Generalized mean function

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ABSTRACT:

In 1969 S.K.Skaff introduced the generalized mean function

In this work we present the theory of an integral mean for generalized GN*-function .We will show under what conditions the mean function is a GN*-function and satisfies a $\Delta -$condition. Moreover, we examine how the minimizing points in the definition of the mean function affect a basic property of the ordinary integral mean.

1.Introduction and Basic Concept:

From the functional analysis as a function space, Orlicz spaces appeared in the first of the 30^{th} by W.R. Orlicz in Orlicz paper [1]. Many theorems and properties about generalized mean function for GN-function is introduced in [5].

we have consolidated the investigation of a new definition generalized mean function for GN*-functions and discussed their properties.

Definition 1.1: [5]

Let $M(t,x)$ be a real valued non-negative function defined on $T \times E^n$ such that:

(i) $M(t,x)=0$ if and only if $x=0$ where for all $t \in T$, $x \in E^n$

(ii) $M(t,x)$ is a continuous convex function of x for each t and a measurable function of t for each x,

(iii) For each $t \in T$, $\lim\limits_{\|x\| = \infty} \dfrac{M(t,x)}{\|x\|} = \infty$, and

(iv)There is a constant $d \geq 0$ such that

$$\inf_{t} \inf_{c \geq d} k(t,c) \rangle 0 \tag{1.1.1}$$

where $K(t,c) = \dfrac{\underline{M}(t,c)}{\overline{M}(t,c)}$,

$, \overline{M}(t,c) = \sup_{|x|=c} M(t,x)$,

$$\underline{M}(t,c) = \inf_{|x|=c} M(t,x)$$

and if $d > 0$, then $\overline{M}(t,d)$ is an integrable function of *t*.We call a function satisfying the properties (i)-(iv) a generalized N-function or a GN-function.

Definition 1.2:

Let $M(t,x,y)$ be a real valued non-negative function defined on $T \times E^n \times E^n$ such that:

(i) $M(t,x,y) = 0$ if and only if *x,y* are the zero vectors $x, y \in E^n$, $\forall t \in T$

(ii) $M(t,x,y)$ is a continuous convex function of x, y for each *t* and a measurable function of *t* for each x, y,

(iii) For each $t \in T$, $\lim_{\substack{\|x\|=\infty \\ \|y\|=\infty}} \dfrac{M(t,x,y)}{\|x\|\|y\|} = \infty$, and

(iv)There are constants $d \geq 0$ and $d_1 \geq 0$ such that

$$\inf_{t} \inf_{\substack{c \geq d \\ c' \geq d_1}} k(t,c,c') > 0 \tag{1.2.1}$$

Where

$$k(t,c,c') = \frac{\underline{M}(t,c,c')}{\overline{M}(t,c,c')},$$

$$\overline{M}(t,c,c') = \sup_{\substack{|x|=c \\ |y|=c'}} M(t,x,y), \underline{M}(t,c,c') = \inf_{\substack{|x|=c \\ |y|=c'}} M(t,x,y)$$

and if $d \rangle 0$ and $d_1 > 0$, then $\overline{M}(t,d,d_1)$ is an integrable function of *t*. We call the function satisfying the properties (i)-(iv) a generalized N*-function or a GN*-function.

Definition 1.3: **[5]**

For each *t* in *T* and *h>0* let

$$M_h(t,x) = \int_{E^n} M(t, x+z) J_h(z) dz$$

where $J_h(z)$ is nonnegative, c^∞ function with compact support in a ball of a radius h such that $\int_{E^n} J_h(z) dt = 1$.

Moreover, let x_0 is any point (depending on *h, t*) which satisfies the inequality

$$M_h(t, x_0) \le M_h(t,x)$$

for all x in E^n. Then the function $\hat{M}_h(t,x)$ defined for each *t* in *T* and $h > 0$ by

$$\hat{M}_h(t,x) = M_h(t, x+x_0) - M_h(t, x_0)$$

is called a **mean function** for $M(t,x)$ relative to the minimizing point x_0.

Definition 1.4:

For each *t* in *T* and *h>0* let

$$M_h(t,x,y) = \int_{E^n} \int_{E^n} M(t, x+z, y+w) J_h(z) J_h(w) dz dw$$

where $J_h(z)$ and $J_h(w)$ are nonnegative, c^∞ function with compact support in a ball of a radius h such that $\int_{E^n} \int_{E^n} J_h(z) J_h(w) dt dt = 1$.

Moreover, let x_0 and y_0 are any point (depending on *h, t*) which satisfies the inequality

$$M_h(t, x_0, y_0) \le M_h(t,x,y)$$

for all x and y in E^n. Then the function $\hat{M}_h(t,x,y)$ defined for each *t* in *T* and $h > 0$ by

$$\hat{M}_h(t,x,y) = M_h(t, x+x_0, y+y_0) - M_h(t, x_0, y_0)$$

is called a **mean function** for $M(t,x,y)$ relative to the minimizing point x_0 and y_0.

The next theorem shows under what condition $\hat{M}_h(t,x,y)$ is a GN*-function.

Definition 1.5:[2]

We say that a GN-function $M(t,x)$ satisfies a $\Delta-$condition if there exist a constant $K \geq 2$ and a non-negative measurable function $\delta(t)$ such that the function $\overline{M}(t, 2\delta(t))$ is integrable over the domain *T* and such that for almost all *t* in *T* we have

$$M(t,2x) \leq KM(t,x) \tag{1.5.1}$$

for all x satisfying $|x| \geq \delta(t)$.

We say that a GN-function satisfies a Δ_0-condition if it satisfies a $\Delta-$condition with $\delta(t) = o$ for almost all *t* in *T*.

In definition 1.5 we could have used any constant $\tau > 1$ in place of the scalar 2 in (1.5.1).

Definition 1.6:

We say that a GN*-function *M(t,x,y)* satisfies a $\Delta-$condition if there exists a constant $K \geq 2$ and non-negative measurable functions $\delta_1(t)$ and $\delta_2(t)$ such that the function $M(t, 2\delta_1(t), 2\delta_2(t))$ is integrable over the domain *T* and such that for almost all *t* in *T* we have

$$M(t,2x,2y) \leq KM(t,x,y) \tag{1.6.1}$$

for all *x* and *y* satisfying $|x| \geq \delta_1(t)$ and $|y| \geq \delta_2(t)$.

We say a GN*-function satisfies a Δ_0-condition if it satisfies a $\Delta-$condition with $\delta_1(t) = 0$ and $\delta_2(t) = 0$ for almost all *t* in *T*.

In definition (1.6) we could have used any constant $\tau > 1$ in place of the scalar 2 in (1.6.1).

Theorem 1.7:[3]

A necessary and sufficient condition that (1.5.1) holds is that if $|x| \le |z|$, then there exists constants $K \ge 1, d \ge 0$ such that $M(t,x) \le KM(t,z)$ for each *t* in *T*, $|x| \ge d$.

Theorem 1.8:

A necessary and sufficient condition that (1.6.1) holds is that if $|x| \le |z|$ and $|y| \le |w|$, then there exists constants $K \ge 1, d \ge 0$ and $d' \ge 0$ such that $M(t,x,y) \le KM(t,z,w)$ for each *t* in *T*, $|x| \ge d$ and $|y| \ge d'$.

Theorem 1.9:[2]

A GN*-function $M(t,x)$ satisfies a $\Delta-$condition if and only if given any $\tau > 1$ there exists a constant $K_\tau \ge 2$ and a non-negative measurable functions $\delta_1(t)$ such that $\overline{M}(t, 2\delta_1(t))$ is integrable over *T* and such that for almost all *t* in *T* we have

$$M(t, \tau x) \le K_\tau M(t,x), \qquad (1.9.1)$$

whenever $|x| \ge \delta_1(t)$.

Theorem 1.10:

A GN*-function $M(t,x,y)$ satisfies a $\Delta-$condition if and only if given any $\tau > 1$ there exists a constant $K_\tau \ge 2$ and a non-negative measurable functions $\delta_1(t)$ and $\delta_2(t)$ such that $\overline{M}(t, 2\delta_1(t), 2\delta_2(t))$ is integrable over *T* and such that for almost all *t* in *T* we have

$$M(t, \tau x, \tau y) \le K_\tau M(t,x,y), \qquad (1.10.1)$$

whenever $|x| \ge \delta_1(t)$ and $|y| \ge \delta_2(t)$.

Theorem 1.11:[5]

If $M(t,x)$ is a GN*-function for which $\overline{M}(t,c)$ is integrable in *t* for each

c, then $\hat{M}_h(t,x)$ is a GN*-function.

Proof:

We will show this result by justifying conditions (i)-(iv) of the definition 1.1. By hypothesis and the choice of x_0, we have for each h, $\hat{M}_h(t,x) \geq 0$ and $\hat{M}_h(t,0) = 0$. On the other hand, if $x \neq 0$, then $M(t,x) > 0$, and hence there are constants h_0 such that

$$a = \inf_{|w| \leq h_0} M(t, x+w) > 0$$

However, since $M(t,x) = 0$ if and only if $x = 0$, the minimizing points x_0 tends to zero as h tends to zero. Therefore, we can choose $g_0 \leq h_0$ such that if $h \leq g_0$,then $M(t, x_0 + r) < a$ for all r for which $|x_0 + r| < h$, For this g_0 we obtain the inequality

$$M(t, x + x_0 + r) \geq \inf_{|w| \leq g_0} M(t, x+w) \geq$$

$$a > M(t, x_0 + r)$$

whenever $|x_0 + r| \leq g_0$.This means for some $h \leq g_0$ we have

$$M(t, x + x_0 + r) > M(t, x_0 + r)$$

$$\int_{E^n} M(t, x + x_0 + r) J_h(r) dr >$$

$$\int_{E^n} M(t, x_0 + r) J_h(r) dr$$

$$M_h(t, x + x_0) > M_h(t, x_0)$$

or $\hat{M}_h(t,x) > 0$ if $x \neq 0$ which proves property (i).

Properties (ii) and (iii) for $\hat{M}_h(t,x)$ follow easily from the same properties for $M(t,x)$. Let us now show (iv). By assumption, there are constants $d \geq 0$ such that

$$\tau(t) \overline{M}(t,c) \leq \underline{M}(t,c) \tag{1.11.1}$$

for all $c \geq d$.Furthermore, it is not difficult to show that for all c we have

$$\overline{M}(t,c) \geq \sup_{|x|\leq c} M(t,x) \tag{1.11.2}$$

and for some fixed w,

$$\inf_{|x|\geq c} M(t,x+w) \leq \inf_{|x|=c} M(t,x+w) \tag{1.11.3}$$

By using (1.11.2), we obtain (for each t in T)that

$$\tau(t)\sup_{|w|=c} M(t,w) \leq \tau(t) \sup_{|r'|<c+|x_0+x_1|} M(t,r') \tag{1.11.4}$$

$$\leq \tau(t) \sup_{|r'|=c+|x_0+x_1|} M(t,r')$$

where $w=x+x_0+r$.On the other hand, by (1.11.1) and (1.11.3), we achieve

$$\tau(t) \sup_{|w|=c+|x_0+x_1|} M(t,w) \leq \inf_{|w|=c+|x_0+x_1|} M(t,w) \tag{1.11.5}$$

$$< \inf_{|x|\geq c} M(t,x+x_0+r).$$

$$< \inf_{|x|=c} M(t,x+x_0+r).$$

If we combine (1.11.4) and (1.11.5), then for all $c \geq d$ we arrive at

$$\tau(t)\sup_{|x|=c} M(t,x+x_0+r) \leq \inf_{|x|=c} M(t,x+x_0+r).$$

From this inequality, we obtain

$$\inf_{|x|=c} \hat{M}_h(t,x) \geq \int_{E^n} \inf_{|x|=c}\{M(t,x+x_0+r)-M(t,x_0+r\}J_h(r)dr$$

$$\geq \int_{E^n} \{\tau(t)\sup_{\substack{|x|=c\\|y|=c'}} M(t,x+x_0+r)-M(t,x_0+r)\}J_h(r)dr, \tag{1.11.6}$$

and

$$\sup_{|x|=c} \hat{M}_h(t,x) \leq \int_{E^n} \sup_{|x|=c} M(t,x+x_0+r)J_h(r)dr. \tag{1.11.7}$$

Moreover, since $\lim\limits_{c=\infty} \sup\limits_{|x|=c} M(t, x + x_0 + r) = \infty$

for fixed x_0, r such that $|r| \leq h$,given

$$K_1(t) = 2\sup_{|r|\leq h} M(t, x_0 + r) / \inf_t \tau(t)$$

there are $d_1 > 0$ such that if $c \geq d_1$,then

$$\sup_{|x|=c} M(t, x + x_0 + r) \geq K_1.$$

Therefore, by using (1.11.6) and (1.11.7), we achieve the inequalities

$$\frac{\inf\limits_{|x|=c} \hat{M}_h(t,x)}{\sup\limits_{|x|=c} \hat{M}_h(t,x)} \geq \tau(t) -$$

$$\frac{\sup\limits_{|r|\leq h} M(t, x_0 + r)}{\inf\limits_{|r|\leq h} \sup\limits_{|x|=c} M(t, x + x_0 + r)} \geq \tau(t) - \frac{1}{2} \inf_t \tau(t) \qquad (1.11.8)$$

for all $c \geq d_0 = \max(d, d_1, |x_0|)$. Taking the infimum of both sides of (1.11.8) over *t*, shows the first part of the property (iv). To show the latter part, assume $d_0 > 0$. Then $\sup\limits_{|x|=d_0} \hat{M}_h(t,x)$ is integrable over *t* in *T* since it is bounded by the integrable function $\overline{M}(t, d_2)$ where $d_2 = d_0 + |x_0| + h$ .This proves property (iv) and the theorem.■

In the next theorem we show under what condition $\hat{M}_h(t,x)$ satisfies a $\Delta-$ condition.

Theorem 1.12:[5]

If $M(t,x)$ is a GN*-function satisfying a $\Delta-$condition and for which $\overline{M}(t,c)$ is integrable in *t* for each *c* , then $\hat{M}_h(t,x)$

satisfies a $\Delta-$condition.

Proof:

It suffices to show that $M_h(t,x)$ satisfies a $\Delta-$condition.

For, $\hat{M}_h(t,x)$ is the sum of a constant and a translation of $M_h(t,x)$ and neither of these operations affects the growth condition. Let us observe first that if $|x|\geq 2$, $|z|\leq h\leq 1$ then $|2x+z|\leq 3|x+z|$.

Hence, by Theorem (1.7), there are constants $K\geq 1$ and $d_1\geq 0$ such that

$$M_h(t,2x)\leq k\int_{E^n} M(t,3(x+z))J_h(z)dz$$

for all x such that $|x|\geq d_2$ and $d_2=\max(d_1,2)$.On the other hand ,by theorem (1.9), there is a constant $K_3\geq 2$, $\delta_1(t)\geq 0$ such that for almost all t in T

$$\int_{E^n} M(t,3(x+z))J_h(z)dz\leq K_3M_h(t,x)$$

for all x,z such that $|x+z|\geq \delta_1(t)$ where $|z|\leq h$ .By combining the above two inequalities, we achieve

$$M_h(t,2x)\leq KK_3M_h(t,x)$$

for all $|x|>\max(d_2,\delta_1(t)+h)=\delta_1'(t)$.

Since $\overline{M}(t,2\delta_1(t))$ is integrable over T ,this yields the integrability of $\overline{M}_h(t,2\delta_1'(t))$ which proves the theorem.■

For each t in T and x in E^n it is known that

$$\lim_{h=0} M_h(t,x)=M(t,x).$$

However, the same property does not hold in general for $\hat{M}_h(t,x)$. This is the point of the next theorem.

Theorem 1.13:**[5]**

For each $h>0$ let x_0^h be the minimizing point of $M_h(t,x)$

defining $\hat{M}_h(t,x)$.Then for each t in T and each x in E^n,there exists $K(t,x)$ such that

$$\lim_{h=0} \hat{M}_h(t,x) = M(t,x) + K(t,x) \lim_{h=0} |x_0^h|$$

Proof:

By the definition of $\hat{M}_h(t,x)$ we can write

$$\left|\hat{M}_h(t,x) - M(t,x)\right| \le$$

$$\int_{E^n} \left|M(t, x + x_0^h + z) - M(t, x_0^h + z) - M(t,x)\right| J_h(z)dz \qquad (1.13.1)$$

However, we know that

$$\left|M(t, x + x_0^h + z) - M(t, x_0^h + z) - M(t,x)\right| \qquad (1.13.2)$$

$$\le \left|M(t, x + x_0^h + z) - M(t,x)\right|$$

$$+ \left|M(t, x_0^h + z) - M(t, z, w)\right| + |M(t,z)|.$$

Moreover , since $M(t,x)$ is a convex function, it satisfies a Lipshitz condition on compact subsets of E^n (see[4Th.5.1]).Therefore ,there exists $K_1(t,x)$ and $K_2(t,x)$ such that

$$\left|M(t, x + x_0^h + z) - M(t,x)\right| \le K_1(t,x)\left|x_0^h + z\right| \qquad (1.13.3)$$

and

$$\left|M(t, x_0^h + z) - M(t,z)\right| \le K_2(t,x)\left|x_0^h\right|. \qquad (1.13.4)$$

If we combine (1.13.3) and (1.13.4) with (1.13.2) and if we substitute the resulting expression into (1.13.1), we achieve the inequality

$$\left|\hat{M}_h(t,x) - M(t,x)\right| \le \left|x_0^h\right|(K_1(t,x,y) + K_2(t,x,y)) +$$
$$\int_{E^n} K_1(t,x)|z| J_h(z)dz + \int_{E^n} |M(t,z)| J_h(z)dz.$$

Since the last two integrals on the right side tend to zero as h tends to zero, we prove the theorem by setting

$$K(t,x) = K_1(t,x) + K_2(t,x).$$

Corollary 1.14: [5]

Suppose $M(t,x)$ is a GN*-function such that $M(t,x)=M(t,-x)$. Then for each t in T and x in E^n ,we have

$$\lim_{h=0} M_h(t,x)=\hat{M}(t,x)$$

Proof:

This result is clear since $\lim_{h=0}\left|x_0^h\right|=0$

if $M((t,x)=M(t,-x)$.In fact, if $M(t,x)$ is even in x then the $x_0^h=0$ for all h.■

For each t in T let A_h denote the set of minimizing points of $M_h(t,x)$ and let B represent the null space of $M(t,x)$ relative to points in E^n , i.e.,

$$B=\{x \quad in \quad E^n : M(t,x)=0\}.$$

If $M(t,x)$ is a GN*-function, then $B=\{0\}$. For the sake of argument, let us suppose that $M(t,x)$ has all the properties of a GN*-function except that $M(t,x)=0$ need not imply $x=0$. We will show the relationships that exist between A_h and B. This is the content of the next few theorems.

Theorem 1.15:[5]

The sets B and A_h are closed convex sets.

Proof:

This result follows from the convexity and continuity of $M(t,x)$ in x for each t in T.■

Theorem 1.16:[5]

Let $B_e=\{x: M(t,x)<e\}$ for each t in T. Then given any $e>0$, there is a constant $h_0>0$. such that $A_h \subset B_e$ for each $h \le h_0$.

Proof:

Since $B \subseteq B_e$, we can choose h_0 sufficiently small so that if x is in B then $x+z$ is in B_e for all z such that $|z|\le h_0$ and $|w|\le h_0$. Let z_1 be arbitrary but fixed points in $A_h, h\le h_0$. Then

$$M_h(t,z_1)\le M_h(t,x)$$

for all x . Therefore, if x in B, we have $M_h(t,z_1)<e$ by our choice of h_0. Letting h tend to zero yields $M(t,z_1)<e$, i.e., z_1, in B_e.

We have commented above that $A_h=\{0\}$ if

$$M(t,x)=M(t,-x).$$

It is also true if *M(t,x)* is strictly convex in *x* for each *t* in *T*.

Theorem 1.17:[5]

Suppose *M(t,x)* is a GN*-function which is strictly convex in *x* for each *t*. Then $h, A_h=\{0\}$ for each *h*.

Proof:

Suppose that there exists $z_0\ne x_0$ such that x_0, z_0 are in A_h. Let $z_1=\frac{(x_0+z_0)}{2}$,. Then, since $M(t,x)$ is strictly convex, $M_h(t,x)$ is strictly convex in *x*, therefore, we have

$$M_h(t,z_1)<\frac{1}{2}M_h(t,x_0)+\frac{1}{2}M_h(t,z_0). \qquad (1.17.1)$$

However, x_0, z_0 are in A_h reduces (1.17.1) to the inequality $M_h(t,z_1)<M_h(t,x)$ for all *x*. This means z_1 is in A_h and x_0, z_0 are not in A_h which is a contradiction. Hence, $x_0=z_0$ .Since $M(t,x)$ is a GN*function, $B=\{0\}$. In this case $x_0=z_0=0$.

٢.Generalized mean function

Theorem 2.1:

If $M(t,x,y)$ is a GN*-function for which $\overline{M}(t,c,c')$ is integrable in *t* for each *c* and *c'*, then $\hat{M}_h(t,x,y)$ is a GN*-function.

Proof:

We will show this result by justifying conditions (i)-(iv) of the definition ٣.1.1. By hypothesis and the choice of x_0 and y_0, we have for each h, $\hat{M}_h(t,x,y) \geq 0$ and $\hat{M}_h(t,0,0) = 0$. On the other hand, if $x \neq 0$ and $y \neq 0$, then $M(t,x,y) > 0$, and hence there are constants h_0 and h_0' such that

$$a = \inf_{\substack{|w| \leq h_0, \\ |w'| \leq h_0}} M(t, x+w, y+w') > 0$$

However, since $M(t,x,y) = 0$ if and only if $x = 0$ and $y = 0$, the minimizing points x_0 tends to zero and y_0 tends to zero as h tends to zero. Therefore, we can choose $g_0 \leq h_0$ and $g_0' \leq h_0'$ such that if $h \leq g_0$ and $h \leq g_0'$, then $M(t, x_0 + r, y_0 + s) < a$ for all r,s for which $|x_0 + r| < h$, $|y_0 + s| < h$ For this g_0 and g_0' we obtain the inequality

$$M(t, x + x_0 + r, y + y_0 + s) \geq \inf_{\substack{|w| \leq g_0 \\ |w'| \leq g_0'}} M(t, x+w, y+w') \geq$$

$$a > M(t, x_0 + r, y_0 + s)$$

whenever $| x_0 + r | \leq g_0$ and $| y_0 + s | \leq g_0'$. This means for some $h \leq g_0$ and $h \leq g_0'$ we have

$$M(t, x + x_0 + r, y + y_0 + s) > M(t, x_0 + r, y_0 + s)$$

$$\int_{E^n} \int_{E^n} M(t, x + x_0 + r, y + y_0 + s) J_h(r) J_h(s) drds >$$

$$\int_{E^n} \int_{E^n} M(t, x_0 + r, y_0 + s) J_h(r) J_h(s) drds$$

$$M_h(t, x + x_0, y + y_0) > M_h(t, x_0, y_0)$$

or $\hat{M}_h(t,x,y) > 0$ if $x \neq 0$ and $y \neq 0$ which proves property (i).

Properties (ii) and (iii) for $\hat{M}_h(t,x,y)$ follow easily from the same properties for $M(t,x,y)$. Let us now show (iv). By assumption, there are constants $d \geq 0$ and $d' \geq 0$ such that

$$\tau(t)\overline{M}(t,c,c') \leq \underline{M}(t,c,c') \tag{2.1.1}$$

for all $c \geq d$ and $c' \geq d'$.Furthermore, it is not difficult to show that for all c and c' we have

$$\overline{M}(t,c,c') \geq \sup_{\substack{|x| \leq c \\ |y| \leq c'}} M(t,x,y) \tag{2.1.2}$$

and for some fixed w and w',

$$\inf_{\substack{|x| \geq c \\ |y| \geq c'}} M(t,x+w,y+w') \leq \inf_{\substack{|x| = c \\ |y| = c'}} M(t,x+w,y+w') \tag{2.1.3}$$

By using (3.3.4), we obtain (for each t in T)that

$$\tau(t) \sup_{\substack{|w| = c \\ |w'| = c'}} M(t,w,w) \leq \tau(t) \sup_{\substack{|r'| < c + |x_0 + x_1| \\ \|s'\| < c' + |y_0 + y_1|}} M(t,r',s') \tag{2.1.4}$$

$$\leq \tau(t) \sup_{\substack{|r'| = c + |x_0 + x_1| \\ |s'| = c' + |y_0 + y_1|}} M(t,r',s')$$

where $w{=}x{+}x_0{+}r$ and $w'{=}y{+}y_0{+}s$.On the other hand, by (2.1.1) and (2.1.3), we achieve

$$\tau(t) \sup_{\substack{|w| = c + |x_0 + x_1| \\ |w'| = c' |y_0 + y_1|}} M(t,w,w') \leq \inf_{\substack{|w| = c + |x_0 + x_1| \\ |w'| = c' + |y_0 + y_1|}} M(t,w,w') \tag{2.1.5}$$

$$< \inf_{\substack{|x| \geq c \\ |y| \geq c'}} M(t,x+x_0+r,y+y_0+s).$$

$$< \inf_{\substack{|x| = c \\ |y| = c'}} M(t,x+x_0+r,y+y_0+s).$$

If we combine (2.1.4) and (2.1.5), then for all $c \geq d$ and $c' \geq d'$ we arrive at

$$\tau(t)\sup_{\substack{|x|=c\\|y|=c'}} M(t,x+x_0+r,y+y_0+s) \le \inf_{\substack{|x|=c\\|y|=c'}} M(t,x+x_0+r,y+y_0+s).$$

From this inequality, we obtain

$$\begin{aligned}\inf_{\substack{|x|=c\\|y|=c'}} \hat{M}_h(t,x,y) \ge \int_{E^n}\int_{E^n} &\inf_{\substack{|x|=c\\|y|=c'}}\{M(t,x+x_0+r,y+y_0+s)\\ &-M(t,x_0+r,y_0+s)\}J_h(r)J_h(s)drds\\ &\ge \int_{E^n}\int_{E^n}\{\tau(t)\sup_{\substack{|x|=c\\|y|=c'}}M(t,x+x_0+r,y+y_0+s)\\ &-M(t,x_0+r,y_0+s)J_h(r)J_h(s)drds,\end{aligned} \tag{2.1.6}$$

and

$$\begin{aligned}&\sup_{\substack{|x|=c\\|y|=c'}}\hat{M}_h(t,x,y)\\ &\le \int_{E^n}\int_{E^n}\sup_{\substack{|x|=c\\|y|=c'}}M(t,x+x_0+r,y+y_0+s)J_h(r)J_h(s)drds.\end{aligned} \tag{2.1.7}$$

Moreover, since $\lim\limits_{\substack{c=\infty\\c'=\infty}} \sup\limits_{\substack{|x|=c\\|y|=c'}} M(t,x+x_0+r,y+y_0+s)=\infty$

for fixed x_0, y_0, r, s such that $|r|\le h$ and $|s|\le h$,given

$$K_1(t) = 2\sup_{\substack{|r|\le h\\|s|\le h}} M(t,x_0+r,y_0+s)/\inf_t \tau(t)$$

there are $d_1 > 0$ and $d_1' > 0$ such that if $c \ge d_1$ and $c' > d_1'$,then

$$\sup_{\substack{|x|=c\\|y|=c'}} M(t,x+x_0+r,y+y_0+s) \ge K_1.$$

Therefore, by using (2.1.6) and (2.1.7), we achieve the inequalities

$$\frac{\inf\limits_{\substack{|x|=c\\|y|=c'}} \hat{M}_h(t,x,y)}{\sup\limits_{\substack{|x|=c\\|y|=c'}} \hat{M}_h(t,x,y)} \ge \tau(t) -$$

$$\frac{\sup\limits_{\substack{|r|\le h\\ |s|\le h}} M(t,x_0+r,y_0+s)}{\inf\limits_{\substack{|r|\le h\\ |s|\le h}} \sup\limits_{\substack{|x|=c\\ |y|=c'}} M(t,x+x_0+r,y+y_0+s)} \ge \tau(t)-\frac{1}{2}\inf_t \tau(t) \tag{2.1.8}$$

for all $c \ge d_0 = \max(d, d_1, |x_0|)$ and $c' \ge d'_0 = \max(d', d'_1, |y_0|)$. Taking the infimum of both sides of (2.1.8) over *t*, shows the first part of the property (iv). To show the latter part, assume $d_0 > 0$ and $d'_0 > 0$. Then $\sup\limits_{\substack{|x|=d_0\\ |y|=d'_0}} \hat{M}_h(t,x,y)$ is integrable over *t* in *T* since it is bounded by the integrable function $\overline{M}(t,d_2,d'_2)$ where $d_2 = d_0 + |x_0| + h$ and $d'_2 = d'_0 + |y_0| + h$. This proves property (iv) and the theorem.■

In the next theorem we show under what condition $\hat{M}_h(t,x,y)$ satisfies a $\Delta-$ condition.

Theorem 2.2:

If $M(t,x,y)$ is a GN*-function satisfying a $\Delta-$condition and for which $\overline{M}(t,c,c')$ is integrable in *t* for each *c* and c', then $\hat{M}_h(t,x,y)$ satisfies a $\Delta-$condition.

<u>Proof:</u>

It suffices to show that $M_h(t,x,y)$ satisfies a $\Delta-$condition.

For, $\hat{M}_h(t,x,y)$ is the sum of a constant and a translation of $M_h(t,x,y)$ and neither of these operations affects the growth condition. Let us observe first that if $|x| \ge 2$, $|y| \ge 2$, $|z| \le h \le 1$ and $|w| \le h \le 1$ then $|2x+z| \le 3|x+z|$ and $|2y+w| \le 3|y+w|$. Hence, by Theorem (1.8), there are constants $K \ge 1$ and $d_1 \ge 0$ such that

$$M_h(t,2x,2y) \le k \int_{E^n}\int_{E^n} M(t,3(x+z),3(y+w)) J_h(z) J_h(w)\,dz\,dw$$

for all x and y such that $|x| \geq d_2$, $|y| \geq d_2$ and $d_2 = \max(d_1, 2)$.On the other hand ,by theorem (1.10), there is a constant $K_3 \geq 2$, $\delta_1(t) \geq 0$ and $\delta_2(t) \geq 0$ such that for almost all t in T

$$\int_{E^n} \int_{E^n} M(t, 3(x+z), 3(y+w)) J_h(z) J_h(w) dz dw \leq K_3 M_h(t, x, y)$$

for all x, y, z, w such that $|x+z| \geq \delta_1(t)$ and $|y+w| \geq \delta_2(t)$ where $|z| \leq h$ and $|w| \leq h$ .By combining the above two inequalities, we achieve

$$M_h(t, 2x, 2y) \leq K K_3 M_h(t, x, y)$$

for all $|x| > \max(d_2, \delta_1(t) + h) = \delta_1'(t)$ and $|y| > \max(d_2, \delta_2(t) + h) = \delta_2'(t)$.

Since $\overline{M}(t, 2\delta_1(t), 2\delta_2(t))$ is integrable over T ,this yields the integrability of $\overline{M}_h(t, 2\delta_1'(t), 2\delta_2'(t))$ which proves the theorem.■

For each t in T and x,y in E^n it is known that

$$\lim_{h=0} M_h(t, x, y) = M(t, x, y).$$

However, the same property does not hold in general for $\hat{M}_h(t, x, y)$. This is the point of the next theorem.

Theorem 2.3:

For each $h > 0$ let x_0^h and y_0^h be the minimizing point of $M_h(t, x, y)$ defining $\hat{M}_h(t, x, y)$.Then for each t in T and each x, y in E^n ,there exists $K(t, x, y)$ such that

$$\lim_{h=0} \hat{M}_h(t, x, y) = M(t, x, y) + K(t, x, y) \lim_{h=0} |x_0^h| \lim_{h=0} |y_0^h|$$

Proof:

By the definition of $\hat{M}_h(t, x, y)$ we can write

$$\left| \hat{M}_h(t, x, y) - M(t, x, y) \right| \leq$$

(2.2.1)

$$\int_{E^n}\int_{E^n}\left|M(t,x+x_0^h+z,y+y_0^h+w)-M(t,x_0^h+z,y_0+w)-M(t,x,y)\right|$$

$$J_h(z)J_h(w)dzdw$$

However, we know that

$$\left|M(t,x+x_0^h+z,y+y_0^h+w)-M(t,x_0^h+z,y_0^h+w)-M(t,x,y)\right| \qquad (2.2.2)$$

$$\leq\left|M(t,x+x_0^h+z,y+y_0^h+w)-M(t,x,y)\right|$$

$$+\left|M(t,x_0^h+z,y_0^h+w)-M(t,z,w)\right|+\left|M(t,z,w)\right|.$$

Moreover , since $M(t,x,y)$ is a convex function, it satisfies a Lipshitz condition on compact subsets of E^n (see[4,Th.5.1]).Therefore ,there exists $K_1(t,x,y)$ and $K_2(t,x,y)$ such that

$$\left|M(t,x+x_0^h+z,y+y_0^h+w)-M(t,x,y)\right|\leq K_1(t,x,y)\left|x_0^h+z\right|\left|y_0^h+w\right|. \quad (2.2.3)$$

and

$$\left|M(t,x_0^h+z,y_0^h+w)-M(t,z,w)\right|\leq K_2(t,x,y)\left|x_0^h\right|\left|y_0^h\right|. \qquad (2.2.4)$$

If we combine (2.2.3) and (2.2.4) with (2.2.3) and if we substitute the resulting expression into (2.2.1), we achieve the inequality

$$\left|\hat{M}_h(t,x,y)-M(t,x,y)\right|\leq\left|x_0^h\right|\left|y_0^h\right|(K_1(t,x,y)+K_2(t,x,y))+$$

$$\int_{E^n}\int_{E^n}\left|x_0^h\right|K_1(t,x,y)|w|J_h(z)J_h(w)dzdw+\int_{E^n}\int_{E^n}\left|y_0^h\right|K_1(t,x,y)|z|J_h(z)J_h(w)dzdw+$$

$$\int_{E^n}\int_{E^n}K(t,x,y)|z||w|J_h(z)J_h(w)dzdw+\int_{E^n}\int_{E^n}M(t,z,w)J_h(z)J_h(w)dzdw$$

Since the last four integrals on the right side tend to zero as *h* tends to zero, we prove the theorem by setting

$$K(t,x,y)=K_1(t,x,y)+K_2(t,x,y)$$

Corollary 2.3:

Suppose $M(t,x,y)$ is a GN*-function such that $M(t,x,y)=M(t,-x,-y)$.

Then for each t in T and x, y in E^n ,we have

$$\lim_{h=0} M_h(t,x,y)=\hat{M}(t,x,y)$$

Proof:

This result is clear since $\lim_{h=0}|x_0^h|=0$ and $\lim_{h=0}|y_0^h|=0$

if $M((t,x,y)=M(t,-x,-y)$.In fact, if $M(t,x,y)$ is even in x and y then the $x_0^h=0$ and $y_0^h=0$ for all h.■

For each t in T let A_h denote the set of minimizing points of $M_h(t,x,y)$ and let B represent the null space of $M(t,x,y)$ relative to points in $E^n \times E^n$, i.e.,

$$B=\{(x,y) \quad in \quad E^n \times E^n : M(t,x,y)=0\}.$$

If $M(t,x,y)$ is a GN*-function, then *B={(0,0)}*. For the sake of argument, let us suppose that $M(t,x,y)$ has all the properties of a GN*-function except that $M(t,x,y)=0$ need not imply $x=0$ and $y=0$. We will show the relationships that exist between A_h and B. This is the content of the next few theorems.

Theorem 2.4:

The sets B and A_h are closed convex sets.

Proof:

This result follows from the convexity and continuity of $M(t,x,y)$ in x and y for each t in T.■

Theorem 2.5:

Let $B_e=\{(x,y): M(t,x,y)<e\}$ for each t in T. Then given any $e>0$, there is a constant $h_0>0$. such that $A_h \subset B_e$ for each $h \le h_0$.

Proof:

Since $B \subseteq B_e$, we can choose h_0 sufficiently small so that if (x, y) is in B then $(x+z, y+w)$ is in B_e for all (z,w) such that $|z| \le h_0$ and $|w| \le h_0$. Let z_1 and w_1 be arbitrary but fixed points in $A_h, h \le h_0$. Then

$$M_h(t, z_1, w_1) \le M_h(t, x, y)$$

for all x and y. Therefore, if *(x,y)* in *B*, we have $M_h(t, z_1, w_1) < e$ by our choice of h_0. Letting *h* tend to zero yields $M(t, z_1, w_1) < e$, i.e., (z_1, w_1) in B_e.

We have commented above that $A_h = \{(0,0)\}$ if

$$M(t, x, y) = M(t, -x, -y).$$

It is also true if *M(t,x,y)* is strictly convex in *x* for each *t* in *T*.

Theorem 2.5:

Suppose *M(t,x,y)* is a GN*-function which is strictly convex in *x* and *y* for each *t*. Then $h, A_h = \{(0,0)\}$ for each *h*.

<u>**Proof:**</u>

Suppose that there exists $z_0 \ne x_0$ and $w_0 \ne y_0$ such that $x_0, y_0, z_0 and w_0$ are in A_h. Let $z_1 = \frac{(x_0 + z_0)}{2}$, $w_1 = \frac{(y_0 + w_0)}{2}$. Then, since $M(t, x, y)$ is strictly convex, $M_h(t, x, y)$ is strictly convex in *x* and *y*, therefore, we have

$$M_h(t, z_1, w_1) < \frac{1}{2} M_h(t, x_0, y_0) + \frac{1}{2} M_h(t, z_0, w_0). \qquad (2.5.1)$$

However, $(x_0, y_0), (z_0, w_0)$ are in A_h reduces (2.5.1) to the inequality $M_h(t, z_1, w_1) < M_h(t, x, y)$ for all *x* and *y*. This means z_1 and w_1 are in A_h and $(x_0, y_0), (z_0, w_0)$ are not in A_h which is a contradiction. Hence, $x_0 = z_0$, $y_0 = w_0$. Since $M(t, x, y)$ is a GN*function, $B = \{(0,0)\}$. In this case $x_0 = y_0 = 0$, $z_0 = w_0 = 0$.

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