,\mu,n}(\sigma,\gamma,\rho,\lambda;\phi)$ and $\mathcal{M}_{\Sigma,q}^{\alpha,\beta,\mu,n}(s,t,r,\lambda;\phi)$ of bi-univalent functions within the open unit disc U, employing quasi-subordination requirements, and ascertain estimates for the coefficients $|a_2|$ and $|a_3|$ for functions belonging to these subclasses. We derived two new theorems with distinct specific cases for our novel subclasses, and these findings differ from prior results presented by other writers. Furthermore, we examine the enhanced outcomes for the relevant classes encompassing numerous new and established implications. The findings presented in the work may stimulate further research, and we have provided opportunities for authors to expand our novel subclasses to obtain other results in bi-univalent function theory.

References

[1] F. M. Al-Oboudi, On univalent functions defined by a generalized Salagean operator, Int. J. Math. Sci., (27) (2004) , 1429–1436.

[2] S. Altinkaya and S. Yalcin, Faber polynomial coefficient bounds for a subclass of bi-univalent functions, C. R. Acad. Sci. Paris, Ser. I. ,(2015), 353, 1075-1080.

[3] S. Altinkaya and S. Yalcin, Coefficient bounds for a subclass of bi- univalent functions, TWMS. Journal of Pure and Applied Mathematics, 6 (2015), 180-185.

[4] S. Altinkaya and S. Yalcin, On a new subclass of bi-univalent functions satisfying subordinate conditions, Acta Universitatis Sapientiae, Mathematica ,(7) (2015), 5-14.

[5] O. Altintas and S. Owa, Majorizations and quasi-subordinations for certain analytic functions, Proc. Jpn. Acad. Ser. A, 68(7) (1992), 181-185.

[6] M. K. Aouf, R. M. El-Ashwah and S. M . El-Deeb, Some inequalities for certain p-valent functions involving extended multiplier transformations, Proc. Pakistan Acad. Sci., 46, (2009), 217–221.

[7] W. G. Atshan, S. Yalcin and R. A. Hadi, Coefficients estimates for special subclass of k-fold symmetric bi-univalent functions, Mathematics for Applications 9(2) (2020), 83-90.

[8] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, Studia Univ. Babes Bolyai Math., 31 (1986), 70-77.

[9] N. E. Cho and H .M . Srivastava, Argument estimates of certain analytic func- tions defined by a class of multiplier transformations, Math. Comp. Mode., 37, (2003), 39–49.

[10] N. E. Cho and T .H . Kim, Multiplier transformations and strongly close-to- convex functions, Bull. Korean Math. Soc., 40, (2003), 399–410.

[11] M. Darus and I. Faisal, Some subclasses of Analytic functions defined by generalized differential operator, Universitatis Apulensis, (29)(2012), 197-215.

[12] E. Deniz, Certain subclasses of bi-univalent functions satisfying subordinate conditions, J. Classical Ana ,2 (1), (2013), 49-60.

[13] P. Duren, Subordination in Complex Analysis, Vol. 599 of Lecture Notes in Mathematics, pp.22-29, Springer, Berlin, Germany, (1977).

[14] P. L.Duren, Univalent Functions, In: Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Hidelberg and Tokyo, (1983).

[15] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett., 24 (2011), 1569-1573.

[16] S. K. Gebur and W. G. Atshan, Second Hankel determinant and Fekete-Szegö problem for a new class of bi-univalent functions involving Euler polynomials, Symmetry, 16(5)(2024),530.

[17] S. P. Goyal and R. Kumar, Coefficient estimates and quasi-subordination properties associated with certain sub- classes of analytic and bi-univalent functions, Math. Slovaca, 65 (2015), No. 3, 533–544.

[18] S. P., Goyal, O. Singh, and R. Mukherjee , Certain results on a subclass of analytic and bi-univalent functions associated with coefficient estimates and quasi-subordination, Palestine J. Math., 5 (2016), No. 1, 79–85.

[19] S. G. Hamidi and J. M. Jahangiri, Faber polynomial coefficient estimates for analytic bi-close to convex functions, C. R. Acad. Sci. Paris, Ser. I. ,(2014) (352), 17-20.

[20] M. Lewin, On a coefficient problem for bi-univalent function, Proceedings of the American Mathematical Society, Vol. 18 (1967), 63-68.

[21] W. Ma and D. Minda, Aunified treatment of some special classes of univalent functions, Proceedings of the conference on complex analysis, Z. Li. F. Ren, L. Yang and S. Zhang, eds., Int. Press (1994), 157-169.

[22] H. Orhan, N. Magesh, and J. Yamini, Coefficient estimates for a class of bi-univalent functions associated with quasi-subordination. Creat .Math. inform. ,Volume 26 (2017), No. 2, 193-199.

[23] A. Patil and U. Naik, Estimates on initil coefficients of certain subclasses of bi-univalent function associated with quasi-subordination, Global Journal of Mathematical Analysis, 5(1) (2017), 6-10.

[24] I. A. R. Rahman and W. G. Atshan, Results on coefficients bounds for new subclasses by using quasi-subordination of analytic function estimates, AIP Conference Proceedings, 2398(1)(2022),060021.

[25] M. S. Robertson, Quasi-subordination and coefficient conjecture, Bull. Amer. Math. Soc., 76 (1970), 1-9.

[26] G. S. Salagean, Subclasses of univalent functions, Lecture Notes in Mathematics 1013, Springer-Verlag, (1983), 362–372.

[27] S. R. Swamy and W. G. Atshan, Coefficient bounds for regular and bi-univalent functions linked with Gegenbauer polynomials, Gulf Journal of Mathematics, 13(2)(2022),67-77.

[28] B. A. Uralegaddi and C. Somanatha, Certain classes of univalent functions, In: Current Topics in Analytic Function Theory, Eds. Srivastava, H.M. and Owa, S., World Scientific Publishing Company, Singapore, (1992), 371–374.

[29] S. Yalcin, W. G. Atshan and H. Z . Hassan, Coefficients assessment for certain subclass of bi-univalent functions related with quasi-subordintion, Novelle serie, tome 108 (122) (2020), 155-162.

[30] Q. A. Shakir, W. G. Atshan, On third Hankel determinant for certain subclass of bi-univalent functions, Symmetry, 16(2) (2024), 239, 1-10.

[31] E. I. Badiwi, W. G. Atshan, A. N. Alkiffai and A. A. Lupas, Certain results on subclasses of analytic and bi-univalent functions associated with coefficient estimates and quasi-subordination, Symmetry, 15(12) (2023), 2208.

[32] P. O. Sabir, H. M. Srivastava , W. G. Atshan, P. O. Mohammed, N. Chorfi and M. Vivas-Cortez, A family of holomorphic and m-fold Symmetric bi-univalent functions endowed with coefficient estimate problems, Mathematics, 11(18)(2023),3970

[33] A. M. Darweesh, W. G. Atshan, A. H. Battor and M. S. Mahdi, on the third Hankel determinant of certain subclass of bi-univalent functions, Mathematical Modelling of Engineering problems, 10(3) (2023), PP.1087-1095.

[34] W. G. Atshan and R. A. Al-sajjad, Some Applications of quasi-subordination for bi-univalent functions using Jackson' convolution operator, Iraqi Journal of Science, 63(10) (2022), pp.4417-4428.